189-251B: Algebra 2
Assignment 6

Due: Wednesday, February 19.

1. Let $T : V \longrightarrow V$ be a nilpotent linear transformation, and let

$$d_j := \dim(\ker(T^j)), \qquad d'_j := d_{j+1} - d_j.$$

(a) Show that the sequence d_j is increasing, so that $d'_j \geq 0$ for all j.

(b) Show that the sequence d'_j is (non-strictly) *decreasing* and tends to zero.

(c) Show that $T^j = 0$ if and only if $d'_j = 0$.

2. Compute the minimal and characteristic polynomials of the linear transformation $T(f) = f'$ acting on the following finite-dimensional vector spaces V of continuous functions on the interval $[0, 1]$.

(a) V= the real vector space of polynomials of degree at most d with real coefficients (with d some fixed integer).

(b) The real vector space spanned by the real-valued functions $\sin(x)$, $\cos(x)$, $\sin(2x)$, $\cos(2x)$.

(c) The complex vector space spanned by the functions $\sin(x)$, $\cos(x)$, $\sin(2x)$, $\cos(2x)$ (viewed here as complex valued functions on $[0, 1]$.)

(d) The real vector space spanned by the functions e^x, xe^x, x^2e^x, e^{2x}.

3. Compute the characteristic and minimal polynomials of the following matrices with entries in the field $\mathbf{Z}_2$ with two elements.

(a) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

4. Write down the eigenspaces and generalised eigenspaces in $\mathbf{Z}_2^4$ for the matrices in question 3.

5. An *linear involution* on a vector space V over a field F is a linear transformation whose square is the identity. Show that any involution is diagonalisable when $2 \neq 0$ in F. Give an example of a non-diagonalisable involution on a vector space over $\mathbf{Z}_2$.