189-251B: Algebra 2
Assignment 5

Due: Wednesday, February 12.

1. Suppose that $T : V \longrightarrow V$ is a linear transformation of vector spaces over $\mathbf{R}$ whose minimal polynomial has no multiple roots. Show that V can be expressed as a direct sum

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$$

of T-stable subspaces of dimensions at most 2. Show that, relative to a suitable basis, T can be represented by an $n \times n$ matrix with at most $2n$ non-zero entries, where $n := \dim(V)$.

2. Let $g(x)$ be a polynomial in $F[x]$ of degree d and let V be the ring $F[x]/(g(x))$.

(a) Show that V is a d-dimensional vector space over F.

(b) Let T be the function on V defined by the rule $T(v) = [x]v$. (Where $[x]$ denotes the equivalence class of x in the quotient ring.) What is the minimal polynomial of T? When is T diagonalisable?

3. Let A be an upper-triangular matrix with entries in a field F. Show that A is diagonalisable if all its diagonal entries are distinct.

4. Let A be an upper-triangular matrix with entries in a field F. Suppose that all the diagonal entries of A are equal. Show that A is diagonalisable if and only if it is diagonal.

5. Let A be an invertible matrix with entries in $F = \mathbf{Z}_p$. Show that A is diagonalisable if and only if its order (i.e., the least t with $A^t = 1$ in the group $\mathbf{GL}_n(\mathbf{Z}_p)$ of $n \times n$ matrices with entries in $\mathbf{Z}_p$) divides $p - 1$.

6. Let V be the (infinite-dimensional) vector space of infinitely differentiable real-valued functions on the real line, and let W be the subset of V consisting

of functions f satisfying the differential equation

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0,$$

where the scalars a_j are real. (Here, $f^{(j)}$ denotes the j-th derivative of f.)

(a) Show that W is a vector subspace of V. (This is why this equation is called a *linear* differential equation!)

(b) Let T be the function from V to V defined by $T(f) = f'$. Show that T is a linear transformation that maps W to itself.

(c) From now on, use T to denote the restriction of T to the space W. Show that the eigenvalues of T are precisely the roots of the polynomial $g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

(d) Show that the eigenspace for T attached to an eigenvalue λ is *always* one-dimensional.

(e) Suppose that $(x-\lambda)^r$ divides $g(x)$ exactly (i.e., no higher power of $(x-\lambda)$ divides it). Show that the generalised eigenspace for T attached to the eigenvalue λ is of dimension r and write down a basis $f_1, \ldots, f_r$ for this generalised eigenspace such that f_j belongs to the kernel of $(T-\lambda)^j$ but not of $(T-\lambda)^{j-1}$. (You may use without proof all that you know about linear differential equations of first order.)