189-251B: Algebra 2
Assignment 4

Due: Wednesday, February 5.

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 15 & -4 & -2 \\ 27 & -8 & -3 \\ 58 & -14 & -9 \end{pmatrix}$ with entries in $\mathbf{R}$. Show that A is diagonalisable. What if the entries of A were taken to be in the field $\mathbf{Z}_p$, where p is a prime number?

2. Suppose T_1 and T_2 are commuting linear transformations on an F-vector space V of dimension n. Suppose that T_2 is diagonalisable and has n *distinct* eigenvalues. Show that T_1 is also diagonalisable. Show with an example that the assumption on the eigenvalues of T_2 being distinct is essential for the statement to be true.

3. Three brands (denoted A, B, and C) of a common consumer product with a weekly purchase cycle vie for shares of a fixed-size market. It has been observed that among the consumers who purchase brand A in a given week, 70% remain loyal to that brand the following week, while 20% switch over to Brand B and 10% to Brand C. Among the purchasers of Brand B, only 40% remain loyal to their brand in the following week, the remaining 60% switching in equal proportions to Brands A and C. Half of the purchasers of Brand C remain loyal to their brand, 20% purchase brand A, and 30% opt for Brand B. This pattern of consumer behaviour remains constant from one week to the next. This (somewhat reductive, and simplistic!) model of consumer behaviour relative to Brands A, B and C is usefully summarised in the so-called *brand transition matrix*

$$M = \begin{pmatrix} 0.7 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}.$$

a. Show that if x, y, and z denote the number of consumers purchasing brands A, B and C in the initial (zeroth) week, the number of consumers purchasing those brands in the nth week will be (x_n, y_n, z_n), where

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

b. Show that the matrix M is diagonalisable. Compute its eigenvalues and eigenvectors.

c. Find a diagonal matrix D and an invertible matrix P with the property that $M = PDP^{-1}$. Use this calculation to show that the entries of the matrix M^n converge to certain limits in $\mathbf{R}$.

d. Show that, as time passes, the shares of Brands A, B and C will stabilise, regardless of the numbers of consumers purchasing those brands in the initial week. Compute the values of the stable market shares.

4. Alice, Bob, Charles, Dan and Eve share a five bedroom appartment in the McGill ghetto. The electrician who was hired to renovate the appartment has completely messed up the wiring in the light switches, and as a result:

1. When Alice switches the light in her room on or off, this also switches the light settings in Bob's, Dan's and Eve's rooms.

2. When Bob switches his light on or off, this also switches the light setting in Dan's room.

3. When Charles switches his light on or off, this also switches the light settings in Alice and Dan's rooms.

4. When Dan switches his light on or off, this also switches the light settings in Alice and Eve's rooms.

5. When Eve switches her light on or off, this also switches the light settings in Alice, Charles and Dan's rooms.

The five roomates all wake up at 7:30 (to be on time for their 8:30 algebra class). Given that all the lights are off at that time, which roommates should flip their switches so that the lights in all five rooms are turned on?
(Hint: Think of a configuration of lights in the appartment as a vector in $\mathbf{Z}_2^5$.)