189-235A: Basic Algebra I
Supplemental Final Exam

Tuesday, May 2, 2006

This exam has ten questions, worth 10 points each. The final grade will be out of 100*.*

1. Solve each of the following congruence equations.
   a) $5x = 3 \pmod{30}$,
   b) $x^2 + 4x + 2 = 0 \pmod{7}$.
   c) How many distinct solutions does the equation $x^{10} = 1 \pmod{101}$ have? (You do not need to write them down.)

2. Let G be a finite group, and let H be a subgroup of G. Show that the cardinality of H divides the cardinality of G.

3. Prove that the ring $\mathbf{R}$ of real numbers and the ring $\mathbf{C}$ of complex numbers are not isomorphic.

4. List the powers $[x]^j$ (for $0 \leq j \leq 7$) of $[x] = x + (x^3 + x + 1)\mathbf{Z}_2[x]$ in the quotient ring $\mathbf{Z}_2[x]/(x^3 + x^2 + 1)$. You should label each result by its unique representative of degree ≤ 2. Using this calculation, write down the multiplicative inverse of $[x]$.

5. Show that there are (up to isomorphism) exactly 3 distinct rings of cardinality 9.

6. Give the definitions of: *maximal ideal* and *prime ideal* in a commutative ring R. Show that if I is a prime ideal then R/I is an integral domain.

7. Write down the class equation for the dihedral group D_7 with 14 elements (i.e., the symmetry group of the regular polygon with 7 sides.) Use this to give a list of all the normal subgroups of D_7.

8. Let n be an integer. Construct a group G containing two elements a and b of order 2 whose product is of order n.

9. Write down three non-isomorphic groups of order 12. (You *should* explain why these groups are non-isomorphic.)

10. Let p be a prime and let G be the set of functions $f : \mathbf{Z}_p \longrightarrow \mathbf{Z}_p$ of the form $f(x) = ax + b$, where $a \in \mathbf{Z}_p^\times$ and $b \in \mathbf{Z}_p$.

a) Show that G is a group under the composition of functions. What is its identity element?

b) Show that G has a *unique* normal subgroup H of order p.

c) Show that the quotient G/H is a cyclic group of order $p - 1$.