189-235A: Algebra I
Assignment 4

Due: Monday, November 4

1. Let F be a (commutative) field. Show that the set R of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, with $a, b, c \in F$, is a subring of the ring $M_2(F)$ of 2×2 matrices with entries in F. Write down a non-trivial two-sided ideal I of R. Describe, in as concrete and explicit a way as you possibly can, the quotient ring R/I. (Your description should be concrete and explicit enough, for example, to make the fact that R/I is a *commutative* ring completely transparent.)

2. Show that the ring $\mathbf{Z}$ of integers has no subrings other than itself. (We recall that according to the conventions in this course, a ring contains neutral elements for addition, and for multiplication, and that these two distinguished elements are supposed to be *distinct*.)

3. Let n be an integer.
(a) Prove that there is a unique homomorphism from $\mathbf{Z}$ to $\mathbf{Z}/n\mathbf{Z}$.
(b) Prove that there is no homomorphism from $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{Z}$.
(c) What conditions do two integers n and m need to satisfy for there to be a homomorphism from $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{Z}/m\mathbf{Z}$?

4. Show that the ideal generated by 3 and $1+\sqrt{-5}$ is not a principal ideal in the ring $\mathbf{Z}[\sqrt{-5}]$.

5. Show that the ideal generated by 5 and $1-8i$ is a principal ideal in the ring $\mathbf{Z}[i]$ of Gaussian integers, and write down a generator.

6. Show that the ring $\mathbf{Z}/2\mathbf{Z}[x]/(x^4+x+1)$ is a field.

7. Find the multiplicative inverse of the element $[x^2+1]$ in the field $\mathbf{Z}/2\mathbf{Z}[x]/(x^4+x+1)$ of the previous exercise.

8. Let F be a field, let $R_1 = F[x]$ be the ring of polynomials with coefficients in F, and let R_2 be the ring of all functions from F to itself, with addition and multiplication defined as the usual operations on functions with values in a ring. Show that the function

$$\varphi : R_1 \longrightarrow R_2$$

which sends the polynomial $f \in F[x]$ to the F-valued function $a \mapsto f(a)$ on F which it induces, is a homomorphism of rings.

9. When $F = \mathbf{Q}$ or $\mathbf{R}$, show that the homomorphism φ of exercise 8 is injective but not surjective[1].

10. When $F = \mathbf{Z}/p\mathbf{Z}$ is the finite field with p elements, show that the homomorphism φ of exercise 8 is not injective. What is its kernel? Show that φ is surjective in this case.

[1] The injectivity of φ holds for any field F of infinite cardinality, and your proof should give this more general statement without too many efforts. The failure of surjectivity is also true for any infinite field F, but is a bit more tricky to establish. You may like to try your hand at doing this!