Math 189-726: Topics in Number Theory

Final Exam

Monday, December 5, 10:00 AM-1:00 PM

The 6 *questions in this exam are each worth* 20 *points, for a maximum possible total of 120, but your final grade will be out of 100.*

This means that you can get full marks in theory by doing 5 out of the 6 problems. But you are advised to attempt all 6, since I will be generous with the partial credit.

Calculators and notes are not allowed – and would probably not be useful in any case!

Notations. Throughout, we will write $z = x+iy$, with $x \in \mathbf{R}$ and $y \in \mathbf{R}^{>0}$ for a variable in the Poincaré upper half-plane $\mathcal{H}$, while s will typically denote a variable in the entire complex plane.

1. Show that, for all $s \in \mathbf{C}$ with $\Re(s) > 1$,

$$\int_0^\infty \frac{t^s}{e^t+1}\frac{dt}{t} = \Gamma(s)(1-2^{1-s})\zeta(s),$$

where

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t}, \qquad \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

are the Gamma-function and the Riemann zeta-function respectively.

Use this to show that $\zeta(s)$ extends to a meromorphic function on $\mathbf{C}$ having a pole of order one at $s = 1$.

2. Let $f \in M_k(\mathbf{SL}_2(\mathbf{Z}))$ be a modular form of weight $k \geq 4$, and denote by

$$f(q) = \sum_{n=0}^{\infty} a_n q^n \in \mathbf{C}[[q]], \qquad q = e^{2\pi i z}$$

its Fourier expansion ("q-expansion".)

(a) If $x \mapsto x^\sigma$ is a (not necessarily continuous!) automorphism of $\mathbf{C}$, show that the q- series

$$f^\sigma := \sum_{n=0}^{\infty} a_n^\sigma q^n$$

is also the q-expansion of a modular form in $M_k(\mathbf{SL}_2(\mathbf{Z}))$. You may use without proof any of the basic facts about modular forms and their q-expansions covered in class.

(b) Use (a) to show that if the coefficients $a_1, \ldots, a_n, \ldots$ are all rational, then the same is true of a_0.

Cultural remark, not relevant for the exam so if you are pressed for time just skip it and read it later: If F is any totally real number field of degree d over $\mathbf{Q}$, it is possible to construct a modular form $f \in M_{dk}(\mathbf{SL}_2(\mathbf{Z}))$ satisfying $a_n(f) \in \mathbf{Q}$ for all $n \geq 1$, and $a_0(f) = \zeta_F(1-k)$, where ζ_F is the Dedekind zeta-function of F. Part (b) applied to such a modular form yields the rationality of the values of the Dedekind zeta-function at negative integers which was originally established by Klingen and Siegel.

3. Let $f \in M_k(\mathbf{SL}_2(\mathbf{Z}))$ be a modular form as in exercise 2, and assume that it is a simultaneous eigenform for all the Hecke operators. Show that, for any prime ℓ, the q-series

$$\tilde{f} := \sum_{(\ell,n)=1} a_n q^n$$

is the q-expansion of a modular form of weight k on the Hecke congruence group $\Gamma_0(\ell^2)$.

4. Let $f = \sum a_n q^n$ be a cusp form of weight k on $\mathbf{SL}_2(\mathbf{Z})$.

(a) Show that the function $z \mapsto y^k |f(z)|^2$ is invariant under Möbius transformations by all matrices in $\mathbf{SL}_2(\mathbf{Z})$.

(b) Show that there is a constant C_f (depending possibly on f) for which

$$|f(z)| \leq C_f y^{-k/2}.$$

(c) Use (b) to show that there is a constant C'_f for which

$$|a_n| < C'_f n^{k/2}.$$

5. Let $E(z,s)$ be the non-holomorphic Eisenstein series defined as a function on $\mathcal{H} \times \{s : \Re(s) > 1\}$ by

$$E(z,s) = \frac{1}{2} \sum_{(m,n) \neq 0} \frac{y^s}{|mz+n|^{2s}}.$$

(a) Show that, for all s with $\Re(s) > 1$,

$$E(z,s) = \zeta(2s) \sum_{\gamma \in T \backslash \mathbf{SL}_2(\mathbf{Z})} y(\gamma(z))^s,$$

where T denotes the subgroup of upper-triangular matrices in $\mathbf{SL}_2(\mathbf{Z})$. Conclude that $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \mathbf{SL}_2(\mathbf{Z})$.

(b) Let $g(z)$ be a smooth function on $\mathbf{SL}_2(\mathbf{Z}) \backslash \mathcal{H}$ of rapid decay at ∞ and having a Fourier expansion of the form

$$g(z) = \sum_{n=0}^{\infty} a_n(y) e^{2\pi i n z},$$

where the coefficients $a_n(y)$ are smooth $\mathbf{C}$-valued functions on $\mathbf{R}^{>0}$. Using the formula in part (a), show that

$$\int_{\mathbf{SL}_2(\mathbf{Z}) \backslash \mathcal{H}} g(z) E(z,s) \frac{dx\,dy}{y^2} = \zeta(2s) \int_0^{\infty} a_0(y) y^{s-2} dy.$$

6. Let μ be a $\mathbf{Q}_p$-valued measure on $\mathbf{Z}_p$.

(a) Write down the definition of the Amice transform A_μ of μ.

(b) If $\mu = \delta_a$ is the Dirac measure at $a \in \mathbf{Z}_p$, show that

$$A_{\delta_a}(T) = (1+T)^a.$$

(c) Given a measure μ, let μ' denote its restriction to $p\mathbf{Z}_p$, defined by

$$\int_{\mathbf{Z}_p} f(x) d\mu'(x) := \int_{\mathbf{Z}_p} \mathbf{1}_{p\mathbf{Z}_p}(x) f(x) d\mu(x),$$

where $\mathbf{1}_{p\mathbf{Z}_p}$ is the characteristic function of $p\mathbf{Z}_p$. Show that

$$A_{\mu'}(T) = \frac{1}{p} \sum_{\zeta^p = 1} A_\mu(\zeta(1+T) - 1),$$

where the sum is taken over all the p-th roots of unity.

(d) Use (b) and (c) to check that

$$A_{\delta'_a}(T) = \begin{cases} 0 & \text{if } a \in \mathbf{Z}_p^\times, \\ (1+T)^a = A_{\delta_a}(T) & \text{if } a \in p\mathbf{Z}_p. \end{cases}$$