189-726A: L-functions and Modular Forms

Assignment 5

Due: Monday, November 21

Note: You may hand in this last assignment one week late, on November 28, but *no later than that.*

1. Let $f = \sum_n a_n q^n$ be a normalised Hecke eigenform of weight k on $\mathbf{SL}_2(\mathbf{Z})$.
a) Show that

$$\sum_n a_n^2 n^{-s} = \zeta(s-k+1) \sum_n a_{n^2} n^{-s}.$$

b) Letting

$$\langle f, g \rangle := \int_{\mathbf{SL}_2(\mathbf{Z})\backslash\mathcal{H}} y^k f(z)\overline{g(z)} \frac{dxdy}{y^2}$$

denote the Petersson scalar product on forms of weight k, show that

$$\langle f, f \rangle = \frac{\pi(k-1)!}{3(4\pi)^k} \sum_{n=1}^{\infty} a_{n^2} n^{-s}|_{s=k}.$$

2. Let K be a finite extension of $\mathbf{Q}_p$ and let R denote its ring of integers. Let

$$\rho : G_{\mathbf{Q}} \longrightarrow \mathbf{GL}_2(\bar{K})$$

be a continuous odd representation, satisfying

$$\text{char.poly}(\rho(\text{frob}_p)) \text{ belongs to } R[x],$$

for all primes p at which ρ is unramified.

Write

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}, \qquad \sigma \in G_{\mathbf{Q}}.$$

a) Show that ρ can be conjugated in such a way that $a(\sigma)$, $d(\sigma)$, and $b(\sigma)c(\tau)$ belong to R for all $\sigma, \tau \in G_{\mathbf{Q}}$.

b) After normalising ρ as in *a*), let B be the ideal of R generated by the expressions $b(\sigma)c(\tau)$ as σ, τ run over $G_{\mathbf{Q}}$. Show that ρ has a conjugate taking values in $\mathbf{GL}_2(R)$ when $B \neq 0$. Show by an example that the conclusion need not hold when $B = 0$.

c) Assume that $B \neq 0$ and that ρ is conjugated as in *b*). Show that the reduction

$$\bar{\rho} : G_{\mathbf{Q}} \longrightarrow \mathbf{GL}_2(R/m_R R)$$

of ρ modulo the maximal ideal m_R of R is irreducible if and only if $B = R$.

d) Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalised Hecke eigenform of weight k, level D and character ϵ, with fourier coefficients in a field K_f. Choose a prime ideal λ of K_f and assume that the function

$$p \mapsto a_p \pmod{\lambda}$$

is not a sum of two characters. Let $K_{f,\lambda}$ be the completion of K_f at λ and let $\mathcal{O}_{f,\lambda}$ be its ring of integers. Show that there is a continuous Galois representation

$$\rho_f : G_{\mathbf{Q}} \longrightarrow \mathbf{GL}_2(\mathcal{O}_{f,\lambda})$$

satisfying

$$\text{char.poly}(\rho_f(\text{frob}_p)) = x^2 - a_p x + \epsilon(p) p^{k-1},$$

for all primes p not dividing $D\text{Norm}(\lambda)$. (You may of course use any of the theorems on the existence of ℓ-adic representations attached to cusp forms that was stated in class.)