189-726A: L-functions and Modular Forms

Assignment 4

Due: Monday, November 7

1. Let $f_i = \sum_n a_n^{(i)} q^n$ (for $i = 1, \ldots, r$) be r Hecke eigenforms on $\mathbf{SL}_2(\mathbf{Z})$ (of possibly different weights.) Show that the Dirichlet series

$$D(f_1, \ldots, f_r, s) := \sum_{n=1}^{\infty} a_n^{(1)} \cdots a_n^{(r)} n^{-s}$$

has an Euler product factorisation of the form $\prod_p R_p(p^{-s})$, where R_p is a rational function. What can you say about the degree of R_p?

2. (*This exercise is taken from p. 41 of Zagier's notes on modular forms*).

For any non-negative integer n and $k > 2$, define the *n-th Poincaré series of weight k* by

$$P_n(z) = \sum_{(c,d) \in (\mathbf{Z}^2)'} (cz + d)^{-k} e^{2\pi i n \frac{az+b}{cz+d}},$$

where $(\mathbf{Z}^2)'$ denotes the set of pairs $(c, d) \in \mathbf{Z}^2$ with $\gcd(c, d) = 1$, and for any such (c, d) the pair (a, b) is chosen (arbitrarily) in such a way that $ad - bc = 1$.

a) Show that this series converges absolutely and defines a modular form of weight k on $\mathbf{SL}_2(\mathbf{Z})$.

b) Show that, if $f = \sum a_n q^n$ is any cusp form of weight k on $\mathbf{SL}_2(\mathbf{Z})$, then

$$\langle P_n, f \rangle_k = (k - 2)!(4\pi n)^{1-k} a_n,$$

where $\langle \ , \ \rangle_k$ denotes the Peteersson scalar product on forms of weight k, as defined in class.