189-726A: L-functions and Modular Forms

Assignment 2

Due: Monday, October 10

1. Let Γ be a congruence subgroup of $\mathbf{SL}_2(\mathbf{Z})$ and let $f(z)$ be a modular form on Γ of weight two.

a) Choose a point z_0 in the complex upper half-plane. Show that the function $m_f : \Gamma \longrightarrow \mathbf{C}$ defined by

$$m_f(\gamma) = \int_{z_0}^{\gamma z_0} f(z)dz$$

does not depend on the choice of z_0, and is a *homomorphism* from Γ to $\mathbf{C}$.

b) Show that the function $f \mapsto m_f$ from $M_2(\Gamma)$ to $\hom(\Gamma, \mathbf{C})$ is injective.

c) Use a) and b) to show that $M_2(\Gamma) = 0$ if Γ has no infinite abelian quotients.

d) Show that the abelianisation of $\mathbf{SL}_2(\mathbf{Z})$ is a finite group of order 12, and conclude that $M_2(\mathbf{SL}_2(\mathbf{Z})) = 0$.

2. Let $f = \sum_n a_n q^n$ be a cusp form of weight k on $\Gamma_0(N) \subset \mathbf{PSL}_2(\mathbf{R})$.

a) Show that the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ belongs to the normaliser of $\Gamma_0(N)$ in $\mathbf{PSL}_2(R)$. Conclude that the transformation

$$f \mapsto N^{-k/2} z^{-k} f(\frac{-1}{Nz})$$

is an involution on $S_k(\Gamma_0(N))$.

b) Assume that f is an eigenvector for this involution, with associated eigenvalue $w_f \in \{1, -1\}$. Show that the L-series $L(f, s)$ extends to an analytic function of $s \in \mathbf{C}$ and satisfies a functional equation relating its values at s and $k - s$. (Hint: follow the approach explained in class in the case where $N = 1$, with the involution of part a) playing the role of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.)

c) Explain how to evaluate the central critical value $L(f, k/2)$ numerically.

d) How many fourier coefficients of f would you need to know in order to evaluate $L(f, k/2)$ with a numerical error of at most $\epsilon > 0$? (Give your answer as a function of N, k, and ϵ.)