189-726A: L-functions and Modular Forms

Assignment 1

Due: Wednesday, September 28

1. Let F be a number field. Show that the Dedekind zeta-function of F and the Artin L-function attached to the induced representation

$$\rho := \mathrm{Ind}_F^{\mathbf{Q}} 1 : G_{\mathbf{Q}} \longrightarrow \mathbf{GL}_d(\mathbf{C}), \qquad d = [F : \mathbf{Q}]$$

are equal.

2. Prove the functional equation $\Gamma(s+1) = s\Gamma(s)$, where $\Gamma(s)$ is the Γ-function, defined for $\Re(s) > 0$ by the integral $\int_0^\infty e^{-t} t^s \frac{dt}{t}$. Show that $\Gamma(s)$ extends to a meromorphic function on $\mathbf{C}$ which is holomorphic except at the non-positive integers, where it has simple poles. What is the residue of $\Gamma(s)$ at $s = -n$, for $n \geq 0$?

3. Show that the integral

$$\int_0^\infty \omega(t) t^{s/2} \frac{dt}{t}$$

converges absolutely when $\Re(s) > 1$.

4. Let $\tau(\chi)$ be the Gauss sum attached to a primitive Dirichlet character χ of conductor q. Show that $|\tau(\chi)| = \sqrt{q}$ and that $\tau(\bar{\chi}) = \chi(-1)\overline{\tau(\chi)}$.