First Example of Theta Correspondence

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Computation of the Weil Representation

Recall that in the local case (i.e. for W any symplectic space defined over a local field) we have the metaplectic group $\widetilde{Sp}(W)$ defined as a the set:

$$\widetilde{Sp}(W) := \{(g,t) \in Sp(W) \times \mathbb{C}^\times\}$$

with operation

$$(g_1,t_1)(g_2,t_2) = (g_1 g_2, \alpha(g_1,g_2)t_1 t_2) \quad , \quad \alpha \in H^2(Sp(W), \mathbb{C}^\times)$$

In the case at hand, $Sp(W)(\mathbb{Q}_p) = SL_2(\mathbb{Q}_p)$, one can realize the metaplectic group as an extension by $\{\pm 1\}$:

$$0 \to \{\pm 1\} \to \widetilde{SL_2}(\mathbb{Q}_p) \to SL_2(\mathbb{Q}_p) \to 0$$

In other words, we have:

$$\widetilde{SL_2}(\mathbb{Q}_p) := \{(g,\epsilon) : g \in SL_2(\mathbb{Q}_p), \epsilon = \pm 1\}$$

and α can be taken to be

$$\begin{aligned} \alpha : SL_2(\mathbb{Q}_p) \times SL_2(\mathbb{Q}_p) &\to \{\pm 1\} \\ (g_1,g_2) &\longmapsto (x(g_1), x(g_2))(-x(g_1)x(g_2), x(g_1 g_2)) \end{aligned}$$

where $(. \ , \ .)$ is the Hilbert symbol on $\mathbb{Q}_p$ and

$$x\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases}$$

Note that in general, unless everything is a square (say in $F = \mathbb{C}$) then this extension is not split, since the cocycle will not be trivial.

We want to describe the Weil Representation on $SL_2(\mathbb{Q}_p)$, for each place p including ∞. In this case $W(\mathbb{Q}_p) \cong \mathbb{Q}_p \times \mathbb{Q}_p$ and $W_1 \cong W_2 \cong \mathbb{Q}_p$ as vector spaces. Fix a charachter $\psi_p : \mathbb{Q}_p^+ \to \mathbb{C}^\times$. Denote by $\omega_p := \omega_{\psi_p}$ the Weil representation of $SL_2(\mathbb{Q}_p)$ on the vector space $S(\mathbb{Q}_p)$ of locally constant, compactly supported functions $\phi : \mathbb{Q}_p \to \mathbb{C}$ (for $p = \infty$, we consider the space of Schwartz functions, i.e. smooth functions on $\mathbb{R}$ such that the derivatives of all orders decrease rapidly at $\pm\infty$).

The group $SL_2(\mathbb{Q}_p)$ is generated by three types of matrices, which we denote by

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} , \ B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} , \ W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

.

Therefore $\widetilde{SL_2}(\mathbb{Q}_p)$ is generated by all the matrices of the form $(A, \pm 1), (B, \pm 1), (W, \pm 1)$. We compute the Weil Representation for each type $(A, 1), (B, 1), (C, 1)$.

Type 1: Elements of the form $(A, 1)$ with $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $a \in \mathbb{Q}_p^\times$. Then if $\phi(x) \in S(\mathbb{Q}_p)$ we have

$$\omega_p \left(\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1 \right) \right) \phi(x) = |a|^{1/2} \phi(ax)$$

Type 2: Elements of the form $(B, 1)$ with $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $b \in \mathbb{Q}_p$. Then

$$\omega_p \left(\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1 \right) \right) \phi(x) = \psi_p \left(\frac{bx^2}{2} \right) \phi(x)$$

Type 3: The element $(W, 1)$. We have

$$\omega_p \left(\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \right) \phi(x) = \gamma \widehat{\phi}(x)$$

where $\widehat{\phi}(x)$ is the Fourier Transform of ϕ:

$$\widehat{\phi}(x) = \int_{\mathbb{Q}_p} \phi(y) \psi_p(x \cdot y) dy$$

(the additive Haar measure dy is chosen so that $\widehat{\widehat{\phi}(x)} = \phi(-x)$) and γ is a certain 8-th root of unity whose explicit computation will not be necessary in what follows.

We want to piece together all these representations over $\mathbb{Q}_p$ and obtain a representation of $\widetilde{SL_2(\mathbb{A})}$. For each finite p, let $\phi_p^0 : \mathbb{Q}_p \to \mathbb{C}$ be the characteristic function of $\mathbb{Z}_p$, i.e.

$$\phi_p^0(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p \\ 0 & \text{if } x \notin \mathbb{Z}_p \end{cases}$$

Note that ϕ_p^0 is locally constant and compactly supported for each p, therefore $\phi_p^0 \in S(\mathbb{Q}_p)$. Moreover, one can show that

$$\omega_p(g) \phi_p^0(x) = \phi_p^0(x) \tag{1}$$

for every $g \in SL_2(\mathbb{Z}_p)$.

The global Weil representation acts on the space $S(\mathbb{A})$ of 'Schwartz-Bruhat' functions $\Phi(X) = \prod_p \Phi_p(X_p)$ where $X = \prod_p X_p$, $\Phi_p \in S(\mathbb{Q}_p)$ for each p and moreover $\Phi_p = \phi_p^0$ for almost all the finite p.

For each finite place p, choose a character $\psi_p : \mathbb{Q}_p \to \mathbb{C}$ such that the conductor of ψ_p is $\mathbb{Z}_p$ for almost all p. For $p = \infty$, let $\psi(x) = e^{2\pi i x}$. We also choose the ψ_p such that $\psi = \prod_p \psi_p$ is trivial on $\mathbb{Q}$, which gives a character on $\mathbb{A}/\mathbb{Q}$.

By tensoring the local Weil representations we get a global representation

$$\omega_\psi(g)\Phi(X) = \otimes_p \omega_{\psi_p}(g_p)\Phi_p(X_p) \quad , \quad g \in SL_2(\mathbb{A})$$

on the space $S(\mathbb{A})$ of Schwarts-Bruhat functions. Note that the product is well-defined: since almost all g_p are contained in $SL_2(\mathbb{Z}_p)$ and almost all Φ_p are equal to ϕ_p^0, it follows from (1) that almost all the components of $\omega_\psi(g)\Phi(X)$ are ϕ_p^0, which is the 'global' requirement of membership in $S(\mathbb{A})$.

Recovering the classical Theta function

We now show how the general theory of theta functions generalizes the construction of the classical theta function $\theta : \mathbb{H} \to \mathbb{C}$ given by:

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$$

The first step is to view the upper-half plane as the quotient $\mathbb{H} = SL_2(\mathbb{R})/SO(2)$. Explicitly, the isomorphism is given by

$$z = x + iy \longmapsto \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} = A(x, y)$$

i.e. z is mapped to the element $A(x, y)$ such that $A(x, y)i = z$.

Define $\Phi(X) \in S(\mathbb{A})$ by

$$\Phi_p(X_p) = \begin{cases} \phi_p^0(X_p) & \text{if } p < \infty \\ e^{-\pi(X_\infty)^2} & \text{if } p = \infty \end{cases}$$

and let $g \in SL_2(\mathbb{A})$ be of the form $g_p = I_2$ for all finite p, and

$$g_\infty = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}$$

with $x, y \in \mathbb{R}$, $y > 0$. Then the theta function

$$\Theta_\Phi(g)(X) = \sum_{X \in \mathbb{A}} \omega_\psi(g)\Phi(X)$$

reduces to $\theta(z)$.

To see this, note that for all finite p,

$$\omega_p(g_p)\Phi_p(X_p) = \Phi_p(X_p)$$

since g_p is the identity.

At $p = \infty$, we have

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$$

so we can use the formulas derived in the previous section. We compute

$$\omega_\infty \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \Phi_\infty(X_\infty) = |y|^{1/4} e^{-\pi y (X_\infty)^2} = \rho(X_\infty)$$

and therefore

$$\begin{aligned} \omega_\infty(g_\infty)\Phi_\infty(X_\infty) &= \omega_\infty \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \rho(X_\infty) \\ &= \psi_\infty \left(\frac{x(X_\infty)^2}{2} \right) \rho(X_\infty) \\ &= |y|^{1/4} e^{\pi i x (X_\infty)^2} e^{-\pi y (X_\infty)^2} \\ &= |y|^{1/4} e^{\pi i (X_\infty)^2 (x+iy)} \end{aligned}$$

We can now rewrite the theta function as

$$\Theta_\Phi(g)(X) = \sum_{X \in \mathbb{A}} \left(\prod_{p<\infty} \phi_p^0(X_p) \right) \cdot |y|^{1/4} e^{\pi i (X_\infty)^2 (x+iy)}$$

Note that

$$\prod_{p<\infty} \phi_p^0(X_p) = \begin{cases} 1 & \text{if } X_p \in \mathbb{Z}_p \quad \forall p \\ 0 & \text{otherwise} \end{cases}$$

which means that the only terms that contribute to the sum are those $X \in \mathbb{A}$ with $X \in \mathbb{Z}$. Therefore we can rewrite the sum as:

$$\Theta_\Phi(g)(X) = |y|^{1/4} \sum_{n \in \mathbb{Z}} |y|^{1/4} e^{\pi i n^2 (x+iy)} = |y|^{1/4} \theta(x+iy)$$

But for each $z = x + iy \in \mathbb{H}$ we can find a suitable g, therefore the function can be pulled back to all of $\mathbb{H}$.