189-346/377B: Number Theory

Assignment 5

Due: Monday, March 12

1. An integer n is said to be *square-free* if its prime factorisation is of the form

$$n = p_1 p_2 \cdots p_r,$$

where $p_1, \ldots, p_r$ are *distinct* primes. Show that for all real $s > 1$,

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n \in S} \frac{1}{n^s},$$

where

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

is the Riemann zeta function, and S is the set of positive square free integers.

2. Using a Sieve argument, show that the number of square-free integers that are less than or equal to x is equal to

$$\zeta(2)^{-1} x + o(x).$$

3. Show that any integer of the form $4n - 1$ always has a prime divisor of the form $4k + 3$. Use this to give a proof that there are infinitely many primes of the form $4k + 3$, analogous to Euclid's proof of the infinitude of primes that was recalled in class. Show by a similar argument that there are infinitely many primes of the form $3k + 2$.

4. Show that any prime p which divides the integer $n^2 + 1$ ($n \in \mathbf{Z}$) is either equal to 2, or is of the form $4k + 1$. Use this to show that there are infinitely many primes of the form $4k + 1$. (Hint: assume otherwise, and study the

asymptotics of $\#\{n^2+1, \quad n \leq \sqrt{x}\}$ as $x \longrightarrow \infty$ in two different ways to derive a contradiction.) **377**: Use a similar approach to prove that there are infinitely many primes of the form $3k+1$. Generalise your method to show that, if q is a fixed prime, there are infinitely many primes which are congruent to 1 modulo q.

The following exercises are taken from the textbook by Leveque.

5. Section 6.2, exercise 7.

6. Section 6.4, exercise 5.

7. Section 6.4, exercise 6.

8. Section 6.4, exercise 9.

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9. Section 6.8., exercise 4.