189-570A: Higher Algebra

Assignment 2

Due: Monday, October 23

1. Let $R = \mathbf{Z}[\sqrt{-5}]$, and let M be the submodule of R generated by 3 and $(1+\sqrt{-5})$. Show that M is not free over R, but is projective.

2. Let k be a field of characteristic $\neq 2$, and let $k[[x]]$ denote the ring of formal power series with coefficients in k. Show that $k[[x]]$ is a complete local ring. Show that the polynomial $(1+x)$ has a square root in $k[[x]]$.

3. Give an explicit formula for the square root of $(1+x)$ whose existence you proved in the previous exercise. What happens if $2 = 0$ in k?

4. Let $\underline{C_1}$ denote the category of $\mathbf{Z}$-modules and $\underline{C_2}$ the category of $\mathbf{F}_p$-vector spaces, where $\mathbf{F}_p$ denotes the finite field with p elements. Consider the rule F which to every object M of $\underline{C_1}$ associates the object $F(M) := M/pM$ of $\underline{C_2}$.

Show that F defines a covariant functor in a natural way.

5. Show that the functor F defined in the previous problem is *right exact*, i.e., that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\underline{C_1}$, then the induced sequence

$$F(M') \longrightarrow F(M) \longrightarrow FM'') \longrightarrow 0$$

is also exact (in the category of $\mathbf{F}_p$-vector spaces). Show that this functor is not left exact.

6. Let G be a group of p-power order, and let $\mathbf{Z}_p$ be the ring of p-adic integers. Show that the ring $\mathbf{Z}_p[G]$ is a complete local ring.

7. Let k be a (commutative) field, and let R be a k-algebra which is finite-dimensional over k. Show that R is semisimple if and only if R contains no non-zero nilpotent elements.

8. Let D be the dihedral group of order 8, and let Q be the quaternion group of order 8, generated multiplicatively by the elements i, j, k in the ring of Hamilton quaternions. Compute the character tables for D and Q. What do you observe?

9. Keeping the notations of the previous exercise, show that the group rings $\mathbf{C}[D]$ and $\mathbf{C}[Q]$ are isomorphic, but that the group rings $\mathbf{R}[D]$ and $\mathbf{R}[Q]$ are *not* isomorphic. (Here, $\mathbf{C}$ and $\mathbf{R}$ denote the complex and real numbers respectively.)

10. Let $G = \mathbf{GL}_3(\mathbf{F}_2)$ be the simple group of order 168 which you encountered in the previous assignment. Write down its character table. (The solution is worked out in a number of textbooks, but obviously you will learn more from this problem if you refrain from peeking, and try to work it out on your own!)