189-570A: Higher Algebra
Assignment 1

Due: Monday, October 2

1. Read through Chapter 1 of Lang's book. The goal here is to familiarise yourself with the basic definitions and results that we did not discuss in class. The material here is sometimes dry, but not terribly difficult.

2. Show that the intersection of two finite index subgroups of a group G is also of finite index.

3. Let G be a non-abelian finite simple group, and let n be an integer satisfying $1 < n! < \#G$. Show that G contains no conjugacy class of size n. If p is a prime dividing the order of G, show that the number of Sylow p-subgroups of G is $> n$.

4. Write down the class equations for the symmetric group S_6 and alternating group A_6, and show that A_6 is a simple group.

5. Let G be a subgroup of A_n of index n. Show that G is abstractly isomorphic to A_{n-1}.

6. Let G be a simple group of cardinality 60. Show that G has exactly 6 Sylow 5-subgroups. Conclude that G is isomorphic to a subgroup of A_6. Using exercise 5, conclude that G is isomorphic to A_5.

7. Let G be a finite group and let X be a G-set consisting of exactly t orbits. Given $g \in G$, let $\mathrm{FP}_X(g)$ be the number of fixed points of g acting on X:

$$\mathrm{FP}_X(g) := \#\{x \in X \quad \text{such that } gx = x.\}.$$

Show that

$$\frac{1}{\#G}\sum_{g\in G}\mathrm{FP}_X(g) = t.$$

8. We keep the notations of exercise 7. The action of G on X is said to be two-fold transitive if for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $g \in G$ with $gx = x'$ and $gy = y'$. Show that this propery holds if and only if

$$\sum_{g\in G}\mathrm{FP}_X(g)^2 = 2(\#G).$$

9. Let $G = \mathbf{GL}_3(\mathbf{F}_2)$ be the group of invertible 3×3 linear transformation with entries in the field $\mathbf{F}_2$ with two elements. What is the cardinality of G? Show that G is simple.

10. We retain the notations of problem 9. Let X_1 be the set of non-zero vectors in $\mathbf{F}_2^3$, and let X_2 be the set of two-dimensional subspaces of $\mathbf{F}_2^3$. Both sets are equipped with their natural structures of G-sets (using the natural action of G on $\mathbf{F}_2^3$ by linear transformations.) Prove the following assertions made by J-P. Serre in his seminar lecture on Thursday.

a) For all $g \in G$, we have $\mathrm{FP}_{X_1}(g) = \mathrm{FP}_{X_2}(g)$.

b) The sets X_1 and X_2 are *not* isomorphic as G-sets