Algebra 1 Assignment 8 Solutions

Problem 1 Let F be a field, we check the group axioms for $G = F - \{1\}$ equipped with the binary operation $*$.

1. For any $a \in G$ we have:

$$0 * a = 0 + a - 0 \times a = 0$$

   G has an identity element, 0.

2. Let $a \in G$ be arbitrary, then we can solve for b such that $a * b = 0$ indeed,

$$\begin{aligned} a * b = 0 & \quad \Leftrightarrow \\ a + b - ab = 0 & \quad \Leftrightarrow \\ b - ab = -a & \quad \Leftrightarrow \\ b = \tfrac{-a}{1-a} & \quad \text{which is well defined because } a \neq 1 \end{aligned}$$

   So it follows that every element has an inverse.

3. We finally verify $(a * b) * c = a * (b * c)$.

$$\begin{aligned} (a * b) * c \quad &= (a + b - ab) + c - (a + b - ab)c \\ &= a + b - ab + c - ac - bc + abc \\ &= a + (b + c - bc) - (ab + ac - abc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a * (b * c) \end{aligned}$$

So $(G, *)$ is a group.

The next two problems are counting problems, the next three facts are key:

1. Any $\sigma \in S_n$ can be essentially uniquely written as a product of disjoint cycles.
2. A k-cycle, i.e. some $\sigma = (123 \ldots k) \in S_n$, has order k, i.e. $\sigma^k = 1$.
3. Disjoint cycles commute

From this we get that the order of an element $\sigma \in S_n$ is determined by the "shape" of its cycle decomposition. e.g.

$$\sigma = \underbrace{(123)}_{=x}\underbrace{(45678)}_{=y}$$

Then since x and y are disjoint we have that they commute so $\sigma^n = (xy)^n = x^n y^n$. It follows that the order of σ is $\operatorname{lcm}(3, 5) = 15$.

Problem 2 It is fairly clear that that the only elements of order 3 in S_3 are three cycles, first we count how many "different looking" three cycles there are

$$\begin{array}{cccccc} (& * & * & * &) \\ & \uparrow & \uparrow & \uparrow & \\ & \text{3 choices} & \text{2 choices} & \text{1 choice} & \end{array}$$

and since every 3-cycle has 3 different presentations e.g.$(123) = (231) = (312)$ then we divide through our result by three (to avoid counting a cycle more than once) we find that there are $3!/3 = 2$ elements of order 3 in S_3.

Problem 3 An element of order six is either a 6-cycle or a product disjoint of 2-cycles and 3-cycles, adopting the convention of writing shorter cycles on the left (to avoid counting things more than once) we have that elements of order 6 in S_5 look like

$$(**)(***)$$

We immediately see that there are $5\times4\times3\times2\times1 = 5!$ different looking products of this form. Now each 2-cycle has 2 presentations and every 3-cycle has three presentations, so there are $5!/(2 \times 3) = 20$ distinct elements of order 6 in S_5.

For the record, counting problems are notoriously tricky, it's easy to forget to divide by something or add things instead of multiplying, but it sure beats checking all 120 elements of S_5.

Problem 4 First notice that any ring is a group under the binary operation $+$. Indeed, zero is our identity element, each element has an inverse (i.e. its additive inverse) and associativity holds. So for any $n \in N$ $(\mathbb{Z}_n, +)$ is a group with n elements.

Now if G and H are groups finite then $G \times H$ the cartesian product is a group with $\text{order}(G) * \text{order}(H)$ elements (count the possibilities). Also the subset $S = \{(g, 1_H) \in G \times H | g \in G\}$ is in fact a subgroup of $G \times H$ that is isomorphic to G (these two claims should be checked).

If a group G is abelian, i.e. for all $x, y \in G, xy = yx$, then all its subgroups must also be abelian.

Now S_3 is a nonabelian group, $(12)(23) = (123) \neq (132) = (23)(12)$, with 6 elements (in fact it is the smallest possible nonabelian group). And the groups $S_3 \times \mathbb{Z}_2$ and $S_3 \times \mathbb{Z}_5$ are groups with orders 12 and 30 respectively and since they have nonabelian subgroups they themselves are nonabelian.

Other examples (with completely different structures) are the groups of symmetries (i.e. reflections and rotations) of a hexagon, D_{12} (though some will write D_6 instead), and a regular polygon with 15 edges, D_{30}. One may check that they have the right number of elements and verify that rotation and reflection do not commute by making the polygons out of construction paper, numbering the vertices and applying the said symmetries.

Problem 5. If every element in G is of order two, then $(ab)^2 = abab = 1$. Multiplying this relation on the left by a and on the right by b gives $ba = ab$ (using the fact that $a^2 = b^2 = 1$.)

A non-abelian group of exponent 3 is the set G of 3×3 upper-triangular matrices with entries in $\mathbb{Z}_3$ and 1's on the diagonal. There are clearly 27 such matrices. The set G is closed under product, and inverses, and therefore forms a subgroup of $\mathbf{GL}_3(\mathbb{Z}_3)$, which can be checked to be non-abelian. Finally, note that if M is in G it can be written as $I + n$, where n is a matrix satisfying $n^3 = 0$, and I is the identity matrix. But then, since $3 = 0$,

$$M^3 = (I + n)^3 = I^3 + 3n + 3n^2 + n^3 = I.$$

(Using the binomial theorem is vaid in this context because I and n commute and generate a commutative subring of $M_3(\mathbb{Z}_3)$.) It follows that every $M \in G$ is of order 3.

Problem 6 To show these two groups are isomorphic, note that any matrix in $\mathbf{GL}_2(\mathbb{Z}_2)$ acts by left multiplication on the 2×1 column vectors with entries in $\mathbb{Z}_2$, and therefore gives rise to a permutation on the set X consisting of the three non-zero such column vectors. This gives a homomorphism from $\mathbf{GL}_2(\mathbb{Z}_2) \longrightarrow S_X$, which is easily seen to be injective (since a matrix which fixes all non-zero vectors can only be the identity matrix.) Since both groups have the same cardinality, 6, the homomorphism is actually an isomorphism.

Problem 7 It follows from the identity between permuations $(i_1 i_2 \cdots i_r) = (i_1 i_2)(i_2 i_3)(i_3 i_4) \cdots (i_{r-1} i_r)$ than any cyclic permutations $(i_1 \cdots i_r)$ can be expressed as a product of transpositions. The general case follows from the fact that any permuation is a product of cyclic permutations.