189-235A: Basic Algebra I
Assignment 7
Due: Wednesday, November 16

1. Let F be a field, and let $f : F \longrightarrow R$ be a homomorphism satisfying $f(0) \neq f(1)$. Show that f is necessarily injective, and that the image of f is isomorphic to F.

2. Let R be the ring $\mathbf{Z}_p[x]$ of polynomials with coefficients in the finite field $\mathbf{Z}_p$, and let $f : R \longrightarrow S$ be a surjective homomorphism from R to a ring S. Show that S is either isomorphic to R, or is a finite ring.

Let R be a commutative ring and let I be an ideal of R. Prove or disprove the following statements.

3. If R is an integral domain, then so is R/I.

4. If every ideal in R is principal, the same is true of R/I.

5. If every ideal in R/I is principal, the same is true of R.

6. Let $R = \mathbf{Z}[x]$, and let I be the ideal $(p, x^2 + 1)$ generated by the integer prime p and the polynomial $x^2 + 1$. Show that R/I is isomorphic to $\mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z}$ is $p = 5$, and is isomorphic to a field with 49 elements if $p = 7$.

7. Let F be a field and let $R = F[[x]]$ denote the ring of formal power series with coefficients in F, i.e., the set of expressions of the form

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in F,$$

where the addition and multiplication are performed by formally expanding out the sums and products (without worrying about issues of convergence, which don't make sense in an arbitrary field F anyways!)

Let $I = (x)$ be the ideal generated by the power series x. Show that R/I is isomorphic to F. Show that any element of R which does not belong to I is invertible. Conclude that any non-trivial ideal of R is contained in I. (A ring with this property is called a *local ring*, a terminology arising from the prototypical example $F[[x]]$, because power series can be thought of as "functions defined only around the value $x = 0$".)

8. Using the isomorphism theorem, identify the quotient ring R/I with a more familiar ring that you already know.

8a. $R =$ the ring of continuous real-values functions on $\mathbf{R}$; $I =$ the ideal of functions $f \in R$ satisfying $f(1) = f(2) = 0$.

8b. $R = \mathbf{Z}[x]$, $I = n\mathbf{Z}[x]$ for some integer $n \in \mathbf{Z}$.

8c. $R = \mathbf{R}[x]$, $I = (x^2 + 1)\mathbf{R}[x]$.

8d. $R = \mathbf{Z}[x]$, $I = (2x - 1)\mathbf{Z}[x]$.