Algebra 1 Assignment 7 Solutions

Problem 1 Let $R = F[X]$ where F is a field and let $I = F$ be the set of constant polynomials. Then I isn't an ideal because $i \in I \Leftrightarrow \deg(i) = 0$ or $i = 0$. So for any $f \in R$ s.t. $\deg(f) > 1$ and $0 \neq i \in I$ we have $\deg(f * i) = \deg(i) + deg(f) > 1 \Rightarrow f * i \notin I$.

Problem 2 Let $R = \mathbb{Z} \times \mathbb{Z}$ and let $I = \{(m, 0) | m \in \mathbb{Z}\}$. Then I is an ideal. Let $(a, 0), (b, 0) \in I$. Then $(a, 0) + (b, 0) = (a + b, 0) \in I$, so I is closed under addition and if $(m, n) \in R$ is some arbitrary element we have that $(m, n) * (a, 0) = (m * a, 0) \in I$ so I is closed under multiplication by arbitrary elements in R. It follows that I is an ideal.

Problem 3 Let I be the set of nilpotent elements of a ring R i.e. $I = \{a \in R | \exists m \in \mathbb{N} \text{ s.t. } a^m = 0\}$. Then I is an ideal. Let $a, b \in I$, then $\exists m, n \in \mathbb{N}$ s.t. $a^m = b^n = 0$. If $r \in R$ is some arbitrary element then $(r * a)^m = r^m * a^m = r^m * 0 = 0$ (the second equality holds because R is commutative) so $r * a$ is also nilpotent so I is closed under multiplication by arbitrary elements. Now let $N = \max(n, m)$ and consider $(a + b)^{2N}$. First note that if $i \leq N$ then $2N - i \geq N$. So by the binomial theorem we have:

$$\begin{aligned} (a+b)^{2N} &= \sum_{i=0}^{2N} \binom{2N}{i} a^i * b^{2N-i} \\ &= \sum_{i=0}^{N} \binom{2N}{i} a^i \underbrace{b^{2N-i}}_{=0} + \sum_{i=N+1}^{2N} \binom{2N}{i} \underbrace{a^i}_{=0} b^{2N-i} \\ &= 0 \end{aligned}$$

It follows that $a + b$ is also nilpotent, so I is closed under addition, it follows that I is an ideal.

For the next two problems, let R be the ring of functions from $\mathbb{Z}$ to $\mathbb{R}$ with addition $+_R$ and multiplication $*_R$. For $f, g \in R$, addition the sum $f +_R g \in R$ is defined as the mapping:

$$(f +_R g)(x) = f(x) + g(x)$$

and the product of two functions f and g is defined as the mapping:

$$(f *_R g)(x) = f(x)g(x)$$

From now on, subscripts for the operation signs will be omitted.

Problem 4 Let I be the set of functions f s.t. $f(0) = f(1)$. I is not an ideal. Indeed, let $f \in I$ be the constant 1 function i.e. for each $n \in \mathbb{Z}, f(n) = 1$ and let g be the function $g(n) = n$. Then we have that $(f * g)(0) = f(0)g(0) = 1 * 0 = 0$ but $(f * g)(1) = f(1)g(1) = 1$. So $f * g \notin I$ so I is not closed under multiplication by arbitrary elements of R so it's not an ideal.

Problem 5 Let $I = \{f \in R | f(0) = f(1) = 0\}$. Let $f, g \in I$ then $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$ and similarly $(f + g)(0) = 0$ so I is closed under addition. Now let $h \in R$ be some arbitrary element. Then we find that $(h * f)(1) = h(1)f(1) = h(1) * 0 = 0$ and $(h * f)(0) = h(0)f(0) = h(0) * 0 = 0$, so $h * f \in I$. It follows that I is closed under multiplication by arbitrary elements of R. It follows that I is an ideal.

Problem 6 Let R be the polynomial ring $F[X]$ with coefficients in a field. Then all of its ideals are principal.

Remark: In class the strategy was to show that an ideal in $\mathbb{Z}$ is generated by its smallest positive element. Recall that in polynomial rings over fields, the notion of size corresponds to the degree of a polynomial.

Proof: Let $I \subset F[X] = R$ be an ideal. Then the set $S = \{n \in \mathbb{N} | f \in I, n = \deg(f)\} \subset \mathbb{N}$ is nonempty so it must have a smallest element. If $0 \in S$ then I contains a constant $\Rightarrow I = R$. Otherwise let $s > 0$ be the smallest element in S. Then there is some $f \in I$ s.t. $\deg(f) = s$, i.e. f is the element of minimal degree in I.

Now, $I = (f)$. On one hand $f \in I \Rightarrow (f) \subseteq I$. On the other hand, suppose $I \nsubseteq (f)$, then there is some $g \in I$ such that $f \nmid g$. So we can apply the division algorithm to get

$$g = fq + r$$

where $r \neq 0$ and $\deg(r) < \deg(f) = s$. But then we get $r = g - fq \in I$ because of closure of ideals under addition and multiplication. And we get that $m = \deg(r) \in S$ and $m < s$ contradiction minimality of s. It follows that $I = (f)$. $\square$