

189-235A: Basic Algebra I

Assignment 8

Due: Wednesday, November 10

1. Let F be a field, and let $f : F \longrightarrow R$ be a homomorphism satisfying $f(0) \neq f(1)$. Show that f is necessarily injective, and that the image of f is isomorphic to F .
2. Show that the image of \mathbf{Z} under any homomorphism is isomorphic either to \mathbf{Z} or to a finite ring.

Let R be a commutative ring and let I be an ideal of R . Prove or disprove the following statements.

3. If R is an integral domain, then so is R/I .
4. If every ideal in R is principal, the same is true of R/I .
5. If every ideal in R/I is principal, the same is true of R .
6. Let $R = \mathbf{Z}[x]$, and let I be the ideal $(p, x^2 + 1)$ generated by the integer prime p and the polynomial $x^2 + 1$. Show that R/I is isomorphic to $\mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z}$ if $p = 5$, and is isomorphic to a field with 49 elements if $p = 7$.

Extra credit.

7. Let F be a field and let $R = F[[x]]$ denote the ring of formal power series with coefficients in F , i.e., the set of expressions of the form

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in F,$$

where the addition and multiplication are performed by formally expanding out the sums and products (without worrying about issues of convergence, which don't make sense in an arbitrary field F anyways!)

Let $I = (x)$ be the ideal generated by the power series x . Show that R/I is isomorphic to F . Show that any element of R which does not belong to I is invertible. Conclude that any non-trivial ideal of R is contained in I . (A ring with this property is called a *local ring*, a terminology arising from the prototypical example $F[[x]]$, because power series can be thought of as “functions defined only around the value $x = 0$ ”.)