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The Theta Body and Imperfection

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1.1 Background and Overview

1.1.1 Perfection and integer programming

Let A be a $0,1$ matrix with columns indexed by V. One approach to solving an integer program

$$(IP) \qquad \max\{w \cdot x : Ax \leq 1, x_v \in \{0,1\} \ \forall v \in V\}$$

is to start by solving the linear programming relaxation (LP) obtained by relaxing the integrality constraint to $1 \geq x_v \geq 0$ for each $v \in V$. If (LP) has an integral optimum, then evidently it is also optimal for (IP). Matrices with this property for each cost vector $w \geq 0$ are called *perfect* and Chvátal [12] showed that each such matrix (without dominated rows) arises as the clique-node incidence matrix of some perfect graph G – Padberg [40] had previously noted that A must correspond to some graph. (Note that if $a \geq a'$ where a, a' are rows of A, then removing a' from A does not affect the problem (IP).) In this case, the space of feasible solutions for (IP) are just the $0,1$ incidence vectors of stable sets in G, and so (IP) amounts to solving a maximum weight stable set problem. Grötschel, Lovász, Schrijver [25] have given a polytime algorithm which given a perfect matrix A (respectively a perfect graph G) together with a weight vector, finds an optimal solution to (IP) (respectively a maximum weight stable set G). We note that their algorithm does not actually solve the problem of recognizing whether a matrix A, or a graph G, is perfect. This remains a beautiful and fundamental question. If, however, one is solving an instance of (IP) for only a single objective function w, then applying an algorithm to determine whether one is holding a perfect matrix A is a bit like a kitten playing with a ball of string. We shall describe, however, why the structure of perfect graphs, or more specifically minimally imperfect graphs, also plays a role in solving larger instances of (IP). For the moment, we digress to review the relevant graph-theoretic progress for the related Strong Perfect Graph Conjecture (SPGC).

Perfect Graphs. Editor J.L. Ramírez Alfonsín and B.A. Reed

1.1.2 Imperfection and partitionability

In the remainder let $G = (V, E)$ be an undirected graph. We let $\alpha(G)$ denote the sizes of a maximum set of mutually nonadjacent nodes, or a *stable set*. Similarly, $\omega(G)$ denotes the size of a maximmum set of pairwise adjacent nodes, or *clique*. Clearly $\omega(G) = \alpha(\bar{G})$, where $\bar{G}$ denotes the graph *complement* of G. We also use n to denote $|V|$ throughout.

We denote by $\chi(G)$, the *chromatic number* of G, i.e., the minimum number of colours needed to paint the nodes so that adjacent nodes do not receive the same colour. It is clear that if K is a clique of G, then any pair of nodes in K must receive distinct colours in any legal colouring of G. Thus for any graph $\chi(G) \geq \omega(G)$, where $\omega(G)$ denotes the size of a largest clique in G. A graph $G = (V, E)$ is *perfect* if for each $S \subseteq V$, this inequality is satisfied with equality: $\chi(G[S]) = \omega(G[S])$.

Perfect graphs were introduced by Berge and were extensively studied before their connection to integer programming was revealed in the 1970's. Perfect graphs, as this book attests, have indeed proved an incredibly fruitful concept in much of combinatorics. Berge offered two perfect graph conjectures in [2]. The first was that a graph G is perfect if and only if its complement $\bar{G}$ is perfect; this was proved by Lovász [34] and is known as the Perfect Graph Theorem. The second conjecture delves into the combinatorial structure of a perfect graph. An *odd hole* is any odd chordless cycle of length at least 5. Note that for any odd hole C we have $\chi(C) = 3 > 2 = \omega(C)$ and hence no odd hole is perfect. Berge asserted that these are the only obstacles to perfection.

Conjecture 1.1 *(The Strong Perfect Graph Conjecture) A graph G is perfect if and only if neither it nor its complement $\bar{G}$ contains an odd hole as an induced subgraph.*

One important consequence of the proof of the Perfect Graph Theorem [34] is that for any minimally imperfect graph (i.e., $G - v$ is perfect for each node v), there exist integers $p, q \geq 2$ such that: (i) $|V| = pq+1$, (ii) for each node v, $G-v$ can be partitioned into q stable sets of size p, and (iii) for each node v, $G - v$ can be partitioned into p cliques of size q. A graph with these properties is called a (p, q)*-graph* and a graph is called *partitionable* if it is a (p, q)-graph for some integers p, q. Partitionable graphs have been widely studied in their own right [3, 8, 11, 14, 20, 45], Chapter **??** and they form one of three key classes pertaining to perfect graphs.

As pointed out in [36], the main difficulty in resolving the SPGC seems to be that any property satisfied by the minimally imperfect graphs is also satisfied by the partitionable graphs. In this sense the non-minimally imperfect partitionable graphs are imposters and resolving the SPGC seems tantamount to driving a wedge between them and the authentic minimally imperfect graphs. One natural class of partitionable graphs are called the webs. These graphs all have the property that the nodes lie in a cyclic order and nodes are adjacent only if they are within some distance in this order. Chvátal [13] proved that any minimally imperfect graph which contains a spanning subgraph which is a web, must be either an odd hole or antihole. In particular, this rules out a large class of potential threats to the veracity of the SPGC. We do not understand as well the esoteric examples described in [14], not to mention classes of partitionable graphs which are yet to be discovered.

The problem of recognizing partitionable graphs is clearly in $\mathcal{NP}$. One needs only

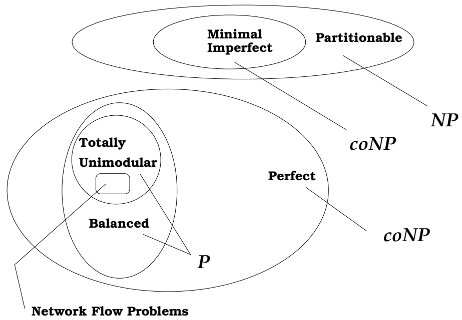

Figure 1.1

exhibit the appropriate colourings and clique covers. One easily deduces that for any (p,q)-graph G, $\omega(G) \leq q$ and that $\alpha(G) \leq p$ and hence that any partitionable graph cannot be perfect. Thus, as observed for instance in [7, 25, 36], this implies that the problem of recognizing perfect graphs is in $co\mathcal{NP}$; one exhibits an appropriate subgraph and a certificate showing that it is partitionable. [1] Similarly, the problem of recognizing minimally imperfect graphs is in $co\mathcal{NP}$; one either exhibits a clique and a colouring of the same size, or one exhibits a **proper** partitionable subgraph. Note that the veracity of the SPGC would immediately imply a polytime recognition algorithm for the minimally imperfect graphs. A priori, however, it implies nothing about whether perfect recognition is in $\mathcal{NP}$, or partitionable recognition is in $co\mathcal{NP}$.

Figure 1.1 presents the status for a number of matrix recognition problems. (We call a matrix *partitionable* (resp. *minimally imperfect*) if it is the maximal clique matrix for a partitionable (resp. minimally imperfect) graph.) Recall that a matrix A is *totally unimodular* if each nonsingular square submatrix has determinant 1 or -1. A matrix is *balanced* if it has no odd cycle submatrix. These last two classes have been shown to be polytime recognizable through works of Seymour [47] and Cunningham, Edmonds [15] (TUM) and Cornuéjols, Conforti, and Rao [10, 9].

The main new algorithmic result of this chapter is the following (see Sections 1.4,1.1.3).

Theorem 1.31 *There is a polytime algorithm which given a graph G determines whether it is partitionable.*

1.1.3 Overview

This chapter's focus is on the theta body and its potential algorithmic applications. Some new results are also described which address the questions of (a) finding an exact separation algorithm for the theta body (Section 1.3.4) (b) recognizing a partitionable graph (Section 1.4.2) and (c) characterizing the perfect graphs (Section 1.5). We

[1] This was shown more directly in [25]; namely, they do not need to "guess" the clique covers and colourings but only the subgraph. They then apply the Ellipsoid Algorithm in order to show that there is a fractional stable set which is too large.

describe each of these briefly below.

Separation for the Theta Body

The theta body of a graph G is a convex body which arises as a projection of a convex set of positive semi-definite matrices. There is a polytime exact separation algorithm for the class of matrices, but we do not know whether theta itself admits such a separation algorithm.

Question 1.2 *Is there a polytime exact separation algorithm for the theta body?*

The results in Section 1.3.4 could be viewed as support for the positive resolution of this question, which is itself a special case of the open question of whether there is a polytime algorithm for solving the exact semi-definite feasibility problem. As discussed in Chapter SDP, another unresolved special case (cf. [21]) of this problem is equivalent to finding a polytime algorithm to solve the following. Given a set of integers $a_1, a_2, \ldots, a_n$ and k, is it the case that $\sum_i \sqrt{a_i} \leq k$? Determining the answer is equivalent to knowing whether the optimum of a certain semi-definite program is greater than k. Of course, there is a polytime algorithm which for a given error $\epsilon > 0$, determines a semi-definite optimum to within ϵ. Thus one could solve the square root sum problem if there was a priori some *bounded gap guarantee*: that is, for any instance, if the optimum value is strictly greater than k, then it is larger by some gap $\epsilon = \frac{1}{(S)^S}$, where S is the size of the instance.

Finding such a bounded gap guarantee is then the key to resolving this number-theoretic question. In Section 1.3.4 we prove a similar bounded gap result for the *theta body* – see Theorem 1.20. Specifically, we show that if a linear function over the theta body is not optimized by a point in its integer hull, the stable set polytope, then the optimum is indeed bounded away from the optimum over the integer hull. This extends an earlier result of Grötschel, Lovász and Schrijver who proved this in the case where G is minimally imperfect and the objective vector is all 1's.

Recognizing Partitionability

The algorithm for recognizing whether a graph is partitionable relies on a simple insight. Namely, if G is a (p,q)-graph, then the intersection of its theta body with the hyperplane $\{x : \bar{\mathbf{1}} \cdot x = p\}$ yields a polytope whose vertices are precisely the n maximum stable sets of G. We call this result 'The Potatope Slicing Proposition'; it is a corollary to Theorem 1.25. We also denote by $TH_p(G)$ the region $\{x : \bar{1} \cdot x = p\} \cap TH(G)$.

A broad overview of the algorithm can be described simply:

- Apply Ellipsoid-Algorithm-based routines to $TH_p(G)$ (and $TH_q(\bar{G})$) in order to find n distinct stable sets of size p (cliques of size q).
- Check that the resulting $2n$ sets satisfy the required partitionable properties.

There are two approaches of carrying out this program. We discuss each briefly.

The more natural approach is to try to q-colour $G - v$ for each node v (and, using the same approach, p-colour $\overline{G} - v$). If we succeed for every node v, then we have indeed obtained all the maximum stable sets (and cliques) of G. This was proved in the last section of Chapter , it also follows from Theorem 1.22. Furthermore, the second phase is now trivial, as these colourings guarantee that G is a (p,q)-graph.

To find the colourings, we focus on the vectors $(0, \frac{1}{q}, ..., \frac{1}{q}), (\frac{1}{q}, 0, \frac{1}{q}, ..., \frac{1}{q}), ..., (\frac{1}{q}, ..., \frac{1}{q}, 0)$. Note first that if $G-v$ has a q colouring using stable sets $S_1, ..., S_q$ with corresponding characteristic vectors $\mathcal{X}_1, ..., \mathcal{X}_q$, then the vector x^v such that $x^v_u = 1/q$ if $u \neq v$ and 0 if $u = v$ can be expressed as $\sum_{i=1}^{q} \frac{1}{q}\mathcal{X}_i$. I.e., x^v is a convex combination of stable sets of G and so it is in $TH_p(G)$. Furthermore, if G is partitionable, then the characteristic vectors of the maximum stable sets of G are linearly independent, and thus this is the only way to express x^v as a convex combination of stable sets of G. So, for partitionable graphs, expressing x_v as a convex combination of vertices of $TH_p(G)$ is equivalent to finding the colouring of $G - v$ with q stable sets.

The only algorithmically non-trivial step carried out under this approach, is finding the expression of a given vector as a convex combination of extreme points of a convex body, namely $TH_p(G)$. In [27], a polynomial-time algorithm (see (6.5.11) of [27]) is described which successfully solves this problem in the case where it is given as input a *well-described polytope* [2] together with a strong optimization oracle. One may check that in the case where the oracle or vertex-complexity of the polyhedron are not as advertised, their algorithm still returns some (possibly bogus) answer. Thus we only need to assure that we may pass a 'mock' strong optimization routine which works properly for partitionable graphs, since for other graphs, no bogus answer from (6.5.11) could induce us to conclude that G is partitionable.

The mock strong optimization oracle which we pass to (6.5.11) is based on the result (6.3.2 (a)) of [27] which shows that a weak optimization routine implies the existence of a strong one if we are given a full-dimensional, well-described, bounded polyhedron. To invoke this result, we work instead with the region $\Sigma(G,p) \doteq conv\{\bar{\mathbf{0}}, TH_p(G)\}$ since it is full-dimensional for any graph G and easily inherits a weak separation algorithm from the theta body (it is easy to see that scaling a q-colouring of $G - v$ by $\frac{1}{q}$ is the only way to express x^v as a convex combination of vertices of $\Sigma_{(G,p)}$, we give the same argument given above for $TH_p(G)$). As with theta, this region is non-polyhedral for general graphs, but it has $0,1$ vertices if G is partitionable and hence it can be well-described in that case. The result (6.3.2) actually describes an algorithm which applies a weak optimization routine up to some prescribed error (depending on the specified polyhedral complexity). It then 'rounds' to the exact answer using Diophantine Approximation techniques (see Chapter 5 in [27]). We note that the algorithm always returns some answer even if it is given a non-polyhedral body. If, however, we pass $\Sigma(G,p)$ for a partitionable graph G together with a vertex-complexity of $\log(n)$, then it is guaranteed to produce an exact optimal solution for a problem.

Section 1.4.2 contains a full description of a partitionable graph recognition algorithm which uses a slightly different algorithm (the DISCOVERY algorithm of Section 1.2.3) for finding the n stable sets (Phase 2). The DISCOVERY algorithm takes as input a convex body K with a weak separation oracle and a set S of $n+1$ affinely independent vectors[3] contained in K. It then either shows that K contains an extreme point which is not integral, or finds a set $n-1$ affinely independent integer valued vectors whose convex hull contains the convex hull of

[2] A polytope is well-described if it comes with an upper bound on the input complexity of any of its extreme points.

[3] A set S of distinct vectors is affinely independent if $S - \bar{\mathbf{0}}$ is linearly independent.

S. In recognizing a partitionable graph, we invoke DISCOVERY with $K = \Sigma(G,p)$ and $S = \{\bar{\mathbf{0}}, (0, \frac{1}{q}, \frac{1}{q}, \dots, \frac{1}{q}), (\frac{1}{q}, 0, \frac{1}{q}, \dots, \frac{1}{q}) \dots (\frac{1}{q}, \frac{1}{q}, \dots, \frac{1}{q}, 0)\}$. It either returns a set S whose elements are $\bar{\mathbf{0}}$ and the incidence vectors of n stable sets, or it concludes that K was not integral.

This approach is more direct in that it does not need to colour $G-v$ for each node v. The DISCOVERY algorithm focuses instead on the simple nature of the characterization given by the Potatope Slicing Proposition: one need only check whether $\Sigma(G,p)$ is a full-dimensional integral simplex. It thus finds the n stable sets one at a time by making n calls to the basic Ellipsoid Algorithm. It also does not then rely on (6.5.11) of [27] (and the subroutines which it in turn calls).

A Continuous Perfect Graph Conjecture

Section 1.5 gives a characterization of perfect graphs based on disallowing a continuous generalization of the partitionable graphs (the (p,q)-*vectors*) in a certain 'theta-ized' convex body.[4] It employs the useful fact that if G is a partionable graph, with clique size q say, then the vector $\frac{1}{q}\chi^V$ is not in the theta body – see Corollary 1.16 which is a direct consequence of a result of Grötschel, Lovász, Schrijver . In contrast, this vector is in the fractional stable set polytope. It is shown that detecting the existence of a (p,q)-vector (and hence recognizing perfect graphs) can be reduced to approximating a quadratic number of (very specialized) norm-maximization problems. Finally, we make a *Continuous Perfect Graph Conjecture* that if G is imperfect, then there does exist a (p,q) vector with either p or q equal to 2. Since an odd hole or anti-hole in G gives rise to such a vector, this would be implied by the Strong Perfect Graph Conjecture.

Layout

The layout of the chapter is as follows. Section 1.2 contains background and some details concerning the Ellipsoid Algorithm. It also contains some algorithmic details used for the complete description of the partitionable graph recognition algorithm. If one is already familiar with some of the proofs in Chapter 6 of [27], then the outline of the algorithm given above probably sufficiently explains the result. Section 1.3 introduces the theta body (as well as the fractional and integral stable set polytopes) and renders explicit some of the main ideas from [35] which will be used in Section 1.3.4. Section 1.4 gives a brief introduction to partitionability and then describes a good characterization of partitionable graphs. This is followed by a detailed description of a recognition algorithm. The final section describes a geometric characterization which has a mildly different flavour than the famous result of Grötschel, Lovász, Schrijver (cf. Theorem 1.7) which asserts that a graph is perfect if and only if its theta body is polyhedral. We also pose a certain *Continuous Perfect Graph Conjecture.*

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[4] We follow A. Sebö who described this as a 'theta-izing'.

1.1.4 Partitionability and branch and cut methods for packing problems

If A is not perfect, then another approach to solving an integer program such as (IP) is through the use of cutting planes. This approach was was born with the first finite-time algorithm for general integer programming due to Ralph Gomory [22, 23]. Later Chvátal laid the theoretical and practical foundations for the use of cutting planes more generally. The development of exact algorithms for integer programming today usually involves identifying a class $\mathcal{C}$ of valid inequalities for the problem ahead of time. Hopefully, this class is well-behaved in the sense that the separation problem can be solved by an efficient algorithm. The strategy is to then solve a linear programming relaxation (LP) and if the optimum x^* is integral, we have the desired optimal solution. Otherwise, we search for an (or many) inequality from $\mathcal{C}$ which x^* violates. This inequality, or *cutting plane*, is then added to the LP and the process is repeated. Normally, the class $\mathcal{C}$ will not contain all necessary facet-inducing inequalities for the integer hull of the problem. Thus eventually, one may reach a solution x^* that satisfies all inequalities in the class. In this case, *branching* is employed, that is, a fractional component v of x^* is chosen, and two subproblems are formed: one with $x^*_v = 0$ and the other with $x^*_v = 1$. In fact, branching normally occurs long before all "useful" cuts in $\mathcal{C}$ are exhausted. A number of ingenious heuristic methods have been devised for determining when (and how) to branch.

Such *branch and cut* algorithms have proved successful in solving many combinatorial optimization problems in practice. For problems of the form (IP), Sewell [46] (work borne of his thesis), and Verweij and Aardal [52] developed branch and cut codes based on the class of *odd cycle inequalities*: $\sum_{v \in C} x_v \leq \frac{|V(C)|-1}{2}$ for each odd cycle C in the graph G. These algorithms use on fast separation algorithms (cf. [28]) for separating over these inequalities. This approach was also recently tested [39] for the problem of finding Steiner designs. There, it met with much less success apparently due to the denseness of the instances. Specifically, after adding a reasonable number of large clique constraints to the LP, a fractional solution is much less likely to violate a constraint as sparse as an odd cycle inequality.

Each odd chordless cycle of length at least 5 is minimally imperfect and each minimally imperfect graph is partitionable. Hence a broader class of cutting planes for the weighted stable set problem is the class of *partitionable inequalities*: $\sum_{v \in H} x_v \leq \alpha(H)$ for each partitionable graph H. Expanding the class of cutting planes in this fashion, would possibly remove the obstacle mentioned above in the studies of Moura. The polytime recognizability of partitionable graphs lends hope to the separation problem being solvable. Specifically, we ask:

Question 1.3 *Is there a polytime algorithm which given a graph G and vector $x^* \in \mathcal{R}^V_+$ determines whether x^* violates either a partitionable inequality or a clique inequality (or even an orthonormal constraint – see Section 1.3.1).*

It is natural to include here the clique inequalities in our class $\mathcal{C}$ at least partially since if we did not, then such a separation routine would also give a polytime algorithm to recognize whether a graph was perfect. Simply input the vector $\frac{1}{p}\chi^V$ to the routine for each $p \geq 2$; the graph is perfect if and only if no violated partitionable inequalities are found. As with the method we describe to recognize partitionable graphs, one might expect such a routine to rely on the semi-definite relaxation theta. We remark that it is only recently that computational experience is being reported in this direction – see [29]. Hopefully, research on such implementations will flourish in the coming years.

1.2 Optimization, Convexity and Geometry

Khachiyan's analysis [32] of the Ellipsoid Algorithm for linear programming led to a general framework for solving linear and convex optimization problems over convex bodies in polynomial time. We briefly discuss the extensions due to Grötschel, Lovász, Schrijver [27] which, stated loosely, asserts that in order to optimize over a class of convex bodies in polytime, it is 'enough' to be able to solve the separation problems in polytime. This was independently shown for polyhedral bodies by Karp and Papadimitriou [31], Padberg and Rao [42] as well as Grötschel, Lovász, Schrijver [24]. We will apply the methods to smooth convex bodies, and so we need to present appropriate error-bounded versions of the problems.

1.2.1 Convexity and encoding conventions

A convex set $K \subseteq \mathcal{R}^n_+$ a *convex corner* (if K is polyhedral, this is also called an *antiblocking polytope*) if for each $x \leq y \in K$ we have also $x \in K$. The *antiblocker* of K, denoted by $\mathcal{A}(K)$, is the set $\{x \in \mathcal{R}^n_+ : x \cdot y \leq 1, \ \forall y \in K\}$. One easily shows that $\mathcal{A}(K)$ is again a convex corner and that $\mathcal{A}(\mathcal{A}(K)) = K$ (see [19], [44] for the case of polyhedra). A convex corner is *polyhedral* if there exists a finite system of inequalities $Ax \leq b$ such that $K = \{x \in \mathcal{R}^N_+ : Ax \leq b\}$. In the remainder, for a convex set K and real value p, we denote by K_p the set $\{x : \bar{\mathbf{1}} \cdot x = p\} \cap K$.

In the remainder, the size of a rational r, denoted $size(r)$, is the value $1 + \lceil \log_2(q) \rceil + \lceil \log_2(p) \rceil$ where $r = \frac{p}{q}$ is in lowest terms. In addition, the size of a vector $a \in \mathcal{Q}^n$ or $n \times n$ matrix will be defined as n times the size of its largest (size) entry. A point x in a convex set C, is an *extreme point* if it is not the midpoint of two distinct elements of C. We let $\mathcal{E}(C)$ denote the set of extreme points of C. A convex set C is a *polyhedron* if there is a finite system of inequalities $Ax \leq b$ such that $C = \{x : Ax \leq b\}$. An extreme point of a polyhedron is also called a *vertex*. The *facet-complexity* of a (closed) convex set $P \subseteq \mathcal{R}^n$ is the smallest number $\varphi \geq n$ such that there is a system $Ax \leq b$ of inequalities which defines P, where each inequality has size at most φ. Note that we leave open the possibility that P is not polyhedral and hence the system is infinite and so φ is also unbounded. The *vertex-complexity* ν is defined similarly. If P is polyhedral, then we have that $\varphi \leq 4n^2\nu$ – see Chapter 10, [44] for the similar details.

We assume that a *convex body* is a convex set K in some space $\mathcal{R}^n$ which is both compact and full-dimensional. Normally we are also equipped with a number R such that K is contained in the ball $S(0, R)$. In this case, the body is said to be *circumscribed* and our input is denoted by (K, R). If we are given $a^0 \in K$ and $r \in \mathcal{R}$ such that $S(a^o, r) \subseteq K$, then the body is said to be a^0*-centered*, or just centered. A body is *well-bounded* if it is both centered and circumscribed in which case its input will be denoted as (K, a^0, r, R). In all cases, we assume that the descriptor K indicates the dimension n as well the name of the body in the class with which we are dealing. Normally, $size(K)$ will be the space needed to encode this name. For instance, it may be defined by a graph or an inequality system. In all cases, we assume that $size(K) \geq n$. If the body is given as circumscribed or well-bounded, then we must add to this the sizes of a^0, r and R accordingly. For a convex body C, given in any manner, we use K_C, r_C, R_C, a_C, n_C to denote the various parameters which may be associated with the input.

For any $\delta > 0$ and convex set K, we denote by $S(K, \delta)$ the set $\{x \in \mathcal{R}^n : \exists y \in K \ s.t. \ \|x - y\| < \delta\}$. We also let $S(K, -\delta) = \{x : S(x, \delta) \subseteq K\}$.

1.2.2 Optimization over a convex body: consequences of the ellipsoid method

In this section we discuss some of the theoretical consequences of applying the Ellipsoid Method to bounded convex sets. We do not describe the algorithm here; the reader is instead referred to [27] for a full treatment. The *weak separation problem* for a class $\mathcal{K}$ of convex bodies (either circumscribed or well-bounded) is:

Weak Separation Problem: *Given $C \in \mathcal{K}$, a vector $y \in \mathcal{Q}^n$ and $\delta > 0$, assert either that $y \in S(K, \delta)$ or display a vector $a \in \mathcal{Q}^n$ such that $\|a\| \geq 1$ and $a \cdot x \leq a \cdot y + \delta$ for each $x \in K$.*

(We remark that it is more usual (cf. [27]) to only require the weaker condition that the separating hyperplane be satisfied only for $x \in S(K, -\delta) \equiv \{x : S(x, \delta) \subseteq K\}$.) We will work primarily with two varieties of the weak separation problem: the *yes-definite* and the *no-definite* versions. For the *no-definite weak separation problem*, we require that the separating hyperplane satisfies the condition $a \cdot x < a \cdot y$ for each $x \in K$. That is, if we end in the 'no' state, then we may definitely declare that y was not in K. If we end in the 'yes' state however, the vector y may or may not be in K. Alternatively, the *yes-definite weak separation problem* ends in the 'yes' state only if $y \in K$. If it ends with a separating hyperplane, however, the vector y may still be in K. The traditional *strong separation problem* could then be defined as a yes- and no-definite separation problem. In the remainder, when we refer simply to weak separation, we shall mean the no-definite weak separation problem.

A *polytime* weak separation algorithm (for any version of the problem) for a class $\mathcal{K}$ of convex bodies is an algorithm which given $C \in \mathcal{C}$ solves (that version of) the weak separation problem and has running time bounded by a polynomial in the size of the body and $size(y)$. [5]

We now consider the following two optimization problems for a class $\mathcal{K}$ of convex bodies.

Weak Optimization Problem: *Given $C \in \mathcal{K}$, an objective function $c \in \mathcal{Q}^n$ and rational* **error** *$\epsilon > 0$, find a vector $y \in S(K_C, \epsilon)$ such that $c \cdot x \leq c \cdot y + \epsilon$ for each $x \in K_C$. (Or if C is not centered, determine that $S(K_C, -\epsilon)$ is nonempty.)*

The **Semi-Strong Optimization Problem** is defined the same way except that it guarantees to find a vector actually in K_C.

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[5] More formally, we should require that the algorithm first recognizes whether C is actually in the given class $\mathcal{C}$. We generally skip such formalities as well as discussions around encoding the input. In calling any routines, however, we will take care throughout to ensure that we only pass valid inputs.

A *polytime* weak (respectively semi-strong) optimization algorithm for a class of convex bodies is an algorithm which solves the weak (respectively semi-strong) optimization problem for each instance in the class and has running time bounded by a polynomial in the size of the body, $size(\epsilon)$ and $size(c)$. The equivalence of weak optimization and weak separation is extensively studied in [24, **?**]. Their arguments (e.g., Theorem 2.4 in [25]) imply a similar result for the semi-strong version. We have:

Theorem 1.4 *(Grötschel, Lovász, Schrijver) Let $\mathcal{K}$ be a class of circumscribed convex bodies. If there is a polytime weak (resp. yes-definite weak) separation algorithm for $\mathcal{K}$, then there is a polytime weak (resp. semi-strong) optimization algorithm for $\mathcal{K}$.*

We also state another result of this flavour. Let $\mathcal{K}$ be a class of convex bodies. The *nonemptiness problem* for $\mathcal{K}$ is: given $C \in \mathcal{K}$ and $\epsilon > 0$, either find $x \in K_C$ or determine that $vol(K_C) < \epsilon$. The *weak membership problem* for $\mathcal{K}$ is: given $C \in \mathcal{K}$, $y \in \mathcal{R}^n$ and $\epsilon > 0$, either (i) assert that $y \in S(K_C, \epsilon)$ or (ii) assert that $y \notin S(K_C, -\epsilon)$.

In general, it is not the case that an algorithm for the membership problem implies the existence of a nonemptiness algorithm. In the case that $\mathcal{K}$ consists of well-bounded convex bodies, this is guaranteed by the following deep theorem.

Theorem 1.5 *(Yudin, Nemirovskiĭ [53]) Let $\mathcal{K}$ be a class of circumscribed convex bodies. If there is a polytime algorithm for the weak membership problem for $\mathcal{K}$, then there is a polytime algorithm for the nonemptiness problem for $\mathcal{K}$.*

1.2.3 The discovery problem

In this section, we describe a subroutine used by our recognition algorithm for partitionable graphs. We need two definitions: a set of vectors $\{v^1, v^2, \ldots, v^q\}$ is *affinely independent* if and only if there is no nontrivial solution to $\sum_i \lambda_i v^i = \bar{\mathbf{0}}, \sum_i \lambda_i = 0$; a full-dimensional *simplex* in $\mathcal{R}^n$ is a body which is the convex hull of $n+1$ affinely independent vectors in $\mathcal{R}^n$.

THE DISCOVERY PROBLEM

The *discovery* problem is as follows. We start with a full-dimensional convex body C together with a full-dimensional simplex (given by its vertices) inside it. We must either return $n+1$ integral vertices of C (as the rows of a matrix) or determine that C has an extreme point which is not integral (in this latter case we return the empty matrix).

More formally, for the discovery problem, we are given a circumscribed convex body $C = (K, R)$ in $\mathcal{R}^n$ and $\epsilon \in (0, \frac{1}{2}]$ and collection $\{z^i : i = 0, 1, \ldots, n\}$ of affinely independent vectors in $S(K, \epsilon)$, either find $n+1$ affinely independent integral vectors in $S(K, \epsilon)$ or assert that K has a non-integral extreme point. This can be visualized as starting with a simplex inside of K (see $conv(x, y, z, 0)$ in Figure 1.2) which we attempt to 'expand' out to discover an integral simplex. This is done by revealing one new vertex at a time. Each new vertex is 'revealed' by making a call to the Ellipsoid Algorithm.

The algorithm below is similar to a number of routines presented in Chapter 6 of [27] (specifically, (6.5.1) – a routine for finding a vertex, (6.5.3) – a routine for determining a basis for the affine hull of a polyhedron (first shown in [16]), and (6.5.11) – a routine for expressing a point as a convex combination of extreme points). Their routines are applied to polyhedra with bounded facet-complexity and are given with strong optimization routines. We are working, however, with bodies which are not necessarily polyhedral, and hence have neither bounded (or even finite) facet-complexity nor strong optimization procedures. Routines such as (6.5.11) also rely on a number of subroutines each of which in turn makes use of the Ellipsoid Algorithm.

DISCOVERY

Input:
A circumscribed convex body $C = (K, R)$, $\epsilon > 0$ and affinely independent $\{z^i : i = 0, 1, \ldots, n\}$ contained in $S(K, \epsilon)$. We are also given weak optimization and separation routines for C, denoted *Opt* and SEP respectively

Output:
Either an $(n+1) \times n$ matrix whose rows are affinely independent integral vectors in $S(C, \epsilon)$ or an empty matrix $[\emptyset]$ (indicating C has a fractional extreme point)

For each $i \in \{0, 1, 2, \ldots, n\}$ with z^i non-integral
    Find a vector c^i which satisfies $c^i \cdot z^j = 0$ for each $j \neq i$ and $c^i \cdot z^i \geq 1$
    $c \leftarrow R^n c^i + (1, R, R^2, \ldots, R^{n-1})$
    Let x^* be the ϵ-optimal output from $Opt(C, c, \frac{\epsilon}{2R^2})$
    Let x be the integral vector such that $x_j = k$ if and only if $|x^*_j - k| < \frac{1}{2}$
    If $(Sep(C, x, \epsilon)$ returns NO) then OUTPUT($[\emptyset]$)
    Else $z^i \leftarrow x$
EndFor
OUTPUT(matrix with rows $z^j : j = 0, 1, 2, \ldots, n$))

Theorem 1.6 *The algorithm* DISCOVERY *correctly solves the discovery problem for a convex body. It also has polynomially bounded running time if* OPT, SEP *are polytime routines.*

Proof. First, note that each cost function c^i can be computed (if it exists) in polytime (by Gaussian Elimination). Thus its size is still bounded by a polynomial in the size of the original input (which includes R). Thus the input to the routines OPT,SEP is bounded by a polynomial in the size of the original input. Since we make at most $2n$ calls to these routines, DISCOVERY is a polytime algorithm as long as OPT,SEP are.

One routinely checks that the existence of the objective vector c^i is guaranteed by the affine independence of a current collection $\{z^j : j = 0, 1, 2 \ldots, n\}$. In turn this implies that the new collection $\{z^j\}$ is also affinely independent since $\{z^j : j \neq i\}$ is contained in the hyperplane $\{x : c^i \cdot x = 0\}$ which will not contain the newly computed z^i. Each new z^i is also guaranteed by SEP to be in $S(K, \epsilon)$. Finally, we only stop if all z^j's are integral. Thus if we return a non-empty matrix, then the output satisfies the required properties.

It remains only to show that if we output $[\emptyset]$, then C does indeed have a fractional extreme point. To see this, set $\beta = c \cdot x^* \geq 1$ for some iteration. If C is integral, there exists integral extreme points $s^1, s^2, \ldots, s^p$ and a convex combination $\sum_{j=1}^{p} \lambda_j s^j$ which yields x^*. Also if C is integral, then by the properties of the perturbation to c^i in obtaining c, there is a unique vector which optimizes c. Let this optimal value be

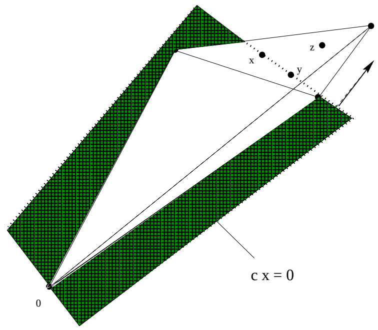

Figure 1.2

$K \in \mathcal{Z}$. Thus for all but at most one j, $c \cdot s^j \leq K-1$. Without loss of generality, the real optimum is achieved by say s^1. Set $\epsilon' = \frac{\epsilon}{2R^2}$, then we have

$$K - \epsilon' < \beta = c \cdot x^* = \lambda_1(c \cdot s^1) + \sum_{i=2}^{p} \lambda_i (c \cdot s^i) \leq \lambda_1 K + (1-\lambda_1)(K-1)$$

from which we deduce that $K - \epsilon' < K - (1-\lambda_1)$ and so we must have $\lambda_1 \geq (1-\epsilon')$. Hence for any component x_u^*, the contribution from the s_u^j's for $j > 1$ is at most $R\frac{\epsilon}{2R^2} \leq \frac{1}{4R}$. It follows that a component x_u^* is within $\frac{1}{2}$ of some integer k if and only if $s_u^1 = k$. We have thus shown that if the new vector x is not in K, then K could not have been integral, and so we are justified in outputting $[\emptyset]$. This completes the proof. □

1.3 The Theta Body

1.3.1 Three convex bodies

First, its *stable set polytope*, denoted by $P(G)$, is the convex hull of the incidence vectors of its stable sets: $conv\{\chi^S : S$ is a stable set of $G\}$. In particular, $P(G)$ lies in $\mathcal{R}_+^V$ and is polyhedral by classical results of Weyl and Minkowski. The antiblocker of $P(G)$ is sometimes called the *fractional clique polytope* of G and is denoted by $\mathcal{Q}(\bar{G})$. One deduces (cf. [19]) that

$$\mathcal{Q}(\bar{G}) = \{x \in \mathcal{R}_+^V : x(S) \leq 1 \text{ for each stable set } S\}$$

where $x(S)$ is shorthand for the value of the sum $\sum_{v \in S} x_v$. One consequence of this is that $\mathcal{Q}(\bar{G})$ is polyhedral. The polytopes $P(\bar{G}), \mathcal{Q}(G)$ are defined analogously and are referred to as the *clique polytope* of G and the *fractional stable set polytope* of G. Since for each stable set S and clique K we have that $\chi^S \cdot \chi^K \leq 1$ (i.e., $|S \cap K| \leq 1$) we see that $P(G) \subseteq \mathcal{Q}(G)$ and $P(\bar{G}) \subseteq \mathcal{Q}(\bar{G})$.

Let $V = \{1, 2, \ldots, n\}$ and d be an arbitrary integer. A set of unit vectors $v^1, v^2, \ldots, v^n \in \mathcal{R}^d$ is an *orthonormal representation of* G if $v^i \cdot v^j = 0$ whenever $ij \notin E$. For a given vector $x \in \mathcal{R}^n$, a *certificate* (in dimension d) is a collection of unit

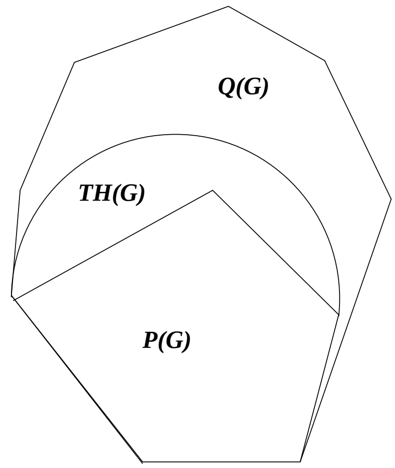

Figure 1.3

vectors $v^1, v^2, \ldots, v^n, c \in \mathcal{R}^d$ such that $v^1, v^2, \ldots, v^n$ is an orthonormal representation of G and $x_i = (v_i \cdot c)^2$ for each $i \in V$. We now define the *theta body of a graph*:

$$TH(G) \equiv \{x \in \mathcal{R}_+^n : \ \exists \text{ a certificate for } x \text{ in some dimension } d\}.$$

The theta body is formally introduced in [26] where the following is shown.

Theorem 1.7 *(Grötschel, Lovász, Schrijver) For any graph G, $TH(G)$ is a convex corner and $\mathcal{A}(TH(G)) = TH(\bar{G})$. Moreover, $P(G) \subseteq TH(G) \subseteq \mathcal{Q}(G)$ and a graph G is perfect if and only if $P(G) = TH(G)$.*

It turns out that working over $TH(G)$ directly can be difficult; as noted in [26], it is not even straightforward to deduce that it is a convex corner. Instead, one works with a class of $n \times n$ matrices (introduced in [35]) of which $TH(G)$ turns out to be a projection. We study this class in the next section.

For a graph G, we denote by $\vartheta \equiv \vartheta(G)$ the value $\max\{\bar{\mathbf{1}} \cdot x : x \in TH(G)\}$. We also define α_w similarly. The idea of using orthonormal representations and ϑ to approximate the size of a maximum stable set α originated in the work of Lovász on the Shannon capacity [35]. There it was first shown that $\alpha(G) \leq \vartheta(G)$ for any graph G. The multiplicative gap in general can be extremely large (the order of n^ϵ) as was first shown by Feige [17]. The reader is referred also to further results in [1, 50].

1.3.2 A semi-definite formulation

For any $w \in \mathcal{Z}_+^V$, we denote by $\vartheta_w(G)$ the value $\max\{w \cdot x : x \in TH(G)\}$. We also denote by G_w, the graph obtained from G by replacing each node v by a set C_v of w_v mutually nonadjacent nodes. We also add an edge between $x \in C_u$ and $y \in C_v$ if $uv \in E(G)$. Note that it is obvious that $\alpha_w(G) = \alpha(G_w)$, but requires some argument to verify that the theta-function also satisfies this property.

We define a convex set $\mathcal{B}(G)$ as the set of $n \times n$ matrices B such that (i) B is symmetric positive semidefinite (ii) $B_{ij} = 0$ if $ij \in E(G)$ and (iii) $trace(B) = 1$. Note that the set of matrices satisfying (i) and (ii) form a cone C in $\mathcal{R}^{n^2}$ and so $\mathcal{B}(G)$ is obtained by restricting to the subset $\{B \in C : \sum_i B_{ii} = 1\}$. In the remainder, for any $w \in \mathcal{R}^n_+$ and $B \in \mathcal{R}^{n \times n}$ we denote by $w(B)$ the quantity $\sum_{i,j} \sqrt{w_i w_j} B_{ij}$.

We now define two operations which take us back and forth between $\mathcal{B}(G)$ and $TH(G)$.

[$\mathcal{B}(G) \to TH(G)$]: Consider $B \in \mathcal{B}(G)$ arising as a Gram-matrix from vectors $z^1, z^2, \ldots, z^n$. For any $w \in \mathcal{R}^n$ we define a vector $\theta(B, w) \in TH(G)$ as follows. We let θ be the vector obtained from the orthonormal representation $\{v^i \equiv \frac{z^i}{\|z^i\|} : i = 1, 2, \ldots, n\}$ and unit vector $c = \frac{\sum_i \sqrt{w_i} z^i}{\|\sum_i \sqrt{w_i} z^i\|}$. Note then that $\sum_i w_i \theta_i = \sum_i w_i (c \cdot v^i)^2 = (\sum_i \|z^i\|^2)(\sum_i (\sqrt{w_i}(c \cdot v^i))^2)$ since $trace(B^*) = 1$. By Cauchy-Schwarz inequality this is at least

$$(\sum_i \sqrt{w_i}(c \cdot v^i)\|z^i\|)^2 = (\sum_i \sqrt{w_i}(c \cdot z^i))^2 = (c \cdot (\sum_i \sqrt{w_i} z^i))^2 =$$

$$((\frac{\sum_i \sqrt{w_i} z^i}{\|\sum_i \sqrt{w_i} z^i\|}) \cdot (\sum_i \sqrt{w_i} z^i))^2 = \|\sum_i \sqrt{w_i} z^i\|^2 = w(B). \quad (1.1)$$

Thus $w \cdot \theta \geq w(B)$.

[$TH(G) \to \mathcal{B}(G)$]: Conversely, let $x \in TH(G)$ arise from an orthonormal representation $v^1, v^2, \ldots, v^n$ and unit vector c. For any $w \in \mathcal{R}^n_+$ we define a matrix $\varpi(x, w)$ as follows. For each $i = 1, 2, \ldots, n$ set $\lambda_i = \sqrt{\frac{w_i}{\sum_i w_i (c \cdot v^i)^2}}(c \cdot v^i)$ and then define $\varpi_{i,j} = \lambda_i \lambda_j (v^i \cdot v^j)$ for each i, j. One sees that $\sum_i \lambda_i^2 = 1 = trace(\varpi)$ and so $\varpi \in \mathcal{B}(G)$. Moreover, the vector $a = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is parallel to $b = (\sqrt{w_1}(c \cdot v^1), \sqrt{w_2}(c \cdot v^2), \ldots, \sqrt{w_n}(c \cdot v^n))$. Since $\|a\|^2 = 1$, $\sum_i w_i x_i = \|a\|^2 \|b\|^2$ which, since $a \parallel b$, is equal to $a \cdot b = (\sum_i \sqrt{w_i} \lambda_i (c \cdot v^i))^2$. The remaining equalities (with λ_i replacing $\|z^i\|$) in (1.1) follow as before and so we deduce that $w(B) = \sum_i w_i x_i$.

The following result shows that optimizing a linear objective over $TH(G)$ is equivalent to the semidefinite programming problem of optimizing over $\mathcal{B}(G)$. It can be deduced by applying the two operations just defined.

Theorem 1.8 *(Lovász[35], Grötschel, Lovász, Schrijver [27])*[6] *Let G be a graph and w be a vector of nonnegative node weights. Then*

$$\vartheta_w(G) = \max\{\sum_{i,j} \sqrt{w_i w_j} B_{ij} : B \in \mathcal{B}(G)\} = \vartheta(G_w). \quad (1.2)$$

Much like Lovász's proof of the Perfect Graph Theorem, the above result also shows that weighted optimization for $TH(G)$ reduces to the cardinality problem in the expanded graph G_w. Let $\Lambda(A)$ denote the largest eigenvalue of a matrix A. The key duality theorem for $\mathcal{B}$ is as follows.

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[6] As described in [26], the proof of the first equality is an extension of the case $w = \bar{1}$ which is proved in [35]. A complete proof for the weighted case is first given in [27]. The result is presented in [25], but the value which they denote by ϑ_w is *defined* to be $\vartheta(G_w)$.

Theorem 1.9 *(Strong Duality Theorem, [35]) For any graph G:*

$$\max\{\sum_{i,j} \sqrt{w_i w_j} B_{ij} : B \in \mathcal{B}(G)\} = \min\{\Lambda(A) : A_{ij} = 1 \textit{ unless } ij \in E\}.$$

One may show that the related parameter $\vartheta'(G) = \max\{\sum_{i,j} \sqrt{w_i w_j} B_{ij} : B \geq 0, B \in \mathcal{B}(G)\}$ also yields an upper bound for $\alpha(G)$. This stronger relaxation was studied in [38] and [43]. The following example, attributed to M. R. Best in [43], shows that ϑ' may be strictly less than ϑ in general. Let G be the graph whose nodes are the binary strings of length 6. We join two nodes by an edge if their Hamming distance is at most three. Then $\vartheta'(G) = 4 < \frac{16}{3} = \vartheta(G)$. Further projection (beyond lifting into $\mathcal{R}^{n^2}$) methods which give even tighter bounds are introduced by Lovász and Schrijver in [37].

1.3.3 Algorithms for the theta body

The class of convex sets $\mathcal{B} = \{\mathcal{B}(G) : G \text{ is a graph}\}$ is algorithmically manageable since the membership problem essentially reduces to a Gaussian Elimination routine (cf. [25]). One may then use Yudin-Nemirovskiĭ's Theorem, Theorem 1.5, to obtain weak separation and optimization. Note that $\mathcal{B}$ is not a class of convex bodies and so some care is needed in obtaining the following. [7]

Theorem 1.10 *(Grötschel, Lovász, Schrijver)*

- *There exists a yes-definite weak separation algorithm $matrix(G, B)$ for $\mathcal{B}$ whose running time is bounded by a polynomial in n and $size(B)$.*
- *There is a polytime algorithm $MatrixOpt(G, w, \epsilon)$ which, given a graph G, $w \in \mathcal{Q}^n$ and $\epsilon > 0$, finds a matrix $B^* \in \mathcal{B}(G)$ such that $w \cdot B^* \cdot w \geq (1 - \epsilon) \max\{w \cdot B \cdot w : B \in \mathcal{B}(G)\}$.*

One easily sees that the class $\mathcal{T} = \{TH(G) : G \text{ is a graph}\}$ does consist of well-bounded convex bodies. A weak optimization algorithm for $TH(G)$ is derived from Theorem 1.10 in [26]. This also implies the existence of a separation algorithm. As we will require it later, we describe a no-definite weak separation algorithm explicitly.

Theorem 1.11 *There is a polytime weak separation algorithm $theta_{Sep}(G, x, \epsilon)$ for $\mathcal{T}$.*

Proof. Note first that by Theorem 1.7, a given vector $x \in \mathcal{Q}^n$ is in $TH(G)$ if and only if $x \geq 0$ and $\max\{x \cdot y : y \in TH(\bar{G})\} \leq 1$. Thus by Theorem 1.8, it would be sufficient to check that $\max\{\sum_{i,j} \sqrt{x_i x_j} B_{ij} : B \in \mathcal{B}(\bar{G})\} \leq 1$. The algorithm first checks that $\bar{\mathbf{1}} \geq x \geq \bar{\mathbf{0}}$ and then calculates values w_i such that $|w_i - \sqrt{x_i}| < \frac{\epsilon}{4n^3}$ for each i. This may be done for instance, by the method of continued fractions. We now run the algorithm $MatrixOpt(\bar{G}, w, \frac{\epsilon}{4(n+\epsilon)})$ and let B^* be the output and σ be the associated

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[7] Note that we must deal with a well-bounded convex body, and so one actually works with a derived class of convex bodies. This new class lies in $\mathcal{R}^{|V|+\binom{n}{2}-|E|-1}$ and is basically obtained by ignoring the entries in a symmetric matrix below the diagonal (cf. [25, 27]).

optimum. Recall that $MatrixOpt$ assures that $B^* \in \mathcal{B}(\bar{G})$ (it is yes-definite) and so B^* is the Gram matrix for some vectors $z^1, z^2, \ldots, z^n$; one may find such a set of vectors in polytime using a Gaussian elimination procedure (cf. [25, 27]). We now have

$$|\sqrt{x} \cdot B^* \cdot \sqrt{x} - w \cdot B^* \cdot w| < \frac{\epsilon}{4n} \tag{1.3}$$

since $|B^*_{ij}| = z^i \cdot z^j = \|z^i\|\|z^j\||cos(z^i, z^j)| \leq 1$ (we need the fact that $B^*_{ii} \leq trace(B^*) = 1$).

Suppose first that $\sigma > 1 + \frac{\epsilon}{2n}$. Let $y^* = \theta(B^*, w)$ and so $\sum_i x_i y^*_i \geq w(B^*) > \sigma - \frac{\epsilon}{4n} > 1 + \frac{\epsilon}{4n}$ by (1.3) where as $\sum_i u_i y^*_i \leq 1$ for each $u \in TH(G)$. Thus y^* defines the desired strict separating hyperplane and so $x \notin TH(G)$.

Conversely, suppose that $\sigma \leq 1 + \frac{\epsilon}{2n}$. By the properties of $MatrixOpt$, we know that $\sigma \geq (1 - \frac{\epsilon}{4(n+\epsilon)}) \max\{w \cdot y : y \in TH(\bar{G})\} \geq (1 - \frac{\epsilon}{4(n+\epsilon)})(\max\{x \cdot y : y \in TH(\bar{G})\} - \frac{\epsilon}{4n})$ from which one deduces that $\max\{x \cdot y : y \in TH(\bar{G})\} \leq 1 + \frac{\epsilon}{n}$. Thus $(\frac{1}{1+\frac{\epsilon}{n}})x \in TH(G)$ and hence $x \in S(TH(G), \epsilon)$ since $\|x\| \leq n$. □

We now have the following by Theorem 1.4, since $\mathcal{T}$ is a class of convex bodies.

Corollary 1.12 *(Grötschel, Lovász, Schrijver [26]) There is a polytime weak optimization algorithm* $theta_{opt}(G, w, \epsilon)$ *for* $\mathcal{T}$.

We remark that one may not eliminate the presence of error terms when optimizing over $TH(G)$ (at least for computational models using binary – or any rational – representation of real numbers). This is because there are instances for which there is a unique optimal solution which is not even rational; for instance, maximizing $\bar{\mathbf{1}} \cdot x$ over $x \in TH(C_5)$ (C_5 denoting the 5-cycle), produces an optimum value of $\sqrt{2}$.

The same may not be true for the separation problem. That is, it may be the case that there is a polytime strong (or yes-definite) separation algorithm for the class of theta bodies – as is the case for $\mathcal{B}$ (see Theorem 1.10). Suppose for instance, that we are given a vector x and a proposed certificate $v^1, v^2, \ldots, v^n, c$. For any $\lambda \in \mathcal{R}^n$ such that $\sum_{i=1}^n \lambda_i^2 = 1$, we may define a Gram matrix $B(\lambda)$, via the vectors $\{z^i = \lambda_i v^i : i = 1, 2 \ldots, n\}$, that is, $B(\lambda)_{ij} = \lambda_i \lambda_j (z^i \cdot z^j)$ for each i, j. One may clearly compute such a vector λ in polytime and then input the matrix $B(\lambda)$ to $matrix$ – x's membership in $TH(G)$ is then attested by the vectors v^i and c if and only if $B(\lambda) \in \mathcal{B}(G)$, and $(v^i \cdot c)^2 = x_i$ for each node i. Unfortunately, one may not be given a proposed certificate for a vector x. Even worse, there may be no such certificate whose size is polynomially bounded in the size of x. If some component of x is, for example, $\frac{1}{2}$, then there can be no certificate for x where both v^i and c are rational vectors!

We call a matrix $B(\lambda)$ a *parent* of x if it is the Gram matrix resulting from some vectors $z^1, z^2, \ldots, z^n$ for which there is a unit vector c such that $(\frac{1}{\|z^i\|} z^i \cdot c)^2 = x_i$ for each i. We believe that for each x there is a polytime computable matrix B^x such that $x \in TH(G)$ if and only if $B^x \in \mathcal{B}(G)$. Moreover, if $x \in TH(G)$, then the computed matrix B^x is in fact a parent of x. We are unable to show this, but we have at least one of its implications.

Theorem 1.13 *For any graph* G*, each* $x \in TH(G)$ *has at least one rational parent* B.

Proof. To see this, let $v^1, v^2, \ldots, v^n, c$ be a certificate for x and let $S_1 = \{\lambda \in \mathcal{R}^n : \|\lambda\| = 1\}$. For each i, j let $Q_{ij} = \{\lambda \in S_1 : B(\lambda)_{ij} \in \mathcal{Q}\}$. One sees that each Q_{ij} is dense in S_1 and hence so is $Q = \cap_{ij} Q_{ij}$. Therefore for any $\lambda \in Q$, we have that $B(\lambda)$ is a rational parent of x. □

We have been unable to show how to 'pre-select' a matrix B^x as described above. We nevertheless conjecture the following, and in the next section we discuss an alternative approach to resolving it.

Conjecture 1.14 *There is a polytime strong (or yes-definite) separation algorithm for the class $\mathcal{T}$.*

1.3.4 Additive gap guarantees and the protrusion of the theta body

In this section we discuss another approach to solving the strong separation problem for theta. In particular, given z, we may first compute an integer K such that $Kz = w$ is an integral vector. Theorem 1.7 implies that $z \in TH(G)$ if and only if $\max\{wx : x \in TH(\bar{G})\} \leq K$. We now seek a polynomially bounded $\epsilon > 0$, such that if if this optimum is strictly greater than K, then it is at least $K + \epsilon$. If $size(\epsilon)$ does not get too large, we may thus use a weak optimization routine to solve the strong separation problem in polytime.

For a graph G, a *slack pair* is any pair $w \in \mathcal{Z}^n, \gamma \in \mathcal{Z}$ such that

$$\vartheta_w \equiv \max\{w \cdot x : x \in TH(G)\} > \gamma.$$

We sometimes call w a *slack direction* to mean that $\vartheta_w(G)$ is non-integral. Any such vector determines a direction in which theta bulges out or protrudes from the stable set polytope. We also refer to $\{x : w \cdot x = \gamma\}$ as a *slack hyperplane*; we are especially interested in such hyperplanes which are supporting to $P(G)$. In this case, we are finding lower bounds for the quantity $\vartheta_w - \alpha_w$, since $\gamma = \alpha_w(G)$. The first such bounded gap result is given in [25] for the case that G is minimally imperfect and $w = \bar{\mathbf{1}}$. In fact, their proof extends identically to partitionable graphs and so we have the following.

Theorem 1.15 *(Grötschel, Lovász, Schrijver) If G is a (p, q)-graph, then*

$$\max\{\bar{\mathbf{1}} \cdot x : x \in TH(G)\} > (1 + \frac{\frac{n}{2}}{(np)^n - n})p > p.$$

We do not include the proof since we describe a more general result (Theorem 1.20) later. Theorem 1.15 implies the following important consequence for partitionable graphs.

Corollary 1.16 *If G is a (p, q)-graph, then $\frac{\bar{\mathbf{1}}}{\mathbf{q}} \notin TH(G)$.*

Proof. Theorem 1.7 asserts that $x \cdot y \leq 1$ for each $x \in TH(G), y \in TH(\bar{G})$. Thus if $\frac{\bar{\mathbf{1}}}{\mathbf{q}} \in TH(G)$, then $\sum_i y_i \leq q$ for each $y \in TH(\bar{G})$ which contradicts Theorem 1.15. □

We now embark on extending the additive gap result of Theorem 1.15 to arbitrary slack directions w. We first introduce some notation.

For an $n \times n$ matrix B and $w \in \mathcal{Z}^n$, we define for each row i, its *weighted row sum:* $W_i(B) \equiv \sum_j \sqrt{w_i w_j} B_{ij}$. Thus $w(B) = \sum_i W_i(B)$. Suppose that B is a Gram-matrix for some vectors $\mathcal{V} = v^{(1)}, v^{(2)}, \ldots, v^{(n)}$. A matrix B' is a *cousin* of B if B' is the Gram-matrix for some vectors $\alpha_1 v^{(1)}, \alpha_2 v^{(2)}, \ldots, \alpha_n v^{(n)}$, where each $\alpha_i \geq 0$ and $trace(B') = trace(B)$.

Lemma 1.17 *Let G be a graph and $w \in \mathcal{Z}^n$. Suppose that $B \in \mathcal{B}(G)$ such that $w(B) \geq M$. Then B has a cousin B^* such that (i) $w(B^*) \geq M$ and (ii) $W_i(B^*) = B^*_{ii} w(B^*)$ for each $i = 1, 2, \ldots, n$. Moreover, if B is a nonsingular, nonnegative matrix such that the graph $H = (V, \{ij : B_{ij} \geq \frac{1}{Q}\})$ is connected, then $det(B^*) \geq (\frac{1}{Q^3 M})^{n(n-1)} det(B)$.*

Proof. Let B arise as a Gram-matrix for some vectors $\mathcal{V} = v^{(1)}, v^{(2)}, \ldots, v^{(n)}$. For some pair $l \neq m$ we consider replacing $v^{(l)}, v^{(m)}$ by $\kappa v^{(l)}$ and $\beta v^{(m)}$ respectively, for some $\kappa, \beta > 0$. Let $B(\kappa, \beta)$ be the positive semi-definite matrix which results as the Gram-matrix for the new collection. First note that $B(\kappa, \beta) \in \mathcal{B}(G)$ only if $g(\kappa, \beta) \equiv (\kappa^2 - 1)B_{ll} + (\beta^2 - 1)B_{mm} = 0$. Next note that the contribution of any entry B_{ij} to $w(B(\kappa, \beta))$ is the same if neither l nor m is in $\{i, j\}$. For each i, j define $a_{ij} = \sqrt{w_i w_j} B_{ij}$ and let $W_i = W_i(B)$. Set $S_l = W_l - a_{ll} - a_{lm}$ and similarly $S_m = A_m - a_{mm} - a_{lm}$. Finding κ, β which maximize $w(B(\kappa, \beta))$ can then be expressed as:

$$
\begin{aligned}
&Maximize \quad F(\kappa, \beta, \lambda) \equiv \kappa^2 a_{ll} + \beta^2 a_{mm} + 2\kappa S_l + 2\beta S_m + 2\kappa\beta a_{lm} - \lambda g(\kappa, \beta) \\
&s.t. \\
&\kappa, \beta \geq 0
\end{aligned}
$$

We know that if the optimum occurs for some $\kappa = 1 = \beta$, then we must have $\nabla F(1, 1, \lambda_0) = 0$. Since $\frac{\partial F}{\partial \kappa} = 2\kappa a_{ll} + 2S_l + 2\beta a_{lm} - 2\lambda_0 \kappa B_{ll}$ we must have $\frac{W_l}{B_{ll}} = \lambda_0$. Similarly $\frac{W_m}{B_{mm}} = \lambda_0$. Hence in general we have that $\lambda_0 B_{ii} = W_i$ for each i. Thus $w(B) = \sum_i W_i = \sum_i \lambda_0 B_{ii} = \lambda_0$ and so each weighted row sum is proportional to its diagonal entry in B^* as required.

We now prove the second assertion. Define $\alpha_i^2 = \frac{B^*_{ii}}{B_{ii}}$. Since $\sum_i \alpha_i^2 B_{ii} = 1$ we have that $\alpha_i^2 \leq Q$ for each i. Thus $w(B^*) \leq Q^2 w(B)$. Without loss of generality we also have that $\alpha_1 \geq 1$.

Claim 1.18 *Any node at distance k from 1 in the graph H satisfies $\alpha_i \geq (\frac{1}{Q^3 M})^k$.*

Proof of Claim. The base case is trivial, so suppose that the shortest path from node i to 1 is of length $k + 1 \geq 1$. Let (P, l, i) be such a path and note by the induction hypothesis $\alpha_l \geq (\frac{1}{Q^2 M})^k$. We have that $w(B^*)\alpha_i^2 B_{ii} = W_i = \sum_j \alpha_i \alpha_j \sqrt{w_i w_j} B_{ij}$. Thus $\alpha_i = (\sum_j \alpha_j \sqrt{w_i w_j} B_{ij})/(w(B')B_{ii}) \geq (\alpha_l \sqrt{w_l w_i} B_{il})/(w(B')B_{ii})$. The claim follows from the induction hypothesis and since $B_{ii} \leq 1, w(B^*) \leq Q^2 M, w_l w_i \geq 1$, and $B_{il} \geq \frac{1}{Q}$. □

Finally, we have that $det(B^*) = (\Pi_i \alpha_i^2) det(B)$. On the other hand, if $d(i1)$ denotes the distance in H between node i and node 1, then a simple proof by induction shows that $\sum_{i=0}^n d(i1) \leq \frac{n(n-1)}{2}$. This completes the proof. □

Lemma 1.19 *Let G be a graph and F be a nontrivial, non-clique facet of $P(G)$ which is induced by an inequality $w \cdot x \leq \gamma$, where w, γ are integral. Then*

$$\max\{w \cdot x : x \in TH(G)\} > (1 + \frac{n-1}{(n^2\gamma^{n-1})^n - n})\gamma > \gamma.$$

Proof. We assume that we have a minimal counterexample in the sense that the inequality $\sum_{i \in S} w_i x_i \leq \gamma$ is not facet-inducing for any proper induced subgraph $G[S]$ for otherwise it is enough to prove the result for the subgraph. In particular, we may assume that $w > 0$. Let S be a matrix whose rows consist of n linearly independent incidence vectors of maximum w-weight stable sets $S_1, S_2, \ldots, S_n$. Set $B = S^T S$ and observe that B is positive definite and that $B_{ij} = n_{ij}$ where n_{ij} is the number of maximum stable sets containing both i and j. Thus $B_{ij} = 0$ if $ij \in E$ and $B_{ii} > 0$. We also claim that $H = (V, \{ij : n_{ij} > 0\})$ is connected. For otherwise, there is some subset S, such that each tight stable set is either contained entirely in S, or entirely in $V - S$. But then, the example is not minimal as we may restrict attention to $G[S]$.

Since B is positive definite and integral, $det(B) \geq 1$. Set $T = trace(B)$ and let B^* be the cousin of B whose existence is assured by Lemma 1.17 (applied to $\frac{1}{T}B$). We then have: $w(B^*) = \sum_{i,j} \sqrt{w_i w_j} B^*_{ij} = w(B^*) \sum_i B^*_{ii} = w(B^*)T \geq \gamma T$. We also have by Lemma 1.17 that $det(B^*) = T^n det(\frac{1}{T}B^*) \geq (\frac{1}{\gamma})^{n(n-1)}$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ be the eigenvalues of B^*. Since $det(B^*) = \Pi_i \lambda_i$ we have that $\lambda_n \geq \frac{det(B^*)}{(\Pi_{i=1}^{n-1} \lambda_i^{-1})}$. But also $\sum_i \lambda_i = T$ so we have that each $\lambda_i \geq \frac{1}{T}$. Thus $\lambda_n \geq \frac{1}{\gamma^{n(n-1)} T^{n-1}}$. Set $Z = (\gamma^{n-1}T)^n$. We thus have that $B' = \frac{1}{T}B$ is a positive definite matrix contained in $\mathcal{B}(G)$ whose smallest eigenvalue is at least $\frac{1}{Z}$ and for which $w(B') = \frac{1}{T}w(B^*) \geq \gamma$.

Construct a matrix B'' by subtracting $\frac{1}{Z}$ from each entry on the main diagonal of B' and then multiplying every entry by $\frac{Z}{Z-n}$. We have that $B'' \in \mathcal{B}(G)$ and: $\sum_{ij} \sqrt{w_i w_j} B''_{ij} = \frac{Z}{Z-n}(w(B') - \sum_i \frac{w_i B'_{ii}}{Z}) \geq \gamma + \frac{n\gamma}{Z-n} - \frac{\gamma}{Z-n}$, where the last inequality follows from the fact that $trace(B') = 1$ and $w_i \leq \gamma$ for each i. Thus

$$w(B'') \geq (1 + \frac{(\gamma - 1)n}{Z - n})\gamma.$$

The result now follows as $T \leq n^2$. □

A slack pair w, γ is *supporting* if $\{x : w \cdot x = \gamma\} \cap P(G) \neq \emptyset$. We now show that any supporting slack pair (not just facet-inducing) satisfies a bounded gap property.

Theorem 1.20 *(Additive Gap Guarantee) There is a polynomial $P(\cdot, \cdot)$ such that for each graph $G = (V, E)$ and each supporting slack pair $w \in \mathcal{Z}^V, \gamma \in \mathcal{Z}$ we have:*

$$\max\{w \cdot x : x \in TH(G)\} > (1 + \frac{1}{N})\gamma > \gamma,$$

where $size(N) \leq P(n, size(w))$.

Proof. Let w, γ define a (rational) supporting slack pair. Then there exists facet-inducing inequalities $w^k \cdot x \leq \gamma$ (the w^k's need not be integral), for $STAB(G)$ and rationals $y_k > 0$ for each $k = 1, 2, \ldots t$, such that $w \leq \sum_k y_k w^k$ and $1 = \sum_k y_k$. Note that at least one of the w^k's is a slack direction, otherwise $w \cdot x \leq \gamma$ is implied by the clique and nonnegativity constraints (i.e., it is valid for $TH(G)$) and so it cannot be a slack direction. Since the vertex complexity of $P(G)$ is at most n, the facet-complexity is at most $4n^3$. Hence if T is the smallest integer such that Tw^k is integral, then $size(T\gamma) \leq 4n^3$. (The reader is referred to [33] for interesting recent directions in the study of the facets of $P(G)$.) Thus by Lemma 1.19, if w^k is a slack direction, then $\vartheta_{w^k}(G) > (1+\frac{1}{N'})\gamma$ where $N' = (n^2 2^{4n^3})^n$.

Next note that given any w^k's and w, one may solve for the y_k's using Gaussian elimination in polynomial time. Thus, there is a polynomial P_1 such that for any given G, w^k's and w, $size(y_k) < P_1(n, size(w))$ for each k.

Now for each k, let $B^{(k)} \in \mathcal{B}(G)$ such that $\sum_{i,j} \sqrt{w_i^k w_j^k} B_{ij}^{(k)} = \vartheta_{w^k}$. Then $B \equiv \sum_k y_k B^{(k)}$ is in $\mathcal{B}(G)$ by convexity. We then have

$$w(B) = \sum_{i,j} \sqrt{w_i w_j} B_{ij} = \sum_{i,j} \sqrt{(\sum_k y_k w_i^k)(\sum_k y_k B_{ij}^{(k)})} \sqrt{(\sum_k y_k w_j^k)(\sum_k y_k B_{ij}^{(k)})} \quad (1.4)$$

$$\geq \sum_{i,j} (\sum_k y_k \sqrt{w_i^k B_{ij}^{(k)}})(\sum_k y_k \sqrt{w_j^l B_{ij}^{(k)}}) \quad (1.5)$$

Apply Cauchy-Schwarz to say $(\sqrt{y_1 w_i^1}, \ldots, \sqrt{y_t w_i^t})$ and $(\sqrt{y_1 B_{ij}^{(1)}}, \ldots, \sqrt{y_t B_{ij}^{(t)}})$

$$\geq \sum_{i,j} \sum_s \sqrt{y_s^2 w_i^s w_j^s} B_{ij}^{(s)} = \sum_s \sum_{i,j} \sqrt{y_s^2 w_i^s w_j^s} B_{ij}^{(s)}$$

$$= \sum_{s=1}^{t} y_s (\sum_{i,j} \sqrt{w_i^s w_j^s} B_{ij}^{(s)}) = \sum_{s=1}^{t} y_s (\vartheta_{w^s}) \geq \gamma.$$

Moreover, since some w^s is slack, we have that $\sum_{i,j} \sqrt{w_i^s w_j^s} B_{ij}^{(s)} \geq (1+\frac{1}{N'})\gamma$ as deduced earlier. Thus (1.4) has a gap of at least $y_s \frac{1}{N'}\gamma$. In other words, $w(B) > (1 + \frac{1}{N' 2^{P_1(n, size(w))}})\gamma$, and this completes the proof. □

The above result does not address the slack hyperplanes which do not support $P(G)$ and hence Question 1.2 remains open.

1.4 Partitionable Graphs

1.4.1 A characterization

Recall that a graph $G = (V, E)$ is a (p,q)-graph if $|V| = pq + 1$, and for each node v, $G - v$ can be partitioned into q stable sets of size p, and into p cliques of size q.

Remark 1.21 *If G is a (p,q)-graph, then*

- *$\alpha(G) = p$ and $\omega(G) = q$*
- *$\chi(G) = \omega(G) + 1$ and $\chi(\bar{G}) = \alpha(G) + 1$.*

The simple proof of the above statements also shows that a graph G is a (p,q)-graph if and only if $|V| = pq + 1$ and for each node v, $G - v$ is both q-colourable and p-clique-coverable, i.e., we impose no restrictions on the sizes.

Given that it is NP-Hard to determine whether $\chi(G) = \omega(G)$ for a graph G, partitionability seems a rather ruly property which should be complicated to check. One sign in the opposite direction was a characterization of such graphs based only on linear programming [49]. The intractability of the LPs involved, however, gave little benefit to recognizing a partitionable graph. Instead we must resort to a characterization based on semi-definite programming.

In the following, a *meshed system* for a graph G is a collection of $n = |V(G)|$ sets $S_i, K_i : i = 1, 2 \ldots, n$ such that (i) each S_i is a stable set of G, (ii) each K_i is a clique of G and (iii) for each i, j we have $|S_i \cap K_j| = \delta_{ij}$. We call K_i, S_i *mates* in this system. A meshed system is *p-regular* if each node is in p stable sets from the system. One easily deduce that each clique in a regular system has the same cardinality q and that therefore $n = pq + 1$; we refer to this has a (p, q)-mesh. Note that one may easily determine whether a given collection of sets forms a (p, q)-mesh.

Theorem 1.22 *(Bland, Huang, Trotter) If G is partitionable, then it has $|V|$ maximum cliques, $|V|$ maximum stable sets, and these form a regular meshed system.*

An older combinatorial characterization (used later) relaxes the definition of partitionability. It is "one-sided" in the sense it requires us to find only the family of stable sets associated with the node-deleted colourings.

Theorem 1.23 *[48] A graph G is a (p, q)-graph if and only if $|V| = pq + 1$ and G has a family of $|V|$ stable p-sets such that (i) each node is in precisely p of these sets and (ii) for each set in the family, there is at least one q-clique disjoint from it.*

We now focus on the extension of a result of Padberg.

Theorem 1.24 *(Padberg $\Rightarrow$, [41]),($\Leftarrow$, [48]) A graph G is minimally imperfect if and only if $P(H) = \mathcal{Q}(H) \cap \{x \in \mathcal{R}^V : x \cdot \bar{\mathbf{1}} \leq \alpha(H)\}$ for both $H = G$ and $\bar{G}$.*

Padberg showed that if G is minimally imperfect, then $\frac{1}{\omega(G)}\chi^V$ is the unique fractional vertex of $\mathcal{Q}(G)$, and it only has as neighbours the incidence vectors of maximum stable sets. Thus adding the constraint $x \cdot \bar{\mathbf{1}} \leq \alpha(G)$, not only slices off $\frac{1}{\omega(G)}\chi^V$, but it does not introduce new fractional extreme points. Somewhat unexpectedly, a similar property also holds for partitionable graphs.

Theorem 1.25 *If G is a (p, q)-graph, then $\mathcal{E}(\mathcal{Q}_p(G)) \cup \mathcal{E}(\mathcal{Q}_q(\bar{G}))$ is the set of incidence vectors of a (p, q)-mesh for G.*

Proof. Let A, B be the $|V| \times |V|$ set-node incidence matrices of a (p, q)-mesh, where A is associated with the stable sets and B with the cliques. Now suppose that $x^* \in \mathcal{E}(\mathcal{Q}_p(G))$ which is not from the meshed system. By nonsingularity of A there is a solution λ^* to the system $A^T \cdot \lambda = x^*$. Moreover, as each column of A^T has p ones and $\bar{\mathbf{1}} \cdot x^* = p$ we must have $\bar{\mathbf{1}} \cdot \lambda^* = 1$. Since $BA^T = J - I$ we may multiply $A^T \cdot \lambda^* = x^*$ by B to obtain: $(J - I)\lambda^* = Bx^* \leq \bar{\mathbf{1}}$ where the last inequality follows from the fact that $x^* \in \mathcal{Q}(G)$. But then we have that for each j: $1 \geq \bar{\mathbf{1}} \cdot \lambda^* - \lambda_j^* = 1 - \lambda_j^*$. Thus $\lambda^* \geq 0$ and so x^* is a convex combination of the stable p-sets of G, a contradiction. □

The geometric interplay between the smooth and polyhedral parts of $TH(G)$ is a bit mysterious. Recall that it is entirely polyhedral if and only if G is perfect.

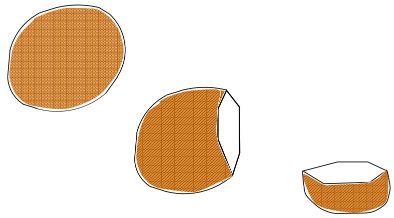

Figure 1.4

Indeed a result of [25] shows that an extreme point of $TH(G)$ is a vertex if and only if it is the incidence vector of a clique; any other extreme point must lie on a 'smooth' portion of the boundary. Thus for a partitionable graph G, the region $\{x \in TH(G) : x \cdot \bar{\mathbf{1}} > \alpha(G)\}$ is a smooth surface. For these graphs, the theta body seems very far from being polyhedral given that there may be multiple copies of imperfect subgraphs strewn about within it. One may visualize $TH(G)$, as described by Vašek Chvátal, the great Celt, as a "potatope" – see Figure 1.4. The following shows that sometimes the intersection of a hyperplane with a smooth portion of a potato may yield a polytope.

Corollary 1.26 *(Potatope Slicing Proposition) If G is a (p,q)-graph, then $TH_p(G)$ is integral, and hence also polyhedral.*

Proof. For any graph G, $P_p(G) \subseteq TH_p(G) \subseteq \mathcal{Q}_p(G)$ and any integral vector of $\mathcal{Q}_p(G)$ also lies in $TH_p(G)$. Thus if $\mathcal{Q}_p(G)$ is integral then $\mathcal{Q}_p(G) = TH_p(G)$. The result now follows immediately from Theorem 1.25. □

The following will yield the necessary $co\mathcal{NP}$ certificates for a partitionable graph recognition algorithm.

Proposition 1.27 *If G is a (p,q)-graph, then each of the following conditions holds:*

1. $|V| = pq + 1$
2. *For each node* $v \in V(G)$, $\frac{1}{q}\chi^{V-\{v\}} \in TH(G)$
3. *For each node* $v \in V(G)$, $\frac{1}{p}\chi^{V-\{v\}} \in TH(\bar{G})$
4. $TH_p(G)$ *has* $|V|$ *integral extreme points*
5. $TH_q(\bar{G})$ *has* $|V|$ *integral extreme points*
6. *Every meshed system of G is regular.*

Proof. Suppose that G is a (p,q)-graph, then (1) holds by definition. Condition (2) holds since for any v, $\chi^{V-\{v\}}$ can be expressed as the sum of the q stable set incidence vectors for the colouring of $G - v$. Condition (3) holds similarly. Next note that $\{\frac{1}{q}\chi^{V-\{v\}}\}_{v\in V}$ is a linearly independent collection of vectors in $TH_p \equiv TH_p(G)$. Hence TH_p has dimension $|V| - 1$. Conditions (4) and (5) follow from the Potatope Slicing Proposition. Finally, (6) holds by Theorem 1.22. □

For a graph G and integer $p \geq 2$ let $\Sigma(G,p) \doteq conv\{\bar{\mathbf{0}}, TH_p(G)\}$. We summarize the previous results as follows.

Theorem 1.28 *A graph G is a (p,q)-graph if and only if $|V| = pq+1$, G has no botched (p,q)-system and both $\Sigma(G,p)$ and $\Sigma(\bar{G},q)$ are integral n-dimensional simplices.*

Proof. Suppose first that G is a (p,q)-graph. Condition (6) of Proposition 1.27 already asserts that G has no botched (p,q)-system. By Theorem 1.25, and Corollary 1.26 we have that $\mathcal{E}(TH_p(G))$ consists of n linearly independent integral vectors and so $\Sigma(G,p)$ is indeed a full-dimensional simplex. Similarly $\Sigma(\bar{G},q)$ is such a simplex. Conversely, suppose that $\Sigma(G,p)$ and $\Sigma(\bar{G},q)$ are integral n-dimensional simplices. After disposing of $\bar{\mathbf{0}}$, either their extreme points form a (p,q)-mesh and hence G is partitionable by, say, Theorem 1.23, or G has a botched system. □

1.4.2 A recognition algorithm

We gave an overview in Section 1.1.3 for the recognition algorithm for the class of partitionable graphs. We now describe the algorithm in more detail. Clearly, it is sufficient to be able to recognize whether a graph $G=(V,E)$ is a $(p,q)-$graph for a given p and q, since we may run through all possible such p,q for which $|V|-1=pq$.

We first introduce some terminology and subroutines in order to give the detailed algorithm. In the following, for an integer p, we call a graph G a *p-contender* if $\bar{\mathbf{1}}\cdot x \le (1+\frac{\frac{n}{2}}{(np)^n-n})p$ is not valid for $\Sigma(G,p)$. Note that Corollary 1.16 shows that if G is a (p,q)-graph, then G is a p-contender and $\bar{G}$ is a q-contender.

The recognition algorithm for (p,q)-graphs first confirms that G is a p-contender. It then applies DISCOVERY (Section 1.2.3) to find the candidate stable p-sets in G. Similarly it finds candidate q-cliques and then finally checks that the resulting families form a meshed system. The remainder of this section is dedicated to describing the remaining details. A proof of correctness is given at the end of this section.

To start, we have that the following polytime routine outputs NO only if a G is not a p-contender and outputs YES only if $\max\{\bar{\mathbf{1}}\cdot x : x\in TH(G)\} > p$.

$Contender(G,p)$
`Set` $\epsilon \leftarrow \frac{\frac{n}{8}}{(np)^n-n}$
`If` $(theta_{opt}(G,\bar{\mathbf{1}},\epsilon)$ `returns an optimum of at most` $(1+\frac{\frac{n}{4}}{(np)^n-n})p)$ `then OUTPUT(NO)`

We also need to apply the Ellipsoid Algorithm to the class of convex sets (not necessarily polyhedral and not necessarily full-dimensional) $\Sigma(G,p)$. We now give a separation algorithm for this class.

Proposition 1.29 *There exists a weak separation algorithm $\Sigma_{SEP}(G,p,y,\epsilon)$ for $\{\Sigma(G,p) : G=(V,E)$ is a graph, $p\in\{2,3,\ldots,n\}\}$ whose running time is bounded by a polynomial in n, $size(y)$ and $size(\epsilon)$.*

Proof. If $\bar{\mathbf{0}}=y$, then the vector is trivially in $\Sigma(G,p)$. If this is not the case, then we first verify that $y\ge 0$. If some $y_v<0$ the non-negativity constraint defines the desired separating hyperplane. Next we check that $\bar{\mathbf{1}}\cdot y\le p$; if this is not the case then we output the violated rank inequality. So suppose that none of these situations occur and set $y'=\frac{p}{\bar{\mathbf{1}}\cdot y}y$. Note that $y\in\Sigma(G,p)$ if and only if $y'\in TH(G)$. Thus if

$theta_{Sep}(G, y', \epsilon)$ asserts that $y' \in S(TH(G), \epsilon)$, one checks that that $y \in S(\Sigma(G,p), \epsilon)$. Otherwise, y' is not in $TH(G)$ and so neither is $\alpha y' \in \Sigma(G,p)$ for any $\alpha \in (0,1]$. In polynomial-time we may then find a non-zero z in $H = \{x \in \mathcal{R}^V : y \cdot x = 0\}$. Clearly, $\bar{\mathbf{0}}$ is the only vector of $\Sigma(G,p)$ in H and so without loss of generality we may assume that $a \cdot z \leq 0$ for each $a \in \Sigma(G,p)$. Setting $c = \frac{1}{\|z\|} z$ yields the desired separating hyperplane. □

Corollary 1.30 *There exists a weak optimization algorithm $\Sigma_{OPT}(G,p,c,\epsilon > 0)$ for $\{\Sigma(G,p) : G = (V,E)$ is a p-contender$\}$ whose running time is bounded by a polynomial in $n, size(\epsilon)$ and $size(c)$.*

Proof. One need only note that if G is a p-contender, then $\Sigma(G,p)$ is a full-dimensional convex body and the result follows from Proposition 1.29 and Theorem 1.4. □

A detailed description of the algorithm now follows.

(p,q)-Recognition Algorithm PARTN

```
Input: A graph G = (V,E) and integers p,q ∈ {2,3,...,n}
Output: YES if G is a (p,q)-graph, NO otherwise

If n ≡ |V| ≠ pq + 1 OUTPUT(NO)

if (InitialCheck(G,p)) or (InitialCheck(Ḡ,q)) outputs NO then OUTPUT(NO)

Find n integral vectors in Σ(G,p) and Σ(Ḡ,q)
A ← Discovery(Σ(G,p), 1/n, {0̄} ∪ {1/q(1̄ − χ^{v}) : v ∈ V})
If (A is empty) OUTPUT(NO)
B ← Discovery(Σ(Ḡ,q), 1/n, {0̄} ∪ {1/p(1̄ − χ^{v}) : v ∈ V})
If (B is empty) OUTPUT(NO)

Drop the 0̄ rows from A,B and mark each row of B as 'unvisited'
For each row a^i of A
    Find an unvisited row b^j of B such that b^j · a^i = 0
    If none exists OUTPUT(NO)
    Else label row a^j as 'visited' and interchange rows a^i, a^j in B
EndFor
If (AB^T ≠ J − I) then OUTPUT(NO)
Else OUTPUT(YES)
```

To complete the description, we include the routine which checks the basic conditions required by the routines Σ_{SEP} and Σ_{OPT} (and hence also by *Discovery*).

```
INITIALCHECK
Input: A graph G = (V,E) and integers p,q such that |V| = pq + 1
Output: YES or NO. Outputs YES only if Σ(G,p) is full-dimensional and (almost)
        contains the affinely independent vectors {(1/q)(1̄ − χ^{v}) : v ∈ V}

If (CONTENDER(G,p) outputs NO) OUTPUT(NO)
For each v ∈ V
     z^v ← (1/q)(1̄ − χ^{v})
     If (theta_Sep(G, z^v, 1/n) returns NO) OUTPUT(NO)
EndFor
OUTPUT(YES)
```

Theorem 1.31 *The algorithm $Partn(G,p,q)$ has running time polynomially bounded in n and outputs YES if and only if G is a (p,q)-graph.*

Proof. Clearly, PARTN is a polytime algorithm since each of DISCOVERY, INITIALCHECK, CONTENDER and the separation routines is. Note that if an invocation of CONTENDER outputs NO, then either G is not a p-contender, or $\bar{G}$ is not a q-contender, and so by Corollary 1.16, G is not a (p,q)-graph. Similarly, if INITIALCHECK ever fails to assert that a vector, say, $\frac{1}{q}\chi^{V-\{v\}}$ is not in $TH(G)$, then by Proposition 1.27, G is not a (p,q)-graph.

Suppose, next that DISCOVERY returns nonempty matrices. Thus the rows of A identify stable p-sets and rows of B identify a q-cliques. The last loop simply determines whether these sets form a regular mesh. If they do (i.e., $AB^T = J - I$), then by Theorem 1.23, G is a (p,q)-graph. If they do not, then by Theorem 1.28, G is not.

Thus it remains only to consider the case when DISCOVERY returns an empty matrix. As described in the proof of Theorem 1.6, one of the bodies $\Sigma(G,p)$ or $\Sigma(G,q)$ contains a non-integral extreme point, and hence again by Theorem 1.28, G is not a (p,q)-graph. This completes the proof. □

1.5 Perfect Graph Characterizations and a Continuous Perfect Graph Conjecture

Corollary 1.16 implies a key property of the semi-definite relaxation for (p,q)-graphs G:

$$\text{The vector } \tfrac{1}{q}\chi^V \notin TH(G), \tag{1.6}$$

whereas this vector is an element of $\mathcal{Q}(G)$. This provides a starting off point for a different type of characterization of perfect graphs.

A *(p,q)-set* for a graph G is a subset $S \subseteq V$ such that

1 $\frac{1}{q}\chi^S \notin TH(G)$
2 $\frac{1}{p}\chi^S \notin TH(\bar{G})$
3 For each node $v \in S$, $\frac{1}{q}\chi^{S-v} \in TH(G)$
4 For each node $v \in S$, $\frac{1}{p}\chi^{S-v} \in TH(\bar{G})$.

Note that we do not demand anything about the cardinality of S. We now have the following.

Theorem 1.32 *A graph G is perfect if and only if it has no (p,q)-sets for integers $p,q \geq 2$.*

Proof. We first prove sufficiency. Suppose that G is not perfect and hence for some $p,q \geq 2$ there is a subset $S \subseteq V$ for which $G[S]$ is a (p,q)-graph by the Perfect Graph Theorem. By (1.6), conditions (1) and (2) are satisfied by S. Moreover, since $G[S]-v$ is q-colourable for any node v, we have that $\frac{1}{q}\chi^{S-v}$ is a convex combination of stable set incidence vectors and hence lies in $P(G) \subseteq TH(G)$. Thus condition (3) holds. The analogous argument shows that (4) holds and so S is indeed a (p,q)-set.

Suppose now that G has (p,q)-set S and set $G' = G[S]$. Suppose that K is a clique of G and let $v \in V - K$. Then since $\chi^K \in TH(\bar{G})$ and $\frac{1}{q}\chi^{S-v} \in TH(G)$ we must have $\chi^K \cdot \frac{1}{q}\chi^{S-v} \leq 1$ as $TH(\bar{G})$ and $TH(G)$ are antiblockers. Thus $|K| \leq q$. It follows that $\omega(G[S]) \leq q$ and similarly $\alpha(G[S]) \leq p$. Consider the problem $\max\{\frac{1}{p}\chi^S \cdot x : x \in TH(G)\}$. This value must be greater than 1 as $\frac{1}{p}\chi^S \notin TH(\bar{G})$ and since $TH(\bar{G}) = \mathcal{A}(TH(G))$. On the other hand if G is perfect, then $TH(G) = P(G)$ and so the optimum is achieved by some stable set Y. We thus deduce $1 < \frac{1}{p}\chi^S \cdot \chi^Y \leq 1$ where the last inequality follows since $|Y| \leq p$. This contradiction completes the proof. □

We now turn this characterization into its continuous version. In the following, a *hook* is any triple (u,v,w) of distinct nodes such that $uv \notin E(G)$ and $vw \in E(G)$. A *root* (or (p,q)-root) is any pair of the form $((u,v,w),(p,q))$, where (u,v,w) is a hook of G and $p,q \geq 2$ are integers.

Let σ be a (p,q)-root. A *σ-vector* is a vector $x \in \mathcal{R}^{V-\{u,v,w\}}$ with the following three properties:

5.1 $(1,0,1,x) \in qTH(G)$
5.2 $(1,0,1,x) \in pTH(\bar{G})$
5.3 $(0,1,1,x) \in qTH(G)$
5.4 $(1,1,0,x) \in pTH(\bar{G})$.

The *weight* of a σ-vector is the value $\|x\|^2$. A vector x is called a *(p,q)-vector* for G if it is a σ-vector for some (p,q)-root σ of G. If in addition, either $(1,1,1,x) \notin qTH(G)$ or $(1,1,1,x) \notin pTH(G)$, then the vector is called *partitionable.* The following lemma shows that any partitionable graph gives rise to a $0,1$ partitionable vector. The proof is immediate from (1.6) and we omit it.

Lemma 1.33 *Let G be a (p,q)-graph and $u,v,w \in V$ such that $uv \notin E, vw \in E$. Then $\chi^{V-\{u,v,w\}}$ is a partitionable (p,q)-vector of weight $pq-2$, for the root $\sigma = ((u,v,w),(p,q))$.*

We now have the following characterization of perfection.

Theorem 1.34 *A graph G is imperfect if and only if it has a partitionable vector.*

Proof. The necessity follows from Lemma 1.33 since any imperfect graph must contain a partitionable subgraph.

So now suppose that σ is a (p,q)-root and x is a σ-vector x. Let $x^1 = (1,1,1,x)$ and first suppose that $x^1 \notin qTH(G)$. Then $\max\{\frac{1}{q}x^1 \cdot y : y \in TH(\bar{G})\} \equiv \zeta > 1$. If G were perfect, then this optimum is obtained by χ^K for some clique K. Thus we have

$$1 < \zeta = \frac{1}{q}x^1 \cdot \chi^K. \tag{1.7}$$

Let $x^2 = (0,1,1,x)$ and $x^3 = (1,0,1,x)$. Now K contains at most one of u or v. If K does not contain v, then $(\frac{1}{q}x^1) \cdot \chi^K = (\frac{1}{q}x^3) \cdot \chi^K$ and this is at most 1 since $\frac{1}{q}x^3 \in TH(G)$. On the other hand, if K does not contain u, then $(\frac{1}{q}x^1) \cdot \chi^K = (\frac{1}{q}x^2) \cdot \chi^K$ and this is also at most 1 as $\frac{1}{q}x^2 \in TH(G)$. But then the right hand side of (1.7) is at most 1, a contradiction. Thus we may assume that the vector $x^1 = (1,1,1,x) \in qTH(G)$. But then we must have that $(1,1,1,x) \notin pTH(\bar{G})$ but then the analogous argument also leads to a contradiction. This completes the proof. □

For a graph G with (p,q)-root σ, let $\tilde{\Psi}(G,\sigma) = \{x : x \text{ is a } \sigma\text{-vector}\}$. We also let $\Psi(G,\sigma)$ denote $\tilde{\Psi}(G,\sigma) \cap H_{V-\{u,v,w\}}$, where H_S denotes the hypercube in $\mathcal{R}^S$.

Fact 1.35 *For any graph G and (p,q)-root σ, $\Psi(G,\sigma)$ is a well-bounded convex body and also a convex corner.*

Note that if G is perfect, then $qTH(G)$ is polyhedral with integral vertices $q\chi^S$ where S is a stable set. We also remark that $qTH(G) \cap H_V$ has an extreme point χ^H for each H which induces a q-colourable subgraph. We now characterize perfection in terms of a quadratic optimization problem. If x is a (p,q)-vector for G, then its *extension* is the vector $(1,1,1,x)$ where the first three components correspond to the hook for x's root.

Theorem 1.36 *The following are equivalent for a graph G.*

1. *G is imperfect*
2. *For some integers $p,q \geq 2$, there is a (p,q)-root σ, such that there exists $x \in \Psi(G,\sigma)$ with $\|x\|^2 > pq-3$*

Proof. The fact that (1) implies both (2) and (3) follows directly from Lemma 1.33. Conversely, suppose that x is a vector for a root with line (u,v,w) which satisfies either (2) or (3). If G is perfect, then by Theorem 1.34 we may derive two convex combinations of stable set incidence vectors and clique incidence vectors as follows: $(\frac{1}{q},\frac{1}{q},\frac{1}{q},\frac{x}{q}) = \sum_i \lambda_i s^i$ and $(\frac{1}{p},\frac{1}{p},\frac{1}{p},\frac{x}{p}) = \sum_i \mu_i k^i$. Let z be the extension of x.

Thus if condition 2 holds, we have

$$1 < \frac{3+\|x\|^2}{pq} = (\frac{1}{q}z)\cdot(\frac{1}{p}z) = (\sum_i \lambda_i s^i)\cdot(\sum_i \mu_i k^i) = \sum_{i,j} \lambda_i\mu_j s^i \cdot k^j \leq \sum_{i,j} \lambda_i\mu_j = 1,$$

which is a contradiction and hence G must have been imperfect. This final contradiction completes the proof. □

Corollary 1.37 *A graph G is imperfect if and only if for some root $\{(u,v,w,),(p,q)\}$ the region $\Psi(G,\sigma) \cap \{x \in \mathcal{R}^{V-\{u,v,w\}} : \bar{\mathbf{1}} \cdot x = pq-2\}$ contains an integral vector.*

The previous characterizations suggest a number of different oracle polytime algorithms for recognizing perfect graphs. Let σ be a root of G. The oracle $norm(G, \sigma)$ is one which outputs YES if there is a σ-vector x such that $||x||^2 > pq - 3$. Similarly $posdef(G, \sigma)$ denotes the oracle which outputs YES if there is a σ-vector whose extension z satisfies $z^T Jz > (pq)^2$. We also let $containment(G, \sigma)$ be the routine which outputs YES if G has a σ-vector x such that $(1, 1, 1, x) \notin qTH(G)$. Any of these oracles can be used to recognize whether a graph is perfect.

Perfection-Oracle-Recognition PERF-OR
Input: *A graph G and oracle F with the properties described above.*

```
For each pair of integers p,q ∈ {2,3,...,n}
    For each u,v,w ∈ V such that uv,uw ∈ E(G), wv ∉ E(G)
        Let σ = ({u,v,w},p,q)
        If (F(G,σ) outputs YES) Output(NO)
    EndFor
EndFor
Output(YES)
```

We note that the norm maximization problem is NP-hard in general – see [18] – even in the case where the body is a parallelotope [4]. Of course for our purposes we need not solve the problem exactly – we only need a $\frac{1}{pq}$-approximation. Even still, the weak version of this problem remains hard as is shown in [6, 5] (for quite simple convex bodies).

We have thus cast the problem of recognizing a perfect graph as that of checking a condition – which is NP-hard in general, but may be easy for perfect graphs – in polynomially many subproblems. This is in contrast to ostensibly having to look at exponentially many subgraphs H to see if each satisfies a condition – $\chi(H) = \omega(H)$ – which is also NP-hard in general but *is* polytime solvable for perfect graphs. In particular, if either oracle NORM or POSDEF is polytime solvable for perfect graphs, then PERF-OR yields a polytime perfect graph recognition algorithm.

A similar approach to recognizing perfection which does not involve fixing a root ahead of time is as follows. Note that it is sufficient to find a $0-1$ vector x which lies in $(qTH(G) \cap H_V) \cap (pTH(\bar{G}) \cap H_V \cap \{x : \sum_i x_i \geq pq+1\})$. Let $C_1 = qTH(G) \cap H_V$ and $C_2 = pTH(\bar{G}) \cap H_V \cap \{x : \sum_i x_i \geq pq + \frac{1}{2}\}$. Given two such convex bodies and $\epsilon > 0$, the Ellipsoid Method can be used (see [27]) to either find a point in $C_1 \cap C_2$ or an Ellipsoid E such that $C_1 \cap C_2 \subseteq E$ and $vol(E) < \epsilon$. If G contains a (p, q)-graph H, then one may show that this intersection has volume at least ϵ where $size(\epsilon)$ is polynomially bounded. Thus we may detect that this intersection is nonempty. The problem is that even if G is perfect, there may be a point in this intersection, for instance, a point which is a convex combination of q-colourable graphs, and a combination of p-clique-coverable graphs.

We close this section with a couple of conjectures. The first is a continuous adaptation of the Strong Perfect Graph Conjecture (SPGC). In the following, let σ be a root in the graph G.

Conjecture 1.38 *(CPGC) If G is imperfect, then it contains a σ-vector of weight*

$> pq - 3$ *where either* p *or* q *is two.*

Note that Theorem 1.32 shows that the SPGC implies the CPGC.

Conjecture 1.39 *If* G *contains a* σ*-vector of weight* $> pq - 3$ *where either* p *or* q *is two, then* G *contains either an odd hole, or an odd antihole.*

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