$\mathrm{Top}^{\mathrm{op}}$ is a quasi-variety

Michael Barr*and M. Cristina Pedicchio

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Abstract

We show that the opposite of the category of topological spaces is a quasi-variety, that is a subobject and product closed subcategory of a varietal category. We identify the varietal category as well as the simple Horn clause that determines the objects of the subcategory.

1 Introduction

The category of topological spaces is usually thought rather poorly of *qua* category. Although it is complete and cocomplete and the underlying set functor even has both adjoints, the free and cofree functors produce spaces without interesting structure and the triple and cotriple on **Set** produced by the adjoints are the identity. The category is far from exact, or even regular. Thus the properties of topological spaces seem rather far removed from those involved in the usual equational theories.

Thus it came as some surprise to us to discover that the situation is quite different when it comes to the dual category. It turns out that that category is at least regular (not that hard to prove, once you suspect it), although not exact and is, in fact quasi-varietal. In this paper we show that it is quasi-varietal, identify the variety and also the simple Horn clause that distinguishes the algebras that are the duals of topological spaces.

2 Quasi-vareties

Definition. Recall that a *varietal category* or *variety* is one that is tripleable over the category of sets. A *quasi-variety* is a full subcate-

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gory of a varietal category that is closed under subobjects and products. Equivalently, it is a surjective-reflective, or regular-epi-reflective subcategory of a variety.

The following theorem is found in [Pedicchio, to appear].

2.1 Theorem *A category $\mathcal{E}$ is quasi-varietal if and only if*

QV–1. *$\mathcal{E}$ is regular;*

QV–2. *$\mathcal{E}$ has coequalizers of equivalence relations;*

QV–3. *$\mathcal{E}$ has a regular projective generator P such that arbitrary (small) sums of copies of P exist in $\mathcal{E}$.*

A regular projective generator is understood to be both regular projective and a regular generator. It is shown in [Barr, 1989] that such a quasi-variety is a full subcategory of a variety consisting of those objects that satisfy a class of generalized Horn clauses that take the form

$$(\bigwedge(\phi_i(x) = \psi_i(x)) \Rightarrow (\phi(x) = \psi(x))$$

where the ϕ_i, ψ_i, ϕ and ψ are operations in the theory and the conjunction may be infinite. These clauses are found by imposing the antecedent equations as an equivalence relation on a free algebra and the consequent is an additional equation necessary to reflect the quotient algebra into the subcategory (of which there may be many; each such gives an additional Horn clause).

As an application, we can see that opposite of the category of topological spaces satisfies these conditions and is thus a quasi-variety. We need a regular injective cogenerator in the category of topological spaces. The simplest such space is the space we call P that has three points, say a, b and c. Aside from P and $\emptyset$, the only open set is $\{a, b\}$. If X is a topological space, a map $f : X \to P$ is determined by three subsets $A = f^{-1}\{a\}$, $B = f^{-1}\{b\}$ and $C = f^{-1}\{c\}$. We must have $A \cup B \cup C = X$ so that such a map is uniquely determined by giving two sets A and B. Continuity is equivalent to $U = A \cup B$ being open. Thus a continuous map $X \to P$ determines a pair (U, A), where $U \subseteq X$ is open and $A \subseteq X$ is arbitrary. Conversely, it is clear that such a pair determines a unique continuous map that takes points of A to a, points of $U - A$ to b and all others to c.

There is no problem in proving directly that P is a regular injective cogenerator, but it will also follow from the results of the next section.

3 Grids

Definition. Recall that a frame is a lattice with arbitrary supremums that are preserved by finite infimums. By a *grid*, we mean a frame with an additional unary operation we denote $'$ satisfying some equations. The equations are best expressed using derived unary operations $x^{\uparrow} = x \vee x'$ and $x_{\downarrow} = x \wedge x'$. The equations (in addition to the frame equations) are:

Gr–1. $u'' = u$.

Gr–2. $^{\uparrow}$ and $_{\downarrow}$ are $\bigvee$ homomorphisms.

Gr–3. $^{\uparrow}$ is a $\wedge$ homomorphism, while $_{\downarrow}$ satisfies $(u \wedge v)_{\downarrow} = u \wedge v_{\downarrow}$.

Gr–4. The interval $[u_{\downarrow}, u^{\uparrow}]$ is a complete atomic boolean algebra with the operations of $\bigvee$ and $'$.

This last requires some explanation to see why it is equational. First, if v is arbitrary, Let v^u denote $(u_{\downarrow} \vee v) \wedge u^{\uparrow} = u_{\downarrow} \vee (v \wedge u^{\uparrow})$. Then $v^u \in [u_{\downarrow}, u^{\uparrow}]$ and $v^u = v$ if $v \in [u_{\downarrow}, u^{\uparrow}]$. It follows immediately that any sentence of the form $v \in [u_{\downarrow}, u^{\uparrow}] \Rightarrow \phi(v) = \psi(v)$ is equivalent to the equation $\phi(v^u) = \psi(v^u)$ and that is true for operations of any arity by replacing v by any string of elements. Finally, a complete atomic boolean algebra is characterized by satisfying the complete distributive law which can be stated, if awkwardly, in terms of $\bigvee$ and $'$ as follows. Let us denote by $\overset{u}{\bigvee}$ and $\overset{u}{\bigwedge}$ the operations of the form

$$\overset{u}{\bigvee_{i \in I}} v_i = \bigvee_{i \in I} v_i^u$$

and

$$\overset{u}{\bigwedge_{i \in I}} v_i = \left(\bigvee_{i \in I} (v_i^u)' \right)'$$

Then we want equations of the form $v^u \vee (v^u)' = u^{\uparrow}$, $v^u \wedge (v^u)' = v_{\downarrow}$ that force $[u_{\downarrow}, u^{\uparrow}]$ to be a boolean algebra, evidently complete, with $\overset{u}{\bigvee}$ is infinite join and, by duality, $\overset{u}{\bigwedge}$ its meet. Then the complete distributive law will state that for all sets I and J and $I \times J$ indexed families v_{ij}

$$\overset{u}{\bigwedge_{i \in I}} \overset{u}{\bigvee_{j \in J}} v_{ij} = \overset{u}{\bigvee_{s : I \to J}} \overset{u}{\bigwedge_{i \in I}} v_{i\, s(i)}$$

This equation is imposed on the whole algebra, but of course is equivalent to the assumption that $[u_{\downarrow}, u^{\uparrow}]$ is a completely distributive complete boolean algebra, which is equivalent to its being atomic (see, for example, [Johnstone, 1982], VII.1.16, page 285).

We denote by **Grid** the category of grids and homomorphisms.

3.1 Proposition *A grid has the following properties:*

1. ${u_\downarrow}' = u^\uparrow$ *and* ${u^\uparrow}' = u_\downarrow$*;*
2. ${u'}_\downarrow = u_\downarrow$ *and* ${u'}^\uparrow = u^\uparrow$*;*
3. $(u \wedge v)_\downarrow = u_\downarrow \wedge v_\downarrow$*;*
4. $u_{\downarrow\downarrow} = {u^\uparrow}_\downarrow = u_\downarrow$ *and* $u^{\uparrow\uparrow} = {u_\downarrow}^\uparrow = u^\uparrow$*;*

Proof.

1. Since $[u_\downarrow, u^\uparrow]$ is a boolean algebra and $'$ is the complement operation, the complement of the top element is the bottom and vice versa.
2. This is immediate since, for example, ${u'}_\downarrow = u' \wedge u'' = u' \wedge u = u_\downarrow$.
3. $(u \wedge v)_\downarrow = u \wedge v_\downarrow$ and also $(u \wedge v)_\downarrow = u_\downarrow \wedge v$ so that $(u \wedge v)_\downarrow = u \wedge v_\downarrow \wedge u_\downarrow \wedge v = u_\downarrow \wedge v_\downarrow$ since evidently $u_\downarrow \leq u$ and $v_\downarrow \leq v$.
4. $$u_{\downarrow\downarrow} = (u \wedge u')_\downarrow = u_\downarrow \wedge {u'}_\downarrow = u_\downarrow \wedge u_\downarrow = u_\downarrow$$
   $${u^\uparrow}_\downarrow = (u \vee u')_\downarrow = u_\downarrow \vee {u'}_\downarrow = u_\downarrow \vee u_\downarrow = u_\downarrow$$
   and similarly for the other two.

□

The following is true because $[u_\downarrow, u^\uparrow]$ is a boolean algebra.

3.2 Proposition *Let G be a grid, $u \in G$ and v, $w \in [u_\downarrow, u^\uparrow]$. Then*

1. $(v \wedge w)' = v' \vee w'$ *and* $(v \vee w)' = v' \wedge w'$*;*
2. $v^\uparrow = u^\uparrow$ *and* $v_\downarrow = u_\downarrow$*;*

□

3.3 Corollary *A grid G is partitioned by sets of the form $[u_\downarrow, u^\uparrow]$.*

4 The main theorem

4.1 Theorem *The category of topological spaces is dual to the full subcategory of grids defined by the Horn clause*

$$(u^\uparrow \vee 1_\downarrow = v^\uparrow \vee 1_\downarrow) \Rightarrow (u^\uparrow = v^\uparrow) \qquad (*)$$

Proof. Define $\Phi : \mathrm{Top}^{\mathrm{op}} \to \mathbf{Grid}$ by letting $\Phi(X)$ be the set of all pairs (U, A) where U is an open subset of X and A is an arbitrary subset of U. The order relation is the restriction of the product order and both $\bigvee$ and $\wedge$ are coordinatewise. $(U, A)' = (U, U - A)$. The derived operations are $(U, A)^\uparrow = (U, U)$ and $(U, A)_\downarrow = (U, \emptyset)$. It is clear that this is a grid and also satisfies the Horn clause. We define a functor $\Psi : \mathbf{Grid}^{\mathrm{op}} \to \mathrm{Top}$. Suppose G is a grid. Then $[1_\downarrow, 1]$ is a complete atomic boolean algebra whose set of atoms we denote by X. Then the interval $[1_\downarrow, 1]$ can be thought of as the set of subsets of X. We will use capitals to denote elements of $[1_\downarrow, 1]$. Say that $U \in [1_\downarrow, 1]$ is open if there is a $u \in G$ such that $u^\uparrow \vee 1_\downarrow = U$. It follows from the fact that $^\uparrow$ commutes with $\bigvee$ that the union of open sets is open and from the fact that $^\uparrow$ commutes with $\wedge$ and the distributivity that an intersection of two open sets is open. Thus we have a topology on X. The set X with this topology is $\Psi(G)$. If $f : G \to G'$ is a grid homomorphism, then $f(1_\downarrow) = f(1)_\downarrow = 1_\downarrow$ so that f takes the interval $[1_\downarrow, 1]$ of G to the corresponding interval of G'. Moreover, since f preserves $\bigvee$ and $'$, it is a morphism of CABAs, which is induced by a function we denote $\Psi(f) : X' \to X$, the set of atoms of $[1_\downarrow, 1]$ in G' and G, resp. Moreover, the duality of CABAs and sets is such that the inverse image function of $\Psi(f)$ is f itself, so that showing that $\Psi(f)$ is continuous is equivalent to showing that f takes open sets to open sets. But if $U = u^\uparrow \vee 1_\downarrow$ is open in X, then $f(U) = f(u)^\uparrow \vee 1_\downarrow$ is open in X'.

It is clear that $\Psi \circ \Phi \cong \mathrm{Id}$ in any case. We finish the argument by letting G be a grid that satisfies $(*)$ and showing that $\Phi(\Psi(G)) \cong G$. Let $X = \Psi(G)$. Define $\phi : G \to \Phi(X)$ by $\phi(u) = (u^\uparrow \vee 1_\downarrow, u \vee 1_\downarrow)$. First we show that ϕ is a grid morphism.

$$\begin{aligned}\phi(\bigvee_{i\in I} u_i) &= ((\bigvee u_i)^\uparrow \vee 1_\downarrow, (\bigvee u_i) \vee 1_\downarrow) \\ &= (\bigvee({u_i}^\uparrow \vee 1_\downarrow), \bigvee(u_i \vee 1_\downarrow)) = \bigvee({u_i}^\uparrow \vee 1_\downarrow, u_i \vee 1_\downarrow)\end{aligned}$$

for arbitrary index sets I, and

$$\begin{aligned}\phi(u \wedge v) &= ((u \wedge v)^\uparrow \vee 1_\downarrow, (u \wedge v) \vee 1_\downarrow) \\ &= ((u^\uparrow \wedge v^\uparrow) \vee 1_\downarrow, (u \wedge v) \vee 1_\downarrow) \\ &= ((u^\uparrow \vee 1_\downarrow) \wedge (v^\uparrow \vee 1_\downarrow), (u \vee 1_\downarrow) \wedge (v \vee 1_\downarrow)) \\ &= (u^\uparrow \vee 1_\downarrow, u \vee 1_\downarrow) \wedge (v^\uparrow \vee 1_\downarrow, v \vee 1_\downarrow)) \\ &= \phi(u) \wedge \phi(v)\end{aligned}$$

To see that ϕ preserves $'$, we use the fact that u' is the complement of u in the lattice $[u_\downarrow, u^\uparrow]$ and show that $\phi(u')$ is the complement of $\phi(u)$

in $[\phi(u)_\downarrow, \phi(u)^\uparrow]$. Then

$$\begin{aligned}\phi(u) \vee \phi(u') &= \phi(u \vee u') = \phi(u^\uparrow)\\ &= (u^{\uparrow\uparrow} \vee 1_\downarrow, u^\uparrow \vee 1_\downarrow) = (u^\uparrow \vee 1_\downarrow, u^\uparrow \vee 1_\downarrow)\\ &= (u^\uparrow \vee 1_\downarrow, u \vee 1_\downarrow)^\uparrow = \phi(u)^\uparrow\end{aligned}$$

since that is how $^\uparrow$ works in $\Phi(X)$.

$$\begin{aligned}\phi(u) \wedge \phi(u') &= \phi(u \wedge u') = \phi(u_\downarrow) = ({u_\downarrow}^\uparrow \vee 1_\downarrow, u_\downarrow \vee 1_\downarrow)\\ &= (u^\uparrow \vee 1_\downarrow, (u \vee 1)_\downarrow) = (u^\uparrow \vee 1_\downarrow, 1_\downarrow)\\ &= (u^\uparrow \vee 1_\downarrow, \emptyset) = \phi(u)_\downarrow\end{aligned}$$

Thus ϕ is a morphism of grids. I claim that ϕ is an isomorphism. In fact, if $\phi(u) = \phi(v)$, then $u^\uparrow \vee 1_\downarrow = v^\uparrow \vee 1_\downarrow$ which implies that $u^\uparrow = v^\uparrow$. Then

$$(u \vee 1_\downarrow) \wedge u^\uparrow = (u \wedge u^\uparrow) \vee (1_\downarrow \wedge u^\uparrow) = u \vee (1 \wedge u^\uparrow)_\downarrow = u \vee {u^\uparrow}_\downarrow = u \vee u_\downarrow = u$$

and similarly $(v \vee 1_\downarrow) \wedge u^\uparrow = v$ so that $u = v$ and ϕ is monic. Let $(U, A) \in \Phi(X)$. Then $U = u^\uparrow \vee 1_\downarrow$ for some $u \in G$, by definition of the topology on X. Then $\phi(A \wedge u^\uparrow) = ((A \wedge u^\uparrow)^\uparrow \vee 1_\downarrow, (A \wedge u^\uparrow) \vee 1_\downarrow)$. We have

$$(A \wedge u^\uparrow)^\uparrow \vee 1_\downarrow = (A^\uparrow \wedge u^{\uparrow\uparrow}) \vee 1_\downarrow = (1 \wedge u^\uparrow) \vee 1_\downarrow = u^\uparrow \vee 1_\downarrow = U$$

and

$$(A \wedge u^\uparrow) \vee 1_\downarrow = (A \vee 1_\downarrow) \wedge (u^\uparrow \vee 1_\downarrow) = A \wedge U = A$$

Thus ϕ is surjective.

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