NOTE ON A THEOREM OF PUTNAM'S

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1. Introduction

In an appendix to his 1981 book, Putnam made the following claim.

1.1. Theorem. *Let L be a language with predicates F_1, F_2, $\ldots$, F_k (not necessarily monadic [unary]). Let I be an interpretation, in the sense of an assignment of an intension to every predicate of L. Then if I is non-trivial in the sense that at least one predicate has an extension which is neither empty nor universal in at least one possible world, there exists a second interpretation J which disagrees with I, but which makes the same sentences true in every possible world as I does.*

In more familiar language, he is claiming that in for any relational theory in a first order language (without equality as a built-in predicate), there are distinct models that satisfy the same first order sentences. This will true if and only if it is true for a single predicate, so we stick to that case here. Thus we can rephrase his assertion as follows.

1.2. Theorem. *Let U be a set and $R \subseteq U^n$ an n-ary relation on U. Then either $R = \emptyset$ or $R = U^n$ or there is a $R' \neq R$ in U^n that satisfies the same first order sentences as R.*

Ed Keenan showed that the equality relation on a two element set is a counter-example and asserted that a similar example could be given on any finite set U ([Keenan, 1995]). He also raised (privately) the question about what was true for infinite sets. The purpose of this note is to show that Theorem 1.2 is true when U is infinite. Of course, Putnam's proof remains invalid because of the finite counter-example. Putnam does not make it clear whether he is limiting himself to finite models or infinite models or not at all.

2. Permutation invariant relations

For a set U and a finite ordinal $n = \{0, 1, \ldots, n-1\}$, we view U^n as the set of all functions $a : n \longrightarrow U$. If $a \in U^n$, the kernel pair of a, denoted $\mathrm{kerp}(a)$ is the equivalence relation on n defined by $\mathrm{kerp}(a) = \{(i, j) \in n \times n \mid a_i = a_j\}$.

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2.1. Proposition. *Let $a, b : n \longrightarrow U$. Then there is a permutation σ of U such that $a = \sigma \circ b$ if and only if* $\mathrm{kerp}(a) = \mathrm{kerp}(b)$.

Proof. Since σ is a bijection, we see from $a = \sigma \circ b$ that $a_i = a_j$ if and only if $(\sigma \circ b)_i = (\sigma \circ b)_j$ if and only if $b_i = b_j$. Thus a and b have the same kernel pair. Conversely, suppose $\mathrm{kerp}(a) = \mathrm{kerp}(b) = \alpha \subseteq n \times n$. From the diagram

$$\begin{tikzcd}
n \arrow[r, two heads] \arrow[d, "="'] & n/\alpha \arrow[r, tail, "\overline{a}"] \arrow[d, "="] & U \\
n \arrow[r, two heads] & n/\alpha \arrow[r, tail, "\overline{b}"'] & U
\end{tikzcd}$$

We see that a and b have isomorphic images. Since n is finite, their images also have isomorphic complements. ∎

For an equivalence relation α on n, let $R_\alpha \subseteq U^n$ denote the set of all functions $n \longrightarrow U$ whose kernel pair is α. Since each function has a unique kernel pair, it follows that $U^n = \sum R_\alpha$, the sum taken over the equivalence relations on n.

Say that a subset $R \subseteq U^n$ is permutation invariant (PI), if $a \in R$ and σ a permutation of U implies that $\sigma \circ a \in R$. The preceding proposition implies that,

2.2. Corollary. *A subset $R \subseteq U^n$ is PI if and only if it is a union of sets of the form R_α.*

Proof. Since a has the same kernel pair as $\sigma \circ a$ for any permutation σ, it follows that R_α is PI. On the other hand, if R is PI, then for any $a \in R$ any $b \in R_\alpha$ has the form $b = \sigma \circ a$ for some permutation σ. Thus if α is the kernel pair of a, we must have $R_\alpha \subseteq R$. Since the R_α partition U^n the subset R is a union of R_α. If we let $\Gamma(R) = \{\alpha \mid R_\alpha \subseteq R\}$ we can write that

$$R = \bigcup_{\alpha \in \Gamma(R)} R_\alpha$$ ∎

2.3. Definable sets. Say that a set $R \subseteq U^n$ is *first order definable* (or simply *definable*) if there is a set of first order sentences (not using equality) that are satisfied by R and by no other subset of U^n. Putnam claimed in his proof that the only definable sets are the empty set and U^n. He then observed that a definable is PI and seems to have assumed that the only PI sets are the empty set and U^n. We have just seen that all the R_α are PI. As mentioned, Keenan showed that equality on a finite set was definable. In fact, on a finite set, every PI relation is definable.

2.4. Theorem. *Every PI relation on a finite set is definable.*

Proof. We first show that equality is definable. In fact, if U is an n element set, then equality is the unique binary relation E such that

Eq–1. $\forall x, xEx$;

Eq–2. $\forall x, \forall y, xEy \Rightarrow yEx$;

Eq–3. $\forall x, \forall y, \forall z (xEy) \wedge (yEz) \Rightarrow xEz$;

Eq–4. $\forall x_1, \ldots, \forall x_{n-1}, \exists x_n, \neg(x_1 E x_n) \wedge \ldots \wedge \neg(x_{n-1} E x_n)$.

Next we see that any R_α is definable. In fact, R_α is definable by the following first order sentence

$$\forall x_1, \ldots, \forall x_n, ((x_1, \ldots, x_n) \in R_\alpha) \iff \left(\left(\bigwedge_{i\alpha j} x_i = x_j\right) \wedge \left(\bigwedge_{\neg i\alpha j} \neg(x_i = x_j)\right)\right)$$

Since each R_α is definable, so is each finite union of R_α, that is, each PI set. ∎

3. The infinite case

We now consider the case of an infinite universe U. We will show that in this case, only the empty and total relations are definable.

Let E be an equivalence relation on U. For an equivalence relation α on n, let $E(R_\alpha)$ denote the subset of U^n consisting of all $a : n \longrightarrow U$ for which the kernel pair of the composite $n \xrightarrow{a} U \longrightarrow U/E$ is exactly α. We note that in general $E(R_\alpha)$ neither includes nor is included in R_α. For example, suppose that E is not the equality. Then when α is the equality relation, R_α consists of the all the injective functions $a : n \longrightarrow U$, while $E(R_\alpha)$ is the proper subset consisting of all those for which the composite $n \longrightarrow U \longrightarrow U/E$ is injective. On the other hand, when α is the trivial relation (all pairs), then R_α is just the set of constant functions, while $E(R_\alpha)$ is the superset consisting of those for which $n \longrightarrow U \longrightarrow U/E$ is constant.

If R is a PI subset of U^n, then $R = \bigcup_{\alpha \in \Gamma(R)} R_\alpha$. Define $E(R) = \bigcup_{\alpha \in \Gamma(R)} E(R_\alpha)$. We do not define $E(R)$ unless R is PI. Then the following two propositions will demonstrate the claim that only the empty set and U^n are first order definable.

3.1. Proposition. *Let $R \subseteq U^n$ be a PI relation and E be an equivalence relation on U such that each equivalence class is infinite and $E(R) = R$. Then $R = \emptyset$ or $R = U^n$.*

3.2. Proposition. *Suppose U is an infinite set and E is an equivalence relation U for which there are infinitely many equivalence classes. Then for any $R \subseteq U^n$, the first order theory of R and of $E(R)$ are the same.*

Assuming these are proven and that U is infinite, let E be an equivalence relation on U such that there are infinitely many equivalence classes and infinitely many elements in each class. Then if $R \in U^n$ is definable, it is PI. Since R and $E(R)$ satisfy the same first order sentences, we must also have that $R = E(R)$ and then either $R = \emptyset$ or $R = U^n$.

4. Proofs

PROOF of 3.1. Let $a : n \longrightarrow U$ be an element of R. Let the image of a consist of the distinct elements $u_1, \ldots, u_k$. Choose distinct elements $v_1, \ldots, v_k \in U$ that are all equivalent mod E. Let σ be any automorphism of U such that $\sigma(u_i) = v_i$, $i = 1, \ldots, k$. Since R is PI, it follows that $\sigma \circ a \in R$. Since $E(R) = R$, then $\sigma \circ a \in E(R)$, which implies that $\sigma \circ a \in E(R_\alpha)$ for some equivalence relation α on n. But since $a_i E a_j$ for all $i, j \in n$, the only equivalence relation on n such that $\sigma \circ a \in E(R_\alpha)$ is the total relation $n \times n$. Thus $R_{n\times n}$ meets $E(R) = R$. But R is the disjoint union of R_α so this implies that $R_{n\times n} \subseteq R$. Let α be an arbitrary equivalence relation on n and suppose that $a : n \longrightarrow U$ belongs to R_α. Repeat the above construction to get a $b : n \longrightarrow U$ such that $b \in R_\alpha$, and all the values of b are equivalent mod E. This means that $b \in E(R_{n\times n}) \subseteq R$ and clearly $b \in R_\alpha$. Thus R_α meets $R_{n\times n}$ so that it meets and hence is included in R. Since α was arbitrary, this shows that $R = U^n$. ∎

PROOF of 3.2. There is a standard method used in logic to prove that two models are isomorphic. It is called the back and forth method since you begin by well-ordering the carriers and then alternate between defining the function and its inverse at the earliest place they are not defined. In this way, you guarantee surjectivity, as well as injectivity. For more details, see [Chang & Keisler], especially the definition on page 114 and Proposition 2.2.4 (ii) on page 115. Define $P_E : U \longrightarrow U/E$ as the projection on the equivalence classes mod E. To apply the back and forth method, we have to define relations $I_m \subseteq U^m \times U^m$ by $a \, I_m \, b$ if $\mathrm{kerp}(a) = \mathrm{kerp}(p_E \circ b)$ and show that

B&F–1. $() \, I_0 \, ()$;

B&F–2. If $f : n \longrightarrow m$ is an injective function and $a \, I_m \, b$, then $a \circ f \; I_n \; b \circ f$

B&F–3. If $a \, I_m \, b$ and $a' : m+1 \longrightarrow U$ extends a, then there is a $b' : m+1 \longrightarrow U$ that extends b for which $a' \, I_{m+1} \, b'$.

B&F–1 is vacuously true. B&F–2 is immediate (even if f is not injective) since both squares of

$$\begin{tikzcd}
\mathrm{kerp}(a\circ f) \arrow[r] \arrow[d] & n\times n \arrow[d, "f\times f"] & \mathrm{kerp}(p_E\circ b\circ f) \arrow[l] \arrow[d] \\
\mathrm{kerp}(a) \arrow[r] & m\times m & \mathrm{kerp}(p_E\circ b) \arrow[l]
\end{tikzcd}$$

are pullbacks.

Finally, we show B&F–3. If a' extends a, there is a new value, $a'(m)$, while $a'(i) = a(i)$ for $i = 0, \ldots, m-1$. It may happen that $a'(m) = a(i)$ for some $i < m$. In that case, just let $b'(m) = b(i)$. Then the kernel pair of a' is generated by that of a and the additional element (i, m) and the same will be true of $p_E \circ b'$. The other possibility is that $a'(m)$ is distinct from the image of a. In that case, the kernel pair of a' is generated by that of a. Since there are infinitely many equivalence classes mod E, we can find an element

$b'(m) \in U$ that is inequivalent mod E to all $b(i)$, $i = 1, \ldots, m$, in which case the kernel pair of $p_E \circ b'$ will also be generated by the kernel pair of b. ∎

References

[1] C. C. Chang and H. J. Keisler (1990), Model Theory, (3rd edition). North Holland.

[2] E. L. Keenan (1995), Logical objects. Preprint.

[3] H. Putnam (1981), Reason, Truth and History, Cambridge University Press.