A NOTE ON COMMUTATIVE ALGEBRA COHOMOLOGY

BY MICHAEL BARR[1]

Communicated by Murray Gerstenhaber, September 29, 1967.

This note consists of three parts. First we give an example to show that the commutative algebra cohomology theory described by Gerstenhaber in [4], and in more generality in [3], does not in general vanish in dimension three even when the coefficient module is injective. This implies that the theory cannot be described as the derived functor of the second cohomology group as was done in [1]. In the second part we show that every element of the third cohomology group can be regarded as an obstruction in the sense of Harrison [5]. Finally in §3 we show that the theory may be restricted, without loss of generality, to algebras with unit (and unitary maps).

1. **An example.** Let k be any field and $R=k[x, y]/(x, y)^2$. Then R has a k-basis consisting of $\{1, x, y\}$ with $x^2=y^2=xy=0$. The module $M=\mathrm{Hom}_k(R, k)$, regarded as an R-module by letting $(\alpha f)(\beta)=f(\beta\alpha)$ for $\alpha, \beta\in R$, is well known to be R-injective (see [2, p. 30]). If $\{\epsilon, \xi, \eta\}$ denotes the basis dual to the given one, then $x\xi=y\eta=\epsilon$, $x\epsilon=x\eta=y\epsilon=y\xi=0$.

Let $f: R\otimes R\otimes R\to M$ be the 3-cochain defined on the basis by

$$f(x\otimes x\otimes y) = \xi = -f(y\otimes x\otimes x),$$
$$f(y\otimes y\otimes x) = \eta = -f(x\otimes y\otimes y),$$

and f on any other combination of basis elements should be 0. Then verifying that f is a commutative cocycle is straightforward. Moreover, for any $g: R\otimes R\to M$,

$$\delta g(x\otimes x\otimes y) = xg(x\otimes y) - g(x^2\otimes y) + g(x\otimes xy) - yg(x\otimes x) \in xM + yM = R\epsilon,$$

which implies that f cannot cobound.

2. **Third cohomology and obstructions.** Let N be a commutative algebra (without unit) and M be its annihilator. Explicitly,

$$M = \{m\in N \mid mN = 0\}.$$

Let N^* be N with a unit adjoined. That is, $N^*=N\times k$ as a k-module

[1] This work was done while the author was an academic guest at the Forschungsinstitut für Mathematik, ETH, Zürich, and was also partially supported by the National Science Foundation grant GP-5478.

with multiplication given by $(n, u)(n', u') = (nn' + un' + u'n, uu')$ for $n, n' \in N$, $u, u' \in K$. Let $EN = \mathrm{Hom}_{N^*}(N, N)$ be the endomorphism ring of N. There is the natural multiplication map $N \to EN$ whose kernel is easily seen to be M and whose image is a central ideal of EN. Let $EN \xrightarrow{\pi} WN$ be the cokernel; then we have an exact sequence

$$0 \to M \to N \to EN \to WN \to 0.$$

If A is a commutative algebra, a map $\alpha: A \to WN$ is called strongly commutative if $\pi^{-1}(\mathrm{Im}\,\alpha)$ is a commutative subalgebra of EN. With such an α we may associate the element ω_α in $\mathcal{E}^3(A, M)$ by applying the map $\mathcal{E}^3(\alpha, M)$ to the element of $\mathcal{E}^3(\mathrm{Im}\,\alpha, M)$ represented by the sequence

$$0 \to M \to N \to \pi^{-1}(\mathrm{Im}\,\alpha) \to \mathrm{Im}\,\alpha \to 0.$$

ω_α is called the obstruction of α.

If $0 \to N \to B \to A \to 0$ is an algebra extension, there is associated a strongly commutative $\alpha: A \to WN$ induced by the diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & B & \to & A & \to & 0 \\ & & 1\downarrow & & \downarrow & & \downarrow\alpha & & \\ 0 & \to & M \to N & \to & EN & \to & WN & \to & 0 \end{array}$$

where $B \to EN$ is the B-module structure map. Using Theorems 4 and 5 of [3] it is very easy to prove

THEOREM 2.1. *A strongly commutative $\alpha: A \to WN$ comes from an extension of A by N if and only if $\omega_\alpha = 0$; in that case the classes of extensions inducing α are in* 1-1 *correspondence with the elements of $\mathcal{E}^2(A, M)$.*

(Also see [5, Theorems 7 and 8], where this is proved using a proof based on a direct cocycle argument.)

THEOREM 2.2. *Let $\omega \in \mathcal{E}^3(A, M)$. Then there is an N whose annihilator is isomorphic to M and a strongly commutative $\alpha: A \to WN$ with $\omega_\alpha = \omega$.*

PROOF. We let ω be represented by a sequence $0 \to M \to N_1 \xrightarrow{\rho_1} B_1 \to A \to 0$. Let B denote the algebra of polynomials with no constant term in the elements of A, and $B \to A$ denote the obvious map $\langle a \rangle \to a$ where $\langle a \rangle$ denotes the generator of B corresponding to $a \in A$. B is free in the category of algebras without unit and thus we may find a map $B \to B_1$ so that

$$\begin{array}{ccc} B & \to & B_1 \\ & \searrow & \downarrow \\ & & A \end{array}$$

commutes. Now if we pull back along $B \rightarrow B_1$ we get that the top row of the diagram

$$\begin{array}{ccccccccc} 0 \rightarrow & M & \rightarrow & N & \xrightarrow{\rho} & B & \rightarrow & A & \rightarrow 0 \\ & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 & \\ 0 \rightarrow & M & \rightarrow & N_1 & \xrightarrow{\rho_1} & B_1 & \rightarrow & A & \rightarrow 0 \end{array}$$

is exact and the diagram commutes. Thus the top row also represents ω. Since $\langle 0\rangle \in \ker(B \rightarrow A)$, we can find $n \in N$ with $\rho n = \langle 0\rangle$. If $n' \in N$ with $n'N = 0$, then $0 = \rho n'\langle 0\rangle$. But $\langle 0\rangle$ cannot be a zero divisor (even if the coefficients were not in a field) and so we conclude that $\rho n' = 0$ or $n' \in M$. Then M is exactly the annihilator of N. Since N is a B-algebra, there is a natural map $\beta: B \rightarrow EN$ which induces $\alpha: A \rightarrow WN$ so that the diagram

$$\begin{array}{ccccccccc} 0 \rightarrow & M & \rightarrow & N & \rightarrow & B & \rightarrow & A & \rightarrow 0 \\ & \downarrow 1 & & \downarrow 1 & & \downarrow \beta & & \downarrow \alpha & \\ 0 \rightarrow & M & \rightarrow & N & \rightarrow & EN & \xrightarrow{\pi} & WN & \rightarrow 0 \end{array}$$

commutes. Moreover since Im β is commutative and N is central, Im $\beta + \pi^{-1}(\text{Im } \alpha)$ is commutative and α is strongly commutative. Then $\omega_\alpha = \omega$, which completes the proof.

3. Algebras with unit.

THEOREM 3.1. *Let A be an algebra with unit and M be a unitary A-module. Then every class of extensions of $\mathcal{E}^3(A, M)$ contains a representative $0 \rightarrow M \rightarrow N \rightarrow B \rightarrow A \rightarrow 0$ in which B has a unit and $B \rightarrow A$ preserves the unit.*

PROOF. Let $0 \rightarrow M \rightarrow N_1 \rightarrow B_1 \rightarrow A \rightarrow 0$ represent an arbitrary element of $\mathcal{E}^3(A, M)$. Let $b \in B_1$ be any pre-image of 1 and let C denote the algebra B_1^* localized at the multiplicative set $\{1, b, b^2, \cdots\}$. Then C is B_1^*-flat and so the sequence

$$0 \rightarrow M \otimes C \rightarrow N_1 \otimes C \rightarrow B_1 \otimes C \rightarrow A \otimes C \rightarrow 0$$

(all tensors over B_1^*) is still exact. From the fact that b is a pre-image of 1, and M is unitary, it follows that the natural maps $A \rightarrow A \otimes C$ and $M \rightarrow M \otimes C$ are isomorphisms. Moreover the element $b \otimes 1/b \in B_1 \otimes C$ is a unit mapping to $1 \in A$. Then with $N = N_1 \otimes C$ and $B = B_1 \otimes C$, the result follows from the commutativity in

$$\begin{array}{ccccccccc} 0 \longrightarrow & M & \longrightarrow & N_1 & \longrightarrow & B_1 & \longrightarrow & A & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & M \otimes C & \rightarrow & N_1 \otimes C & \rightarrow & B_1 \otimes C & \rightarrow & A \otimes C & \rightarrow 0. \end{array}$$

PROPOSITION 3.2. *The categories of A-modules and unitary A^*-modules are naturally equivalent.*

PROOF. Trivial.

THEOREM 3.3. *There is a* 1-1 *correspondence between extensions* $0 \rightarrow M \rightarrow N \rightarrow B \rightarrow A \rightarrow 0$ *and unitary extensions* $0 \rightarrow M \rightarrow N \rightarrow B^* \rightarrow A^* \rightarrow 0$.

PROOF. Trivial.

The meaning of these statements is that for computing cohomology it suffices to restrict attention to those sequences

$$0 \rightarrow M \rightarrow N \rightarrow B \rightarrow A \rightarrow 0$$

in which $B \rightarrow A$ is a map of unitary k-algebras and M and N are unitary A and B modules respectively.

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UNIVERSITY OF ILLINOIS