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Coequalizers and Free Triples

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Introduction

This paper is concerned with two problems which, although not apparently closely related, are solved in part by the same methods. The first problem is: given a bicomplete (=complete and cocomplete) category $\mathscr{X}$ and a triple $\mathbf{T}$ on $\mathscr{X}$, is $\mathscr{X}^{\mathbf{T}}$ also bicomplete? The second is: given a category $\mathscr{X}$ and a functor $R: \mathscr{X} \to \mathscr{X}$, does R generate a free triple?

This paper began as an attempt to show that the category of contramodules over a coring is cocomplete (see (4.4)). Many people, too numerous to mention, have contributed materially to the results and their applications.

All notation and terminology not explicitly defined below may be found in the introduction to [2].

The first section of this paper gives the main definitions used and in section two we give the fundamental lemma on which the proofs are based. The next two sections prove and give applications of the cocompleteness theorem. Section five gives the construction of free triples and in section six we apply this to show that if $\mathscr{T}\!h$ is a small theory, then under certain conditions the category of $\mathscr{T}\!h$ algebras in $\mathscr{X}$ is tripleable over $\mathscr{X}$. In the next section we apply these results to the category of sets and we show that for a certain large full subcategory of endofunctors on sets there is a "free triple triple". The last section gives another cocompleteness theorem, not related to that of section three. This latter is a generalization of the result that every category of algebras over sets is cocomplete.

1. Notation and Definitions

(1.1) Throughout this paper $\mathscr{X}$ denotes, without further mention, a bicomplete category – one in which any functor $E: \mathscr{E} \to \mathscr{X}$ with $\mathscr{E}$ small has a limit (=projective limit), denoted $\lim R$, and a colimit (=inductive limit), denoted $\operatorname{colim} R$. We also suppose that $\mathscr{X}$ is locally small – there is only a set of morphisms between any two objects.

(1.2) If n is a limit ordinal number, we also use n to denote the ordered category of ordinals $<n$. A functor $n \to \mathscr{X}$ is called an n-sequence in $\mathscr{X}$. If we say that n is a cardinal number, this means that it is an ordinal which is the smallest of that cardinality.

(1.3) Let $\mathscr{M}$ be a class of monomorphisms and n a limit ordinal. An n-sequence $D: n \to \mathscr{X}$ is called an $(\mathscr{M}, n)$-sequence of subobjects of $X \in \mathscr{X}$ if there is a natural transformation ζ from D to the constant functor X such that for each $i \in M$, $\zeta i\colon Di \to X$ is a morphism in $\mathcal{M}$.

(1.4) If $T\colon \mathcal{X} \to \mathcal{X}$ is a functor, we say that T is $(\mathcal{M}, n)$-small if whenever $D\colon M \to X$ is an $(\mathcal{M}, n)$-sequence, the natural map $\operatorname{colim} TD \to T \operatorname{colim} D$ is an isomorphism. It will be called n-small if this holds for $\mathcal{M}$ = class of all monomorphisms.

(1.5) We say that $\mathcal{X}$ has small $\mathcal{M}$ factorizations if for every $X \in \mathcal{X}$ there is a set ΓX of objects of $\mathcal{X}$ such that any $X \to Y$ in $\mathcal{X}$ factors as $X \to Z \to Y$ with $Z \in \Gamma X$ and $Z \to Y$ in $\mathcal{M}$. If $\mathcal{M}$ = class of all monomorphisms, we say that $\mathcal{X}$ has small factorizations.

(1.6) We say that $\mathcal{X}$ is $(\mathcal{M}, n)$-small if it has small $\mathcal{M}$ factorizations and if given $D\colon M \to \mathcal{X}$ an $(\mathcal{M}, n)$-sequence of subobjects of X, the induced map $\operatorname{colim} D \to X$ is also in $\mathcal{M}$.

(1.7) We say that $T\colon \mathcal{X} \to \mathcal{Y}$ has rank $\leq m$ for a cardinal m if whenever M is an m complete cardinal T is n-small.

(1.8) If $T\colon \mathcal{X} \to \mathcal{X}$ is a functor, we define a category $(T{:}\mathcal{X})$ whose objects are pairs (X, x) where $X \in \mathcal{X}$ and $x\colon TX \to X$. A morphism in this category between (X, x) and (Y, y) is an $f\colon X \to Y$ such that $f \cdot x = y \cdot Tf$.

2. The Main Lemma

(2.1) **Lemma.** *Suppose the category $\mathcal{X}$ has small $\mathcal{M}$ factorizations and the functor $T\colon \mathcal{X} \to \mathcal{X}$ is $(\mathcal{M}, n)$-small. Then for each object $X \in \mathcal{X}$ there is a set ΛX of objects of $\mathcal{X}$ such that any diagram*

$$\begin{array}{ccc} & & TY \\ & & \downarrow{\scriptstyle y} \\ X & \xrightarrow{f} & Y \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} & & TZ & \xrightarrow{Tb} & TY \\ & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle y} \\ X & \xrightarrow{a} & Z & \xrightarrow{b} & Y \end{array}$$

with $Z \in \Lambda X$. Moreover if $w\colon TW \to W$ and $g\colon W \to X$ are such that fg is a morphism in $(T{:}\mathcal{X})$, then $a \cdot g$ is also.

Proof. We first define ΛX. It is defined by means of an inductively defined sequence $\{\Lambda_m X\}_{m<n}$, as follows. $\Lambda_0 X = \Gamma X$ (see 1.5). If $\Lambda_m X$ is defined,

$$\Lambda_{m+1} X = \bigcup \{\Gamma(Z * TZ) \mid Z \in \Lambda_m X\},$$

where $*$ denotes coproduct. If m is a limit ordinal and $\Lambda_p X$ is defined for all $p < m$, let

$$\Lambda_m X = \bigcup \Gamma\Big(\coprod_{p<m} Z_p\Big),$$

where the union is taken over all coproducts of families $\{Z_p \mid p<m,\ Z_p \in \Lambda_p X\}$. Finally we define ΛX to consist of all objects of the form colim Z_m, where $\{Z_m\}$ is an $(\mathcal{M}, n)$-sequence such that $Z_m \in \Lambda_m X$. Let Z_0 be chosen in $\Gamma X = \Lambda_0 X$ so that there is a factorization of f as $X \xrightarrow{a_0} Z_0 \xrightarrow{b_0} Y$ with $b_0 \in \mathcal{M}$. Now factor the map

$$(Y, y) \cdot (b_0 * Tb_0)\colon Z_0 * TZ_0 \to Y$$

as

$$Z_0 * TZ_0 \xrightarrow{a_1} Z_1 \xrightarrow{b_1} Y$$

with $b_1 \in \mathcal{M}$ and $Z_1 \in \Gamma(Z_0 * TZ_0) = \Lambda_1 X$. Let a_1 have components $e_{01}\colon Z_0 \to Z_1$ and $c_0\colon TZ_0 \to Z_1$. Then clearly $b_1 \cdot e_{01} = b_0$ and $b_1 \cdot e_0 = y$. Tb_0. We will now define for all $m<n$, Z_m, $b_m\colon Z_m \to Y$, $c_m\colon TZ_m \to Z_{m+1}$; and for $p<m<n$, $e_{pm}\colon Z_p \to Z_m$ such that $Z_m \in \Lambda_m X$, $b_m e_{pm} = b_p$, $b_{m+1} \cdot c_m = y \cdot Tb_m$ and if $r<p<m<n$, $b_{pm} b_{rp} = b_{rm}$. This is done by induction, as follows. Given Z_p, b_p, c_p for all $p<m$ satisfying the above, first suppose m has a predecessor. Then factor

$$(Y, y) \cdot (b_{m-1} * Tb_{m-1})\colon Z_{m-1} * TZ_{m-1} \to Y$$

as

$$Z_{m-1} * TZ_{m-1} \xrightarrow{a_m} Z_m \xrightarrow{b_m} Y$$

with

$$b_m \in \mathcal{M} \quad \text{and} \quad Z_m \in \Gamma(Z_{m-1} * TZ_{m-1}) \subset \Lambda_m X.$$

If a_m has components $e_{m-1,m}$ and c_{m-1}, it is clear that $b_m e_{m-1,m} = b_{m-1}$ and $b_m c_{m-1} = y \cdot Tb_{m-1}$. If $p<m-1$, define $e_{pm} = e_{m-1,m} e_{p,m-1}$. Then $b_m \cdot e_{pm} = b_m \cdot e_{m-1,m} \cdot e_{p,m-1} = b_{m-1} \cdot e_{p,m-1} = b_p$, and for $r<p<m-1$, $e_{mp} \cdot e_{rp} = e_{m-1,m} \cdot e_{p,m-1} \cdot e_{rp} = e_{m-1,m} e_{r,m-1} = e_{r,m}$. On the other hand, if m is a limit ordinal, factor the map $\coprod_{p<m} Z_p\ Y$ whose p^{th} coordinate is b_p as $\coprod_{p<m} Z_p \xrightarrow{a_m} Z_m \xrightarrow{b_m} Y$. If the p^{th} coordinate of a_m is e_{pm}, then by definition $b_m \cdot e_{pm} = b_p$. Also for $r<p<m$, $b_m \cdot e_{pm} \cdot e_{rp} = b_p e_{rp} = b_r = b_m \cdot e_{rm}$, from which we can cancel the monomorphism b_m and conclude that $e_{pm} \cdot e_{rp} = e_{rm}$. When this is done for all $m<n$, the $\{Z_m\}$ and $\{e_{pm}\}$ constitute an n-sequence and the $\{b_m\}$ show that it is an $(\mathcal{M}, n)$ sequence of subobjects of X (1.3). Also $Z_m \in \Lambda_m X$. Thus if we let $Z = \operatorname{colim} Z_m$, $Z \in \Lambda X$. Let $d_m\colon Z_m \to Z$ be the natural maps to the colimit and $b\colon Z \to Y$ be the unique map such that $b \cdot d_m = b_m$. By hypothesis, we also have that $TZ = \operatorname{colim} TZ_m$ and so we can define $z\colon TZ \to Z$ by requiring that $z \cdot Td_m = d_{m+1} \cdot c_m$. This is a compatible family, for if $p<m$, $b_{m+1} \cdot e_{p,m+1} \cdot c_p = b_{p+1} \cdot c_p = y \cdot Tb_p = y \cdot Tb_m \cdot Te_{pm} = b_{m+1} \cdot c_m \cdot Te_{pm}$, and we may cancel b_{m+1} and get $e_{p,m+1} \cdot c_p = c_m \cdot Te_{pm}$. Then $y \cdot Tb \cdot Td_m = y \cdot Tb_m = b_{m+1} \cdot c_m = b \cdot d_{m+1} \cdot c_m = b \cdot z \cdot Td_m$, and thus by uniqueness of a map from a colimit, $y \cdot Tb = b \cdot z$. If $a = d_0 \cdot a_0$, $b \cdot a = b \cdot d_0 \cdot a_0 = b_0 \cdot a_0 = f$, which completes the proof of the first assertion.

For the second, first observe that $b_1 \cdot c_0 \cdot T(a_0 \cdot g) = b_1 \cdot c_0 \cdot Ta_0 \cdot Tg = y \cdot Tb_0 \cdot Ta_0 \cdot Tg = y \cdot Tf \cdot Tg = y \cdot T(f \cdot g) = f \cdot g \cdot w = b_0 \cdot a_0 \cdot g \cdot w = b_1 \cdot e_{01} \cdot a_0 \cdot g \cdot w$, and b_1 is a monomorphism, so $c_0 \cdot T(a_0 \cdot g) = e_{01} \cdot a_0 \cdot g \cdot w$. Then $z \cdot T(a\, g) = z \cdot Td_0 \cdot T(a_0 \cdot g) = d_1 \cdot c_0 \cdot T(a_0 \cdot g) = d_1 \cdot e_{01} \cdot a_0 \cdot g \cdot w = d_0 \cdot a_0 \cdot g \cdot w = a \cdot g \cdot w$, which completes the proof.

3. Coequalizers in Categories of Algebras

(3.1) Before giving our main results we require a preliminary result. Although it is stated for coequalizers, it would remain true if that were replaced by any colimit. The result seems to be well known in folklore and was no doubt first observed by Freyd, but I have been unable to find a reference.

(3.2) **Proposition.** *Suppose $\mathscr{Y}$ is a complete category and*

$$Y_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} Y_0$$

is a diagram in $\mathscr{Y}$. Then it has a colimit if and only if there is a solution set for the colimit, i.e. a set of maps $\{d_i\colon Y_0 \to Y_i\}_{i\in I}$ such that $d_i\, d^0 = d_i\, d^1$ and any $d\colon Y_0 \to Y$ such that $d\, d^0 = d\, d^1$ factors as $Y_0 \to Y_i \to Y$ for some $i \in I$.

Proof. The proof is an easy application of Freyd's adjoint functor theorem ([3], problem 3-J). Form the category $\mathscr{Z}$ whose objects are maps $d\colon Y_0 \to Y$ with $d \cdot d^0 = d \cdot d^1$ and maps are commutative triangles

$$\begin{tikzcd} & Y_0 \arrow[dl, "d"'] \arrow[dr, "d'"] & \\ Y \arrow[rr, "f"] & & Y' \end{tikzcd}$$

$\mathscr{Z}$ is easily seen to be complete when $\mathscr{Y}$ is. Then the set $\{d_i\colon Y_0 \to Y_i\}$ is a pre-initial set in that category (meaning every object of $\mathscr{Z}$ admits a map from at least one d_i). The map $d_*\colon Y_0 \to \prod Y_i = Y_*$, whose i^{th} coordinate is d_i, is a pre-initial object (meaning it has at least one map to every object of $\mathscr{Z}$), and finally, the equalizer of all the endomorphisms of $d_*\colon Y_0 \to Y_*$ is an initial object of $\mathscr{Z}$, which is easily seen to be the coequalizer of d^0 and d^1.

(3.3) **Theorem.** *Suppose $\mathscr{X}$ is a category and $\mathbf{T} = (T, \eta, \mu)$ is a triple on $\mathscr{X}$. If there is a class $\mathscr{M}$ of monomorphisms and a cardinal n such that $\mathscr{X}$ has small $\mathscr{M}$ factorizations and T is $(\mathscr{M}, n)$-small, then the category $\mathscr{X}^{\mathbf{T}}$ of algebras has coequalizers.*

Proof. Suppose $(V, v) \underset{e^1}{\overset{e^0}{\rightrightarrows}} (W, w)$ are two maps in $\mathscr{X}^{\mathbf{T}}$. Let X be the coequalizer in $\mathscr{X}$ of $V \underset{e^1}{\overset{e^0}{\rightrightarrows}} W$. If $h\colon (W, w) \to (Y, y)$ is a morphism in $\mathscr{X}^{\mathbf{T}}$ with $h \cdot e^0 = h \cdot e^1$, it factors in $\mathscr{X}$ as $W \xrightarrow{g} X \xrightarrow{f} Y$. If we apply (2.1) to this, we get a commutative diagram

$$\begin{tikzcd} TW \arrow[rr, "T(f\cdot g)"] \arrow[d, "w"] & & TZ \arrow[r, "Tb"] \arrow[d, "z"] & TY \arrow[d, "y"] \\ W \arrow[r, "g"] & X \arrow[r, "f"] & Z \arrow[r, "b"] & Y \end{tikzcd}$$

with $Z \in \Lambda X$. Since g already coequalizers e^0 and e^1, so does $f \cdot g$. Thus every map coequalizing e^0 and e^1 factors through an object (Z, z) with $Z \in \Lambda X$. Thus the set of such objects forms a solution set for the coequalizer, and since $\mathscr{X}^{\mathbf{T}}$ is complete, the coequalizer exists.

Note added in proof. Professor H. Schubert has pointed out that there is a gap in the proof of (3.3). Namely, it is not shown that (Z, z) is a $\mathbf{T}$-algebra. That it does indeed satisfy the conditions of an algebra may be readily worked our from the commutativity of the following diagrams together with the fact that the b's are monomorphisms.

$$\begin{tikzcd}
T^2 Z_m \arrow[r, shift left, "T e_{m,m+1} \cdot \mu Z_m"] \arrow[r, shift right, "T c_m"'] \arrow[d, "T^2 b_m"'] & T Z_{m+1} \arrow[r, "c_{m+1}"] \arrow[d, "T b_{m+1}"'] & Z_{m+2} \arrow[d, "b_{m+2}"] \\
T^2 Y \arrow[r, shift left, "\mu Y"] \arrow[r, shift right, "T y"'] & T^2 Y \arrow[r, "y"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
Z_m \arrow[rr, bend left, "e_{m,m+1}"] \arrow[r, "\eta Z_m"'] \arrow[d, "b_m"'] & T Z_m \arrow[r, "c_m"'] \arrow[d, "T b_m"'] & Z_{m+1} \arrow[d, "b_{m+1}"] \\
Y \arrow[r, "\eta Y"] & T Y \arrow[r, "y"] & Y
\end{tikzcd}$$

(3.4) **Corollary.** *Under the same hypothesis, $\mathscr{X}^{\mathbf{T}}$ has all small colimits, i.e. is cocomplete.*

Proof. This follows directly from a theorem of Linton ([8], corollary 2, p. 81).

(3.5) *Remark.* It has recently come to may attention that H. Schubert has obtained results very similar to the above. The main differences are: no condition like the existence of $\mathscr{M}$ or the ΓX is assumed; the functor T must commute with all n-sequences, not just of monomorphisms.

4. Applications of (3.2)

(4.1) Let C be an additive group equipped with homomorphisms ε: $C \to \mathbf{Z}$ and δ: $C \to C \otimes C$ such that $\delta \otimes C \cdot \delta = C \otimes \delta \cdot \delta$ and $\varepsilon \otimes C \cdot \delta = C \otimes \varepsilon \cdot \delta = C$. This is called an associative coring with counit. The functor T: $\mathscr{A}\mathscr{b} \to \mathscr{A}\mathscr{b}$ ($\mathscr{A}\mathscr{b}$ is the category of abelian groups) given by $TA = \mathrm{Hom}(C, A)$ can be given the structure of triple in an obvious way, using ε and δ. An algebra for that triple is called a contramodule. It is an abelian group A together with a map $\mathrm{Hom}(C, A) \to A$ satisfying appropriate identities. It was previously unknown for any ordinal except ω nor for any directed colimit except simply ordered.

(4.2) **Proposition.** *The category $\mathscr{A}\mathscr{b}$ has small factorizations ($\mathscr{M}$ = class of all monomorphisms).*

Proof. Trivial.

(4.3) **Proposition.** *If α is the cardinal of C and n is any infinite ordinal which is α complete, then* $\mathrm{Hom}(C, -)$ *is n-small.*

Proof. If $A \in \mathscr{A}\mathscr{b}$ and $\{A_m\}_{m \leqq n}$ is an n-sequence of subobjects of A, then their colimit is their set union B. Any map of $C \to B$ takes each $c \in C$ to some $A_{m(c)} \subset B$. Since the indices are at most α in an α-complete lattice, there is some $m < n$ such that the map factors through A_m. This shows that $\mathrm{colim}(C, A_m) \to (C, \mathrm{colim}\, A_m)$ is onto and it is clearly 1-1.

(4.4) **Corollary.** *For any coring C the category of C-contramodules is cocomplete.*

Proof. It is only necessary to find an α-complete limit ordinal. Just take the first ordinal whose cardinal is $>\alpha$. Clearly the sup of α or fewer ordinals, each $\leqq\alpha$, is $\leqq\alpha$ also.

(4.5) More generally, if $\mathscr{X}$ is any category which is tripleable over sets, $\mathscr{X}$ has small factorizations, and if $\mathbf{T}=(T,\eta,\mu)$ is any triple on $\mathscr{X}$ with T being n-small, then $\mathscr{X}^{\mathbf{T}}$ is cocomplete.

(4.6) The same argument remains true if the category $\mathscr{S}$ of sets is replaced by a category $\mathscr{S}^A$ where A is a set. If $\mathscr{A}$ is a small category with underlying set of objects A, then $\mathscr{X}=(\mathscr{A},\mathscr{S})$ is tripleable over $\mathscr{S}^A$; and if $\mathbf{T}=(T,\eta,\mu)$ is any triple on $\mathscr{X}$ with T n-small for some n, then $\mathscr{X}^{\mathbf{T}}$ is cocomplete.

(4.7) A topological space X is called a K-space if $Y\subset X$ is closed whenever its intersection with every compact subset of X is closed. The inclusion of K-spaces into all topological spaces has a left adjoint which retopologizes a space by adding all such sets to the list of closed sets. We call this the K-ification of X. Now let $\mathscr{K}$ denote the category of Hausdorff pointed K-spaces. Let $\Sigma\colon\mathscr{K}\to\mathscr{K}$ and $\Omega\colon\mathscr{K}\to\mathscr{K}$ denote the usual suspension and loop space functors, K-ified. Then Σ is left adjoint to Ω, so the composite functor $T=\Omega\Sigma$ has the natural structure of a triple $\mathbf{T}$. If $X\in\mathscr{K}$, a monomorphism $Y\to X$ is called a K-subspace of X if it is an ordinary subspace K-ified.

(4.8) **Lemma** (Dold-Thom). *If $X\in\mathscr{K}$ and $\{X_n\}_{n\in\omega}$ is a countable ascending chain of K-subspaces of X, then any compact subset of* colim X_n *is already contained in some X_n.*

Proof. See [1], 2.14.

(4.9) **Proposition.** *If $\{X_n\}_{n\in\omega}$ is a countable ascending chain of K-subspaces of X, then the natural map* $\operatorname{colim}\Omega X_n\to\Omega\operatorname{colim}X_n$ *is a homeomorphism.*

Proof. It is clear that the natural map is 1-1. Let C denote the circle. Then $\Omega=\operatorname{Hom}(C,-)$. A map of C to $\operatorname{colim}X_n$ lands in some compact subset of $\operatorname{colim}X_n$ which by (4.8) is contained in some X_n. This shows that the natural map is onto. To show that it is a homeomorphism, we must show that if $A\subset\operatorname{colim}(C,X_n)$ is closed, then its image in $(C,\operatorname{colim}X_n)$ is also. But for this it is only necessary to show that its intersection with every compact subset $B\subset(C,\operatorname{colim}X_n)$ is compact (recall everything is Hausdorff). But such a subset is represented by a function (the "name of" the inclusion) $B\times C\to\operatorname{colim}X_n$ which, as above, actually lands in some X_n. Thus every compact subset of $(C,\operatorname{colim}X_n)$ is a compact subset of some (C,X_n). Then if $A\subset\operatorname{colim}(C,X_n)$ is closed, $A\cap(C,X_n)$ is also closed and $A\cap B\cap(C,X_n)$ is compact. Since B is an arbitrary compact subset of $(C,\operatorname{colim}X_n)$, it follows that A is closed.

(4.10) **Corollary.** *$\mathscr{K}^{\mathbf{T}}$ is cocomplete.*

Proof. Take $n=\omega$ and $\mathscr{M}$ to be the class of K-subspace inclusions. It is clear that every map factors as a map which is onto and a K-subspace inclusion.

It is interesting to observe that the Dold-Thom argument does not work for any ordinal except ω nor for any directed colimit except simply ordered.

This example was worked out in collaboration with J. Beck.

5. Free Triples

(5.1) Let $R: \mathscr{X} \to \mathscr{X}$ be a functor. By the free triple generated by R we mean a triple $\mathbf{T} = (T, \eta, \mu)$ and a natural transformation $\rho: R \to T$ such that if $\mathbf{T} = (T_1, \eta_1, \mu_1)$ is another triple and $\rho_1: R \to T_1$ is a natural transformation, then $\rho_1 = \tau \cdot \rho$, where $\tau: \mathbf{T} \to \mathbf{T}_1$ is a map of triples.

We shall derive a necessary and sufficient condition for the existence of a free triple which together with (2.1) will yield a useful sufficient condition.

(5.2) **Proposition.** *If R is an endofunctor and $\mathbf{T}$ a triple on $\mathscr{X}$, then there is a natural* 1-1 *correspondence between natural transformations $R \to T$ and functors $\mathscr{X}^{\mathbf{T}} \to (R:\mathscr{X})$ which commute with the underlying functors to $\mathscr{X}$. (Such functors are said to be over $\mathscr{X}$.)*

Proof. Suppose $\lambda: R \to T$ is a natural transformation. Define a functor $S(\lambda)$: $\mathscr{X}^{\mathbf{T}} \to (R:\mathscr{X})$ by $S(\lambda)(X, x) = (X, x \cdot \lambda X)$. This evidently becomes a functor over $\mathscr{X}$. To go the other way, if S: $\mathscr{X}^{\mathbf{T}} \to (R:\mathscr{X})$ is a functor over $\mathscr{X}$, let $S(TX, \mu X) = (TX, \theta X) \in (R:\mathscr{X})$. Then let $\lambda(S)$: $R \to T$ by $\lambda(S)X = \theta X \cdot R\eta X$. To show naturality, suppose f: $X \to Y$. Then Tf: $(TX, \mu X) \to (TY, \mu Y)$ is a morphism in $\mathscr{X}^{\mathbf{T}}$, so STf: $(TX, \theta X) \to (TY, \theta Y)$ is a morphism in $(R:\mathscr{X})$. This means $\theta Y \cdot RTf = Tf \cdot \theta X$. But also η is natural, so $Tf \cdot \lambda(S)X = Tf \cdot \theta X \cdot R\eta X = \theta Y \cdot RTf \cdot R\eta X = \theta Y \cdot R\eta Y \cdot Rf = \lambda(S)Y \cdot Rf$. Now $S(\lambda)(TX, \mu X) = (TX, \mu X \cdot \lambda TX)$, so $\lambda(S(\lambda)) = \mu \cdot \lambda T \cdot R\eta = \mu \cdot T\eta \cdot \lambda = \lambda$. On the other hand, if S: $\mathscr{X}^{\mathbf{T}} \to (R:\mathscr{X})$ is a functor over $\mathscr{X}$, suppose $S(X, x) = (X, x^{\#})$. Now x: $(TX, \mu X) \to (X, x)$ is a morphism in $\mathscr{X}^{\mathbf{T}}$ and hence x: $(TX, \theta X) \to (X, x^{\#})$ is also a morphism, which means $x^{\#} \cdot Rx = x \cdot \theta X$. But then $x^{\#} = x^{\#} \cdot Rx \cdot R\eta X = x \cdot \theta X \cdot R\eta X$, and so $S(X, x) = (X, x \cdot \theta X \cdot R\eta X) = (X, x \cdot \lambda(S)X) = S(\lambda(S))(X, x)$.

(5.3) **Proposition.** *Suppose* $\mathbf{T} = (T, \eta, \mu)$ and $\mathbf{T}' = (T', \eta', \mu')$ *are triples on $\mathscr{X}$. Then there is a* 1-1 *correspondence between triple maps $\mathbf{T} \to \mathbf{T}'$ and functors $\mathscr{X}^{\mathbf{T}'} \to \mathscr{X}^{\mathbf{T}}$ over $\mathscr{X}$.*

This may be easily proved by making minor modifications in the above proof. In any case, it has long been well-known, going all the way back to Lawvere's thesis [7].

(5.4) **Theorem.** *Suppose R: $\mathscr{X} \to \mathscr{X}$ is a functor,* $\mathbf{T}(T, \eta, \mu)$ *is a triple on $\mathscr{X}$, and there is an isomorphism $\mathscr{X}^{\mathbf{T}} \to (R:\mathscr{X})$ over $\mathscr{X}$. Then $\mathbf{T}$ is the free triple generated by R.*

Proof. Just put together (5.2) and (5.3).

We will show later that this condition is also necessary.

(5.5) **Theorem.** *Suppose $\mathscr{X}$ is a category and R an endofunctor. If there is a class $\mathscr{M}$ of monomorphisms and limit ordinal n such that X has small $\mathscr{M}$ factorizations and R is $(\mathscr{M}, n)$ small, then the underlying functor $(R:\mathscr{X}) \to \mathscr{X}$ is tripleable. Thus R generates a free triple.*

Proof. The underlying functor U: $(R:\mathscr{X}) \to \mathscr{X}$ is easily seen to creat limits and coequalizers of U-contractible coequalizer pairs. Hence, by the *PTT*, it is only necessary to show that U has an adjoint. Since $\mathscr{X}$ is complete and U creates limits, $(R:\mathscr{X})$ is also complete, so it is only necessary to find a solution set. By (2.1) the set of all algebras whose underlying object is in ΛX is a solution set at X.

(5.6) **Proposition.** *Suppose that $\mathscr{Z}$ is a complete category, $\mathscr{Z}_1$ is a full subcategory closed under limits, and $\mathscr{Z}_1$ has a cogenerator (i.e. cogenerating set). Suppose $U: \mathscr{Z} \to \mathscr{X}$ is a limit preservving functor with the property that for every $Z \in \mathscr{Z}$ there is a $Z_1 \in \mathscr{Z}_1$ and an $f: Z_1 \to Z$ such that Uf is a split epimorphism. Then U has a left adjoint.*

Proof. By assumption, $\mathscr{Z}_1 \to \mathscr{Z}$ is full and thus preserves limits and then so does the composite $\mathscr{Z}_1 \to \mathscr{Z} \to \mathscr{X}$. Since $\mathscr{Z}_1$ has a cogenerator this composite has an adjoint F (see [3], problem $3-M$). Now if $g: X \to UZ$ is a morphism, choose $f: Z_1 \to Z$ a morphism in $\mathscr{Y}$ whose domain is in $\mathscr{Z}_1$ such that Uf is a split epimorphism. Then g factors through Uf, and then by adjointness there is a map $FX \to Z_1$ such that the composite morphism

$$X \to UFX \to UZ_1 \to UZ$$

is f. But then $\{FX\}$ is a solution set at X and the adjoint exists by the functor theorem.

(5.7) **Definition.** Let $U: \mathscr{Z} \to \mathscr{X}$ be a functor. A full subcategory $\mathscr{Z}_1 \subset \mathscr{Z}$ is called a Birkhoff subcategory of $\mathscr{Z}$ with respect to U if $\mathscr{Z}_1$ is closed under products, subobjects, and U-split quotients. The last means that if $f: Z_1 \to Z$ with $Z_1 \in \mathscr{Z}_1$ and Uf a split epimorphism, then $Z \in \mathscr{Z}_1$ also. If U is clearly understood we will simply call $\mathscr{Z}_1$ a Birkhoff subcategory of $\mathscr{Z}$.

Notice that a subcategory closed under products and subobjects is closed under products and equalizers and hence under all limits.

(5.8) **Proposition.** *Suppose $U: \mathscr{Y} \to \mathscr{X}$ creates limits and U-contractible coequalizers. If Δ is any set of objects of $\mathscr{Y}$, there is a Birkhoff subcategory $\mathscr{Z} \subset \mathscr{Y}$ such that the composite $\mathscr{Z} \subset \mathscr{Y} \to \mathscr{X}$ is tripleable and such that $\Delta \subset \mathscr{Z}$.*

Proof. Let $\mathscr{Z}_0$ be the full subcategory whose objects are all products of objects of Δ, and $\mathscr{Z}_1$, be the full subcategory whose objects are all subobjects of objects of $\mathscr{Z}_0$. Since both a product of monomorphisms and a composite of monomorphisms are monomorphisms, $\mathscr{Z}_1$ is closed under products and subobjects and has a cogenerator, namely Δ. Finally, let $\mathscr{Z}$ be the full subcategory whose objects are the codomains of U split epimorphisms with domain in $\mathscr{Z}_1$. We claim that $\mathscr{Z}$ is closed under products and subobjects also. If for each $i \in I$, $Z_{1,i} \to Z_i$ is a U-split epimorphism, so is $\prod Z_{1,i} \to \prod Z_i$. If $Z_1 \to Z$ is a U-split epimorphism and $Z' \subset Z$, let

$$\begin{tikzcd} Z_1' \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z_1 \arrow[r] & Z \end{tikzcd}$$

be a pullback. Since $\mathscr{Z}_1$ is closed under subobjects, $Z_1' \in \mathscr{Z}_1$. Since U preserves limits,

$$\begin{tikzcd} UZ_1' \arrow[r] \arrow[d] & UZ' \arrow[d] \\ UZ_1 \arrow[r] & UZ \end{tikzcd}$$

is also a pullback, and it is easily shown that a pullback of a split epimorphism is a split epimorphism. Thus $\mathscr{Z}$ is a Birkhoff subcategory of $\mathscr{Y}$, and by (5.6), $\mathscr{Z} \subset \mathscr{Y} \to \mathscr{X}$ has an adjoint and is then clearly tripleable.

This proposition was suggested by an *ad hoc* argument of J. Isbell in the case of complete boolean algebras.

(5.9) **Theorem.** *Suppose $U: \mathscr{Y} \to \mathscr{X}$ creates limits and U contractible coequalizers, and suppose $\mathbf{T}$ is a triple on $\mathscr{X}$ and $V: \mathscr{X}^{\mathbf{T}} \to \mathscr{Y}$ a functor over $\mathscr{X}$ which is universal in the sense given below. Then V is an equivalence.*

The sense of universal is the following. If $\mathbf{T}'$ is another triple on $\mathscr{X}$ and $V': \mathscr{X}^{\mathbf{T}'} \to \mathscr{Y}$ is another functor over $\mathscr{X}$, then there is a unique (up to natural equivalence) functor $\mathscr{X}^{\mathbf{T}'} \to \mathscr{X}^{\mathbf{T}}$ such that

$$\begin{tikzcd} X^{\mathbf{T}'} \arrow[rr] \arrow[dr, "V'"'] & & \mathscr{X}^{\mathbf{T}} \arrow[dl, "V"] \\ & \mathscr{Y} & \end{tikzcd}$$

commutes (up to a natural equivalence). This functor is necessarily over $\mathscr{X}$, and by (5.3) is induced by a unique triple morphism $\mathbf{T} \to \mathbf{T}'$.

Proof of (5.9). Let $Y \in \mathscr{Y}$. By (5.8), there is a tripleable Birkhoff subcategory $\mathscr{Z} \subset \mathscr{Y}$ which contains Y. The inclusion must factor through V, which implies that Y is in the image of V and then that V is onto objects. Now we show that if FX is the free $\mathbf{T}$-algebra on the object X, then $\{VFX\}$ is a solution set at X. In fact, if $f: X \to UY$ is a map and $Y = VA$, then f factors as

$$X \to TX \xrightarrow{UVg} UVA = UY$$

for some $g: FX \to A$. Then $h = Vg: VFX \to Y$ is such that Uh factors f. Thus U has an adjoint and the other properties imply that U is tripleable itself. But then the usual uniqueness properties of universal objects imply that V must be an equivalence.

(5.10) **Corollary.** *If R generates a free triple $\mathbf{T}$, then $(R:\mathscr{X}) \to \mathscr{X}$ is tripleable.*

Proof. Just put together (5.5), (5.6) and (5.9).

(5.11) **Proposition.** *If $X \in \mathscr{X}$ is such that $FX \in (R:\mathscr{X})$ together with $\eta X: X \to UFX$ are a partial adjoint at X and $x: RUFX \to UFX$ is the structure of UFX, then $(\eta X, x): X * RUFX \to UFX$ is an isomorphism.*

Proof. Writing TX for UFX we know that it is characterized by the following universal mapping property given $f: X \to Y$ and $y: RY \to Y$, there is a unique $g: TX \to X$ such that $g \cdot \eta X = f$ and $y \cdot Rg = g \cdot x$. This may be reworded as follows. Given $(f, y): X * RY \to Y$ there is a unique $g: TX \to X$ such that $g \cdot (\eta X, x) = (f, y) \cdot (X * Rg)$. But this is precisely the solution to the problem of finding an initial object in the category $(X * R : \mathscr{X})$ where $X * R$ denotes the functor whose value at an object Y is $X * RY$ (with the obvious extension to morphisms). Consequently, the following proposition, suggested by Lambek, completes the proof.

(5.12) **Proposition.** *If $R: \mathscr{X} \to \mathscr{X}$ is a functor and if (Z, z) is initial in $(R : \mathscr{X})$, then $z: RZ \to Z$ is an isomorphism.*

Proof. Since $Rz: R^2 Z \to RZ$ gives also an $(R : \mathscr{X})$ object, there is a unique $g: Z \to RZ$ such that $g \cdot z = Rz \cdot Rg$. Then $z \cdot g \cdot z = z \cdot Rz \cdot Rg - z \cdot R(z \cdot g)$, which is the assertion that $z \cdot g: (Z, z) \to (Z, z)$ is a morphism, which since (Z, z) is initial, is necessarily the identity. Thus $z \cdot g = Z$, and then $g \cdot z = Rz \cdot Rg = R(z \cdot g) = RZ$, so that $g = z^{-1}$.

(5.13) The question of the existence of free triples was initiated by J. Beck in unpublished work.

6. Triples on Sets

(6.1) In this section we study special properties of triples on the category $\mathscr{S}$ of sets. Like so many other things, these same results will hold in any "set-like" category such as pointed sets or modules over a semi-simple ring in which (almost) all monomorphisms and all epimorphisms split and hence these properties (of being mono or epi) are preserved by functors. In [8], example on p. 89, Linton shows that for $\mathbf{T} = (T, \eta, \mu)$, any triple on $\mathscr{S}$, T preserves all monos (even those with empty domain).

(6.2) **Proposition.** *Let $\mathscr{E}$ be a small category. Then there is a cardial m depending only on $\mathscr{E}$ such that the functor* $\lim: (\mathscr{E}, \mathscr{S}) \to \mathscr{S}$ *has rank $\leqq m$.*

Proof. Suppose $\{E_p: \mathscr{E} \to \mathscr{S} \mid p < m\}$ is an ascending chain of functors. Let E be the union ($=$colimit) of the E_p. A point $x \in \lim E$ is represented by a sequence (x_e), $e \in \mathscr{E}$ where $x_e \in Ee$. Since $Ee = \bigcup E_p e$, for each $e \in \mathscr{E}$, $\exists\, p_e < m$ such that $x_e \in E_{p_e} e$. Then $\{p_e\}$ is a set of n ordinals each $< m$, and since m is n-complete, $p = \sup p_e < m$ also. Then $x_e \in E_p e$ for all $e \in \mathscr{E}$ and so $x \in \bigcup \lim E_p$. This shows that the natural map.

$$\operatorname{colim} \lim E_p \to \lim \operatorname{colim} E_p$$

is onto and since it is clearly 1-1, the result follows.

(6.3) **Definitions.** Let End $^*\mathscr{S}$ denote the full category of these endofunctors of $\mathscr{S}$ which have rank $\leqq m$ for some m. Also let Trip $^*\mathscr{S}$ denote the full category of triples whose underlying functor lies in End $^*\mathscr{S}$ and

$$U^*: \text{Trip } {}^*\mathscr{S} \to \text{End } {}^*\mathscr{S}$$

denote the underlying functor functor.

(6.4) **Theorem.** End $^*\mathscr{S}$ *is complete and cocomplete. Moreover the inclusion* End $^*\mathscr{S} \to$ End $\mathscr{S}$ *preserves limits and colimits.*

Proof. To see that it is cocomplete, let $E: \mathscr{E} \to \mathrm{End}^*\mathscr{S}$ be a functor with $\mathscr{E}$ small. Choose m sufficiently large that every value of E has rank $\leqq m$. Now letting e denote some object of $\mathscr{E}$, we have for any m-complete cardinal n and any $D: n \to \mathscr{X}$,

$$\operatorname*{colim}_{i\in n}(Ee\cdot D) \simeq Ee(\operatorname*{colim}_{i\in n} D).$$

Thus, letting e range over $\mathscr{E}$,

$$\begin{aligned}\operatorname*{colim}_{e\in\mathscr{E}}(\operatorname*{colim}_{i\in n}(Ee\cdot D)) &\simeq \operatorname*{colim}_{e\in\mathscr{E}}(Ee(\operatorname*{colim}_{i\in n} D)) \\ &\simeq (\operatorname*{colim}_{e\in\mathscr{E}} Ee)(\operatorname*{colim}_{i\in n} D),\end{aligned}$$

the latter being an isomorphism because colimits in functor categories are computed "pointwise". Now since colimits commute with colimits,

$$\begin{aligned}\operatorname*{colim}_{e\in\mathscr{E}}(\operatorname*{colim}_{i\in n}(Ee\cdot D)) &\simeq \operatorname*{colim}_{i\in n}(\operatorname*{colim}_{e\in\mathscr{E}}(Ee\cdot D)) \\ &\simeq \operatorname*{colim}_{i\in n}((\operatorname*{colim}_{e\in\mathscr{E}} Ee)\cdot D),\end{aligned}$$

where again the latter isomorphism is an instance of a pointwise computation. Putting these together, we get

$$(\operatorname*{colim}_{e\in\mathscr{E}} Ee)(\operatorname*{colim}_{i\in n} D) \simeq \operatorname*{colim}_{i\in n}((\operatorname*{colim}_{e\in E} Ee)\cdot D)$$

which, since D is arbitrary, implies that $\operatorname*{colim}_{e\in\mathscr{E}} Ee$ has rank $\leqq m$. The same argument works for $\lim_{e\in\mathscr{E}} Ee$ provided m is also taken sufficiently large that limits over $\mathscr{E}$ have rank $\leqq m$.

(6.5) **Proposition.** *U^* has a left adjoint F^* and is tripleable.*

Proof. A functor $R: \mathscr{S} \to \mathscr{S}$ which has rank m is n-small for some n (e.g. the first $n > m$) and by (5.5) generates a free triple $\mathbf{T} = (T, \eta, \mu)$. If $D: n \to \mathscr{S}$ is an n-sequence of subobjects of X, then $TD: n \to \mathscr{S}$ is an n-sequence of subobjects of TX (this innocent-seeming point that T preserves monos is essential and is the only reason for restricting attention to $\mathscr{S}$ or "set-like" categories). Thus

$$\operatorname{colim} RTD \to R \operatorname{colim} TD$$

is an isomorphism. Now, for $i\in n$ let $(TX_i,\ x_i: RTX_i \to TX_i)$ denote the free algebra generated by D_i and let

$$x: R \operatorname{colim} TX_i \to \operatorname{colim} TX_i$$

be the inverse of the above isomorphism composed with $\operatorname{colim} x_i$. Then $(\operatorname{colim} TX_i, x)$ is easily seen to be the colimit of the diagram $\{(TX_i, x_i)\}$ in $(R:\mathscr{X})$. Applying the underlying functor, we have $\operatorname{colim} TX_i \simeq T \operatorname{colim} X_i$. Thus the free triple on a functor in $\mathrm{End}^*\mathscr{S}$ is in $\mathrm{Trip}^*\mathscr{S}$. The remainder of the proof that U^* is tripleable is quite easy and is left to the reader.

(6.6) **Theorem.** Trip $^*\mathscr{S}$ *is bicomplete.*

Proof. That it is complete follows immediately from the fact that it is tripleable over a complete category. To show that it is cocomplete, there are two approaches. One is to use (3.2) with $\mathscr{M}$ = all monomorphisms and $n = \omega$

(which requires showing that End $*\mathscr{X}$ is locally small). The other is a direct argument as follows. Given a functor $E: \mathscr{E} \to \mathrm{Trip}\ *\mathscr{X}$, let $R = \mathrm{colim}\ U^* E$ in End $*\mathscr{X}$. Then if $\mathbf{T}$ is the free triple generated by R, $(R:\mathscr{X}) = \mathscr{X}^{\mathbf{T}}$. Now for $e \in \mathscr{E}$, Ee is a triple, and there is induced a natural transformation $U^* Ee\ R$ which gives a natural transformation λ: $U^* Ee \to U^* \mathbf{T}$. Write $Ee = (T_e, \eta_e, \mu_e)$. Then the full subcategory of $\mathscr{S}^{\mathbf{T}}$ consisting of all (X, x) such that $(X, x \cdot \lambda X)$ is an Ee algebra is easily seen to be a Birkhoff subcategory of $\mathscr{X}^{\mathbf{T}}$. The algebras which have this property for all $e \in \mathscr{E}$ are still a Birkhoff subcategory, hence tripleable (by [9], 3.6), and by (5.2), (5.3), and (5.4), the resultant triple is the colimit of E.

(6.7) Next we give some examples and results to show that some such restriction as rank is necessary to guarantee the existence of free triples.

If one supposes the existence of a proper class of strongly measurable cardinals, it is possible to show that the functor which assigns to a set the set of ω measures on it lacks rank but the free triple exists (e.g. by (5.5) with $n = \omega$). However no assumption about the existence of such measurable cardinals is known to be consistent. For example Kiesler has pointed out to me that even if there is one measurable cardinal it is consistent to suppose that it is the only one.

(6.8) *Example.* If $R: \mathscr{S} \to \mathscr{S}$ is the covariant power set functor, it does not generate a free triple.

Proof. In fact, by (5.12) $(R:\mathscr{S})$ does not even have an initial object (free object on the empty set). Then by (5.10) the free triple does not exist.

(6.9) Let $\mathbf{T}_1 = (T_1, \eta_1, \mu_1)$ be the triple on $\mathscr{S}$ in which T_1 is the covariant power set functor, $\eta_1 X: X \to T_1 X$ takes $p \in X$ to $\{p\}$, and $\mu_1 X: T_1 T_1 X \to T_1 X$ is union, i.e. $\mu_1 X(\{A_i | i \in I\}) = \bigcup \{A_i | i \in I\}$ for subsets $A_i \subset X$. This is easily seen to be a triple. The algebras are the category whose objects are complete sup lattices and complete sup preserving maps. (Warning: of course a complete sup lattice has infs, but maps needn't preserve them.) Let $\mathbf{T}_2 = (T_2, \eta_2, \mu_2)$ be the triple such that $T_2 X = \mathbf{N} \times X$ where $\mathbf{N}$ is the monoid of natural numbers and η_2 and μ_2 come from the monoid structure of $\mathbf{N}$. Then the $\mathbf{T}_2$-algebras consist of $\mathbf{N}$-sets, i.e. sets-with-endofunction and morphisms which preserve the endofunction.

(6.10) **Proposition.** *In Trip $\mathscr{S}$, $\mathbf{T}_1$ and $\mathbf{T}_2$ do not have a coproduct.*

Proof. Suppose that $\mathbf{T}$ were the coproduct. Then $\mathbf{T}$ would have the universal property that given two maps $\mathbf{T}_1 \to \mathbf{S}$, $\mathbf{T}_2 \to \mathbf{S}$ for some triple, there would be induced $\mathbf{T} \to \mathbf{S}$. In view of (5.3), this is equivalent to asserting that $\mathscr{S}^{\mathbf{T}}$ is the product of $\mathscr{S}^{\mathbf{T}_1}$ and $\mathscr{S}^{\mathbf{T}_2}$ in the category of tripleable categories over $\mathscr{S}$. If we define a category $\mathscr{Y}$ whose objects are 3-tuples (X, x_1, x_2) such that $(X, x_1) \in \mathscr{S}^{\mathbf{T}_1}$ and $(X, x_2) \in \mathscr{S}^{\mathbf{T}_2}$ and whose morphisms preserve both structures, it is easily seen that $\mathscr{Y}$ is the product of $\mathscr{S}^{\mathbf{T}_1}$ and $\mathscr{S}^{\mathbf{T}_2}$ in the category of categories over $\mathscr{S}$. Moreover the underlying set functor $(X, x_1, x_2) \rightsquigarrow X$ is easily seen to satisfy the hypotheses of (5.8). Thus the naturally induced functor $\mathscr{S}^{\mathbf{T}} \to \mathscr{Y}$ would be an equivalence and $\mathscr{Y}$ would be tripleable.

Now over $\mathscr{S}$, a Birkhoff subcategory of tripleable category is tripleable (see [9], 3.6). Moreover, it is known that the category of complete boolean algebras is not tripleable over $\mathscr{S}$ (see [4]). Thus the proof is reduced to the following observation of Lawvere (unpublished).

(6.11) **Proposition.** *If $\mathscr{Y}$ is as above, then the category of complete boolean algebras is a Birkhoff subcategory of $\mathscr{Y}$.*

Proof. An object of $\mathscr{Y}$ may be viewed as a pair $(L, ')$ where L is a complete sup-lattice and $': L \to L$ is a set function. If we ask that these satisfy the following equations

$$x''=x; \quad x \vee x'=1; \quad 1' \vee x = x;$$
$$((x \vee y)' \vee z')' = (x' \vee z')' \vee (y' \vee z')';$$
$$(x' \vee y')' \vee z = ((x \vee z)' \vee (y \vee z)')'$$

for all $x, y, z \in L$, the result is a boolean algebra, necessarily complete. Moreover, morphisms which preserve $\vee$ and $'$ preserve the boolean operations. Since the category of all such algebras is equationally defined, it is a Birkhoff subcategory.

7. Algebras in Categories

(7.1) In this section we consider a condition on a category $\mathscr{X}$ that allows us to conclude that such things as groups in $\mathscr{X}$ or rings in $\mathscr{X}$ are tripleable over $\mathscr{X}$.

(7.2) A varietal theory is a category $\mathscr{T}\!h$ with a functor $\Phi: \mathscr{S} \to \mathscr{T}\!h$ which preserves coproducts and is an isomorphism on objects. (This is dual to what is usually called a theory.) It is said to have rank n if the objects $\{\Phi(m) \mid m<n\}$, together with all maps $\Phi(f)$, form a right adequate subcategory. This means that for any p, $\Phi(p)$ is the colimit of $\Phi(m)$ for maps $\Phi(f): \Phi(m) \to \Phi(p)$ with $m<n$. The category of $\mathscr{T}\!h$ algebras in $\mathscr{X}$, denoted $\mathscr{X}^{\mathscr{T}\!h}$, consists of all functors $X: \mathscr{T}\!h^{\mathrm{op}} \to \mathscr{X}$ such that $X(\Phi(n)) = X(\Phi(1))^n$. Equivalently, if X also denotes $X(\Phi(1))$, an algebra consists of an object X together with a map $X^p \to X$ for each $\Phi(1) \to \Phi(p)$ in $\mathscr{T}\!h$ such that corresponding to any commutative diagram

$$\begin{tikzcd} & \Phi(p_1) \arrow[dr] & \\ \Phi(1) \arrow[ur] \arrow[dr] & & \Phi(p) \arrow[dl] \\ & \Phi(p_2) & \end{tikzcd}$$

in $\mathscr{T}\!h$ the diagram

$$\begin{tikzcd} & X^{p_1} \arrow[dr] & \\ X^{p} \arrow[ur] \arrow[dr] & & X \arrow[dl] \\ & X^{p_2} & \end{tikzcd}$$

commutes. Here it should be added that the map $\Phi(p_i) \to \Phi(p)$ is determined by a p_i-fold of maps $1 \to \Phi(p)$, $i=1,2$. The corresponding p_i-fold of maps $X^p \to X$ then induces a map $X^p \to X^{p_i}$ and these are the maps used in the above diagram. All this depends on the fact that $\Phi(p)$ is the p-fold coproduct of $\Phi(1)$. If $\mathcal{T}\!h$ has rank n, then all this data need only be given for $p<n$. By a pre-algebra we will mean an object X together with $X^m \to X$ for each $\Phi(1) \to \Phi(m)$ in $\mathcal{T}\!h$ but not necessarily satisfying the commutativity condition. If we define $R: \mathcal{X} \to \mathcal{X}$ by

$$RX = \coprod_{p<n} \coprod_{\Phi(1) \to \Phi(p)} X^p,$$

then the following is clear.

(7.3) **Theorem.** *The category of $\mathcal{T}\!h$ prealgebras is the same as the category $(R:\mathcal{X})$.*

Of course, we have omitted saying what maps are but this should be clear from the theorem.

(7.4) **Theorem.** *Suppose $\mathcal{X}$ has small $\mathcal{M}$ factorizations and R is $(\mathcal{M}, n)$ small. Then $\mathcal{X}^{\mathcal{T}\!h} \to \mathcal{X}$ is tripleable.*

Proof. It is only necessary to show that the obvious inclusion $\mathcal{X}^{\mathcal{T}\!h} \to (R:\mathcal{X})$ has an adjoint, for by (5.5) the underlying functor $(R:\mathcal{X}) \to \mathcal{X}$ does. Now the inclusion obviously preserves limits, so the only thing needed is a solution set, given an object $(X,x) \in (R:\mathcal{X})$, construct ΛX as in (2.1) and consider all objects $(Z,z) \in \mathcal{X}^{\mathcal{T}\!h}$ such that $Z \in \Lambda X$. Now any map $(X,x) \to (Y,y)$ factors through an object $(Z,z) \in (R:\mathcal{X})$ with $Z \in \Lambda X$, so it is sufficient to show that $(Z,z) \in \mathcal{X}^{\mathcal{T}\!h}$. Now Z is given as a colimit of subobjects Z_m of Y for $m<n$. For any $\Phi(1) \to \Phi(p)$ the corresponding map $Z^p \to Z$ is given as a colimit of maps $Z_m^p \to Z_{m+1}$. Corresponding to any commutative diagram

$$\begin{tikzcd} & \Phi(p_1) \arrow[dr] & \\ \Phi(1) \arrow[ur] \arrow[dr] & & \Phi(p) \\ & \Phi(p_2) \arrow[ur] & \end{tikzcd}$$

in $\mathcal{T}\!h$ we will have a diagram

$$\begin{tikzcd} & Z_{m+1}^{p_1} \arrow[dr] & \\ Z_m^p \arrow[ur] \arrow[dr] & & Z_{m+2} \\ & Z_{m+1}^{p_2} & \arrow[l, from=2-3, to=3-2] \end{tikzcd}$$

which, since Z_{m+2} is a subobject of Y and $(Y,y) \in \mathcal{X}^{\mathcal{T}\!h}$, will commute. By taking colimits the result follows.

(7.5) **Proposition.** *A necessary and sufficient condition that the functor R above be $(\mathcal{M}, n)$-small is that each power function $X \to X^m$ be $(\mathcal{M}, n)$-small for $m<n$.*

Proof. Trivial, since coproducts commute with colimits.

(7.6) In categories $\mathscr{S}^{\mathbf{T}}$ it is easy to see that if $\mathbf{T}$ has rank p, then the underlying functor creates colimits of n sequences when n is p-complete. Then if n is $\max(m, p)$-complete, m^{th} power will be n-small and (7.4) applies. This gives another proof that the tensor product of two triples with rank exists, for the $\mathbf{T}_1$ algebras ($=$algebras for the theory of $\mathbf{T}_1$) in $\mathscr{S}^{\mathbf{T}_2}$ are precisely the $\mathbf{T}_1 \otimes \mathbf{T}_2$ algebras.

8. Another Cocompleteness Theorem

(8.1) Notice that (3.4) does not imply the well known result of Linton that every category tripleable over $\mathscr{S}$ is complete ([8], example on p. 89). In this section we present a different cocompleteness theorem which does imply that result.

(8.2) By a factorization system on $\mathscr{X}$ we mean a pair $(\mathscr{I}, \mathscr{P})$ of subcategories of $\mathscr{X}$ satisfying the following conditions.

1. Every isomorphism is in $\mathscr{I} \cap \mathscr{P}$. In particular $\mathscr{I}$ and $\mathscr{P}$ each contain all the objects of $\mathscr{X}$.

2. $\mathscr{X} = \mathscr{I}\mathscr{P}$; that is, every map factors as a map in $\mathscr{P}$ followed by a map in $\mathscr{I}$.

3. If $f: X_0 \to X_1 \in \mathscr{P}$ and $g: Y_0 \to Y_1 \in \mathscr{I}$, then $\mathrm{Hom}(f, g) = \mathrm{Hom}(X_1, Y_0)$. That is, for every commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{a} & Y_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_1 & \xrightarrow{b} & Y_1 \end{array}$$

there is a unique $c: X_1 \to Y_0$ making both triangles commute.

These factorization systems (invented by Isbell under the name "bicategory structure", see [5]) have been studied by Kennison [6] and Kelly (unpublished). If the class $\mathscr{I}$ consists of monomorphisms, then $(\mathscr{I}, \mathscr{P})$ will be called a right factorization system. This is known to be equivalent (in the presence of kernel pairs) to each of the following statements.

a) $fg \in \mathscr{P} \Rightarrow f \in \mathscr{P}$.

b) Every split epimorphism is in $\mathscr{P}$.

We say that $\mathscr{X}$ is $\mathscr{P}$-co-well powered if for any $X \in \mathscr{X}$ there is, up to isomorphism, only a set of $f \in \mathscr{P}$ whose domain is X. This can turn out to be a serious restriction if $\mathscr{P}$ is large.

(8.3) **Theorem.** *Suppose $(\mathscr{I}, \mathscr{P})$ is a right factorization system on $\mathscr{X}$ and $\mathscr{X}$ is $\mathscr{P}$-co-well powered. If $\mathbf{T}$ is a triple on $\mathscr{X}$ with $T(\mathscr{P}) \subset \mathscr{P}$, then $\mathscr{X}^{\mathbf{T}}$ is cocomplete.*

Proof. Exactly as in (3.3), it is sufficient to find a solution set for coequalizers. If

$$(W, w) \underset{d^1}{\overset{d^0}{\rightrightarrows}} (X, x) \xrightarrow{d} (Y, y)$$

is any diagram in $\mathscr{X}^{\mathbf{T}}$ with $d \cdot d^0 = d \cdot d^1$, factor the map d in $\mathscr{X}$ as $X \xrightarrow{f} Z \xrightarrow{g} Y$ with $f \in \mathscr{P}$, $g \in \mathscr{I}$. In the diagram

$$\begin{tikzcd} TX \arrow[r, "x"] \arrow[d, "Tf"'] & X \arrow[r, "f"] & Z \arrow[d, "g"] \\ TZ \arrow[r, "Tg"] & TY \arrow[r, "y"] & Z \end{tikzcd}$$

we have $Tf \in \mathscr{P}$ and $g \in \mathscr{I}$, so by (8.2.3) there is a unique z: $TZ \to Z$ such that $g \cdot z = y \cdot Tg$ and $z \cdot Tf = f \cdot x$. Thus (Z, z) is a prealgebra at least, and f and g are maps of prealgebras. We must now show that (Z, z) is an algebra. But since $(\mathscr{I}, \mathscr{P})$ is a right factorization system, g is a monomorphism, so that we can cancel it from

$$g \cdot z \cdot Tz = y \cdot Tg \cdot Tz = y \cdot Ty \cdot T^2 g = y \cdot \mu Y \cdot T^2 g = y \cdot Tg \cdot \mu Z = g \cdot z \cdot \mu Z$$

and conclude $z \cdot Tz = z \cdot \mu Z$. Similarly, $g \cdot z \cdot \eta Z = y \cdot Tg \cdot \eta Z = y \cdot \eta Y \cdot g = g$, so that $z \cdot \eta Z = Z$. Also $g \cdot f \cdot d^0 = g \cdot f \cdot d^1$, so that $f \cdot d^0 = f \cdot d^1$. Since $\mathscr{X}$ is $\mathscr{P}$-co-well-powered, a solution set for the coequalizer of d^0 and d^1 can be constructed to consist of all algebras (Z, z) whose underlying Z is in a representative set of P-morphisms with domain $\mathscr{X}$.

(8.4) *Remark.* We may observe in passing that the above proof contains two other statements which are of independent interest.

a) Under the conditions of the theorem, $(\mathscr{I}, \mathscr{P})$ lifts to a factorization system in $\mathscr{X}^{\mathbf{T}}$ consisting of maps in $\mathscr{X}^{\mathbf{T}}$ whose underlying $\mathscr{X}$-morphism are in $\mathscr{I}$ and $\mathscr{P}$ respectively.

b) If $(\mathscr{I}, \mathscr{P})$ is a right factorization system in a complete category $\mathscr{Y}$ which is $\mathscr{P}$-co-well powered, then $\mathscr{Y}$ has coequalizers.

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