THE CHU CONSTRUCTION: HISTORY OF AN IDEA

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In 1975, I began a sabbatical leave at the ETH in Zürich, with the idea of studying duality in categories in some depth. By this, I meant not such things as the duality between Boolean algebras and Stone spaces, nor between compact and discrete abelian groups, but rather self-dual categories such as complete semi-lattices, finite abelian groups, and locally compact abelian groups. Moreover, I was interested in the possibilities of having a category that was not only self dual but one that had an internal hom and for which the duality was implemented as the internal hom into a "dualizing object". This was already true for the complete semi-lattices, but not for finite abelian groups or locally compact abelian groups. The category of finite abelian groups has an internal hom, but lacks a dualizing object, while locally compact groups have a dualizing object, but not an internal hom that is defined everywhere. Although you could define an abelian group of continuous homomorphisms between locally compact abelian groups, there was no way of systematically putting a locally compact topology on the hom set that would lead to the desired properties.

The desired properites were what I subsequently called $*$-autonomy. I do not want to go into the technical details here (see [Barr, 1999], for example). Basically, a $*$-autonomoous category has an internal hom, denoted $\multimap$, a symmetric monoidal structure, , along with the usual coherent isomorphisms $\mathrm{Hom}(AB, C) \simeq \mathrm{Hom}(A, B \multimap C)$. In addition, there should be a "dualizing object", $\perp$, such that if we define $A^* = A \multimap \perp$, the subsequent canonical map $(A \multimap B) \longrightarrow (B^* \multimap A^*)$ should be an isomorphism.

There is a certain redundancy in this definition. For instance, the most efficient (and perhaps the most natural) way is to assume just $\multimap$ and $\perp$ and define $A^* = A \multimap \perp$ and then $AB = (A \multimap B^*)^*$, since the latter two are provably naturally isomorphic.

By the end of the year, I had in fact produced a moderate number of examples of $*$-autonomous categories. One of them was a full subcategory of topological abelian groups that included all the locally compact abelian (LCA) groups in such a way that the duality restricted to them was the well-known duality of LCA groups. As one would expect, the the circle group was the dualizing object. In addition, the category was complete and cocomplete. In fact, it was built, essentially, by completing the LCA groups.

Another example was a full subcategory of the category of locally convex topological vector spaces. In the process of studying that category, I read a number of books on that subject, for example [Schaefer, 1970] and [Pietsch, 1972]. Each one contained a certain construction of "pairs". A pair is a pairing of topological vector spaces (E, E') along with a bilinear map[1] $< -, - >: E \times E' \longrightarrow \mathbf{C}$, the complex numbers. This device was used

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[1] It was usually supposed that the pairing was non-singular. As a first approximation, I had to omit that hypothesis. Later, a way was found to reinstate if desired, at least in the familiar examples.

because of the large number of possible topologies on the dual space. Basically, you were to think of E and E' as being dual to each other with certain topologies.

As I was preparing an early draft of the subsequent monograph [Barr, 1979], I began to be concerned about the fact that I had constructed this elaborate theory and had only a bare handful of examples. Aside from the complete semi-lattices the other examples were constructed with some difficulty using quite *ad hoc* methods. I had noticed the pairs and wondered if there were a way of defining an internal hom among pairs. It turned out to be surprisingly easy. I began with the obvious interpretation that in a pair $\mathbf{E} = (E, E')$, E was the set of points and E' the set of functionals. (Of course, in the dual pair E' is the set of points and E the set of functionals.) So if $\mathbf{F} = (F, F')$ is another such object, the points of $\mathbf{E} \multimap \mathbf{F}$ should certainly be the set of maps between the pairs and a map was a continuous linear map $f : E \longrightarrow F$ such that for all $\phi \in F'$, $\phi.f \in E'$. Call this set $[\mathbf{E}, \mathbf{F}]$. Then we wanted

$$\mathbf{E} \multimap \mathbf{F} = ([\mathbf{E}, \mathbf{F}], ?) \qquad (*)$$

with ? an unknown to be determined. A similar guess was that the tensor product of two objects might be given by

$$(E, E')(F, F') = (EF, ??) \qquad (**)$$

where ?? is another unknown to be determined. But if we put these together with the fact that in any $*$-autonomous category,

$$\mathbf{EF} = (\mathbf{E} \multimap \mathbf{F}^*)^*$$

(or, equivalently, that $\mathbf{E} \multimap \mathbf{F} = (\mathbf{EF}^*)^*$), we can compute

$$\begin{aligned} \mathbf{EF} &= (\mathbf{E} \multimap \mathbf{F}^*)^* \\ &= ((E, E') \multimap (F', F))^* \\ &= ([(E, E'), (F', F)], ?)^* \\ &= (?, [(E, E'), (F', F)]) \end{aligned}$$

Comparing this with $(**)$ we see that

$$\mathbf{EF} = (EF, [(E, E'), (F', F)])$$

Similarly, we find that

$$\mathbf{E} \multimap \mathbf{F} = ([\mathbf{E}, \mathbf{F}], EF')$$

These formulas are suggestive, but they don't prove anything. But, amazingly, it all works!

Thus I ended up with a category whose objects were pairs $\mathbf{E} = (E, E')$ of vector spaces equipped with a pairing $EE' \longrightarrow \mathbf{C}$. A map from $\mathbf{E}$ to $\mathbf{F} = (F, F')$ is a pair of linear maps (f, f') in which $f : E \longrightarrow F$ and $f' : F' \longrightarrow E'$ (note the direction reversal) such that $< fv, w > = < v, f'w >$ whenever $v \in E$ and $w \in F'$. There is no topology assumed and no continuity on the linear maps.

At this point, it seemed clear that no special properties of vector spaces was being used and this construction was an instance of a general one. Let $\mathscr{V}$ be an autonomous category, meaning one which had a coherently associative, unitary, symmetric tensor product, $--$, an internal hom $- \multimap -$ with the usual adjunction

$$\mathrm{Hom}(AB, C) \simeq \mathrm{Hom}(A, B \multimap C)$$

The only additional assumption we need to make on $\mathscr{V}$ is that it have pullbacks. Fix an object of $\mathscr{V}$, which we will call $\perp$. Form the category of pairs $\mathbf{E} = (E, E')$ equipped with a morphism $EE' \longrightarrow \perp$. If $\mathbf{F} = (F, F')$ is another object, then a morphism $\mathbf{E} \longrightarrow \mathbf{F}$ is a pair (f, f') where $f : E \longrightarrow F$ and $f' : F' \longrightarrow E'$ such that the diagram

$$\begin{array}{ccc}
EF' & \xrightarrow{fF'} & FF' \\
{\scriptstyle Ef'}\downarrow & & \downarrow{\scriptstyle <-,->} \\
EE' & \xrightarrow[<-,->]{} & \perp
\end{array}$$

commutes. This condition can be interpreted as saying that a map is a pair consisting $E \longrightarrow F$ and a map $F' \longrightarrow E'$ that induce the same map $EF' \longrightarrow \perp$ which can made into a pullback diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathbf{E}, \mathbf{F}) & \longrightarrow & \mathrm{Hom}(E, F) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(F', E') & \longrightarrow & \mathrm{Hom}(EF', \perp)
\end{array}$$

This definition gives a category that is called $\mathrm{Chu}(\mathscr{V}, \perp)$. Using the pullbacks, we can internalize the above definition, by defining $[\mathbf{E}, \mathbf{F}]$ so that

$$\begin{array}{ccc}
[\mathbf{E}, \mathbf{F}] & \longrightarrow & E \multimap F \\
\downarrow & & \downarrow \\
F' \multimap E' & \longrightarrow & EF' \multimap \perp
\end{array}$$

is a pullback. Then define

$$\mathbf{E} \multimap \mathbf{F} = ([\mathbf{E}, \mathbf{F}], EF')$$

The pairing is either of the two equal maps

$$[\mathbf{E}, \mathbf{F}]EF' \longrightarrow (E \multimap F)EF' \longrightarrow FF' \longrightarrow \perp$$

or

$$[\mathbf{E},\mathbf{F}]EF' \longrightarrow (F' \multimap E')EF' \longrightarrow E'E \longrightarrow \perp$$

It seemed clear that this gave a $*$-autonomous category, but there were a number of unpleasant details to be verified. Since my student, Po-Hsaing Chu needed a master's project, so I asked him to verify them, which he did [Chu, 1978, 1979]. I now had expanded from six to infinity the repertory of known $*$-autonomous categories. At this point, the construction, and in fact the whole idea of $*$-autonomous categories, was more-or-less forgotten, not least by me.

One interesting thing is that I never looked to see if there was any connection between the $*$-autonomous categories that I had carefully constructed and related $*$-autonomous categories. For example, I had constructed a $*$-autonomous category that was a full subcategory of topological abelian groups and made no attempt to relate it to $\mathrm{Chu}(\mathbf{Ab}, K)$, where K is the circle group. This was a mistake.

Eight years later, in 1987, I became aware of Jean-Yves Girard's linear logic. I do not recall how I became aware of it, but at the meeting at The University of Colorado that summer, I knew enough to bring Girard a copy of [Barr, 1979]. Since $*$-autonomous categories give models of linear logic and since the Chu construction gives easy examples of $*$-autonomous categories, this was reasonable, but I must have known that before going to Boulder. I think Robert Seely must have lectured on this in our seminar that spring. At any rate, knowledge of $*$-autonomous categories in general and the Chu construction in particular had penetrated the linear logic and theoretical computer science community.

1. Further work

Vaughan Pratt is convinced that Chu categories are an important adjunct to the study of a number of questions in theoretical computer science. He has written a series of papers expanding on this perception. A small selection of them is [Pratt, 1993a, 1993b, 1995].

In the meantime, I became interested in the connection between the categories constructed in my 1979 monograph and the Chu construction. In that paper, I had constructed a small number of $*$-autonomous categories. In all but one case I had begun with an incomplete, but $*$-autonomous category that was embedded as a full subcategory of a complete autonomous category that lacked a duality. I then showed how there was a kind of completion of the $*$-autonomous subcategory inside the category of topological (or uniform space) objects of the complete autonomous category that was $*$-autonomous. The construction was fairly difficult and it was not easy to understand which objects are in the subcategory. Eventually, it occurred to me that if $\mathscr{V}$ is the category of discrete objects and $\perp$ is the dualizing object, then $\mathrm{Chu}(\mathscr{V}, \perp)$ is a candidate for the $*$-autonomous category that I had constructed.

For example, in the category of abelian groups, there is a full $*$-autonomous subcategory of the category of topological abelian groups that includes all the locally compact groups. Let $\mathrm{chu}(\mathbf{Ab}, K)$ denote the full subcategory of $\mathrm{Chu}(\mathbf{Ab}, K)$ consisting of those pairs (A, A') for which the pairing is non-singular: for any $a \in A$, if $a \neq 0$, then there is an $a' \in A'$ for which $< a, a' > \neq 0$ and vice versa. The definitions of the internal hom and tensor have to be modified, but the modifications are essentially obvious. It is now known that $\mathrm{chu}(\mathbf{Ab}, K)$ is equivalent to a full subcategory of topological abelian groups that includes all the locally compact group. Is this the same category as constructed "by hand"? I don't know and I have no idea how to even approach the question (and, to be sure, there is not any reason to know the answer).

2. Mackey

I had imagined that the source of the idea of pairs of vector spaces was Grothendieck. It looked like the kind of construction he would have done, although he might have made a category of it. But in fact, it goes back further than that. The source seems to have been George Mackey's thesis, published as [Mackey, 1945]. In it he defines pairs as above, except that his pairs are not assumed to be non-singular in the second variable. So the dual of (E, F) is $(F, E/E_0)$ where E_0 consists of those vectors that are annihilated of every functional in F.

One interesting point about Mackey's paper is that it looks as if he was proposing that the set of functionals be considered as a replacement for the topology, not just an adjunct to it. After all, the topology on a set is a equivalent to a set of functions from the set into the Sierpinski space 2. Not any set of functions will do; it must be a $\bigcup$-$\bigcap$-closed sublattice of the set of all functions. In the case of the vector spaces, the condition is even simpler; it must be a linear subspace of the space of all functionals. Note that I am not suggesting that this is the same thing as a topological vector space. It is not, but it might be a subject of more interest, properly studied on its own, than topological vector spaces. At all events, if this was Mackey's intention, it has not been followed up. Instead, the various authors have used these pairs as a tool for the study of duality, rather than as the objects of study. And the dual is, in turn, used as a tool in the study of the deeper properties of these spaces.

3. A rational—but incorrect—reconstruction

Here is how the Chu construction *should* have been discovered. That it wasn't just shows how sometimes mathematics is two steps forward, then one back.

Suppose you want to make a self-dual category that contains a given category $\mathscr{C}$. Assuming that $\mathscr{C}$ has a terminal object 1, the category $\mathscr{C}^{\mathrm{e}} = \mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ is obviously self-dual and embeds $\mathscr{C}$ as the objects of the form $(C, 1)$. If $\mathscr{C}$ should happen to be (symmetric) closed monoidal, then you might not expect $\mathscr{C}^{\mathrm{e}}$ to be a closed monoidal category, but it is, at least when $\mathscr{C}$ has finite products. The reason you might not expect it is that $\mathscr{C}^{\mathrm{op}}$ is not generally closed monoidal when $\mathscr{C}$ is. For instance, the category of finite dimensional coalgebras over a field is closed monoidal (in fact, cartesian closed), but the opposite category of algebras certainly is not. What is true (and it is crucial) is that when $\mathscr{C}$ is closed monoidal, $\mathscr{C}^{\mathrm{op}}$ is enriched over $\mathscr{C}$.

We begin with

$$\mathrm{Hom}((C,D),(C',D')) = \mathrm{Hom}(C,C') \times \mathrm{Hom}(D',D)$$

which is the definition of the homfunctor in $\mathscr{C}^e$. This leads one to try

$$(C,D) \multimap (C',D') = ((C \multimap C') \times (D' \multimap D), X) \qquad (*)$$

where $X = X(C,D,C',D')$ is to be determined. Similarly, you might guess that

$$(C,D)(C',D') = (CC', Y)$$

where $Y = Y(C,D,C',D')$ is to be determined. Naturally, you want more than just self dual and closed monoidal, you want $*$-autonomous, which implies that

$$\begin{aligned}(C,D) \multimap (C',D') &\simeq ((C,D)(C',D')^*)^* = ((C,D)(D',C'))^* \\ &= ((C,D)(D',C'), Y(C,D,D',C'))^* = (Y(C,D,D',C'), CD')\end{aligned}$$

Comparing this to $(*)$, we see that $X(C,D,C',D') = CD'$ and $Y(C,D,D',C') = (C \multimap C') \times (D' \multimap D)$. This leads directly to the formulas

$$(C,D) \multimap (C',D') = ((C \multimap C') \times (D' \multimap D), CD')$$

and

$$(C,D)(C',D') = (CC', (C \multimap D') \times (C' \multimap D))$$

Amazingly, this all works! In fact, the reader might have noticed that it is just $\mathrm{Chu}(\mathscr{C},1)$ since an object of the latter category is a pair (C,D), together with a morphism $CD \longrightarrow 1$, of which there is exactly one. This category could thus have been discovered with no motivation from Mackey's pairs. Could have been but wasn't.

Moreover, this leads directly to the Chu construction. If M is a commutative monoid object in any $*$-autonomoous category, an M-action is an object A, together with an morphism $MA \longrightarrow A$ that satisfying the usual associative and unitary identities. It is not hard to prove that the category of M-actions is again a $*$-autonomous category. Any object $\bot$ of $\mathscr{C}$ gives a monoid object $M = (\top, \bot)$ of $\mathscr{C}^e$. Since the unit for the tensor product of $\mathscr{C}^e$ is $(\top, 1)$ and the unit map for M is $(\mathrm{id}, !)$, where $! : \bot \longrightarrow 1$ is the unique arrow. The multiplication is

$$(\top,\bot)(\top,\bot) = (\top\top, (\top \multimap \bot) \times (\top \multimap \bot)) = (\top, \bot \times \bot) \xrightarrow{(\mathrm{id},\Delta)} (\top,\bot)$$

It then turns out that the category of M-actions is precisely $\mathrm{Chu}(\mathscr{C},\bot)$.

Moreover, this constructions works with certain changes even when is not symmetric. In that case, two internal homs are needed in $\mathscr{C}$, one to be right adjoint to $C-$ and the other to be right adjoint to $-C$. In that case, M need not be commutative (in fact, the concept is undefinable) and instead of left actions of M, we must use the category of two-sided actions. This is studied in detail in [Barr, 1995], where the $*$-autonomous structure of $\mathscr{C}^e$ was first described. The simplification that this construction offers is more important in the asymmetric case than in the symmetric one.

References

M. Barr (1979), $*$-Autonomous categories. Lecture Notes in Mathematics **752** Springer-Verlag, Berlin, Heidelberg, New York.

M. Barr (1995), Non-symmetric $*$-autonomous categories. Theoretical Computer Science, **139**, 115-130.

M. Barr (1996), The Chu construction. Theory and Applications of Categories, **2**, 17–35.

M. Barr (1999), $*$-Autonomous categories: once more around the track. Theory and Applications of Categories, **6**, 5–24.

M. Barr (2000), On $*$-autonomous categories of topological vector spaces. Cahiers de Topologie et Géométrie Différentielle Catégorique, **39** 243–254.

P.-H. Chu (1978), *Constructing $*$-autonomous categories.* M. Sc. thesis, McGill University.

P.-H. Chu (1979), *Constructing $*$-autonomous categories.* Appendix to [Barr, 1979].

G. Mackey (1945), *On infinite dimensional vector spaces.* Trans. Amer. Math. Soc. **57**, 155–207.

A. Pietsch, Nuclear Locally Convex Spaces. Springer-Verlag, 1972.

V.R. Pratt (1993a), *Linear Logic for Generalized Quantum Mechanics.* In Proceedings Workshop on Physics and Computation, Dallas, IEEE Computer Society.

V.R. Pratt (1993b), *The Second Calculus of Binary Relations.* In Proceedings of MFCS'93, Gdańsk, Poland, 142–155.

V.R. Pratt (1995), *The Stone Gamut: A Coordinatization of Mathematics.* In Proceedings of the conference Logic in Computer Science IEEE Computer Society.

H.H. Schaefer, Topological Vector Spaces. Third printing, Springer-Verlag, 1970.

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