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CLOSED CATEGORIES AND TOPOLOGICAL VECTOR SPACES

by Michael BARR *

INTRODUCTION

In the paper [1], henceforth referred to as DVS, we considered two duality theories on the category $\mathfrak{V}$ of topological vector spaces over a discrete field K. They were each described by a certain topology on the set of linear functionals. The first, the weak dual, led to a category of reflexive spaces (i.e. isomorphic to their second dual) which gave a closed monoidal category when the *hom* sets are topologized by pointwise convergence.

The second, strong duality, was based on uniform convergence on linearly compact (LC) subspaces. This led to a nicer duality theory (now the discrete spaces are reflexive) but we did not describe there any closed monoidal category based on that strong *hom*. In this paper we fill that gap.

It is clear that one cannot expect an internal *hom*-functor which behaves well on all spaces or even all reflexive ones. It is a consequence of the closed monoidal structure that the tensor product of two LC spaces must be LC (see Section 1). Such a product is totally bounded (in a suitable generalized sense which is, together with completeness, equivalent to linear compactness). This fact suggests looking at a subcategory of spaces which satisfy same completeness condition. If the category is to have a self-duality theory, a dual condition is imposed as well. When this is done the result is indeed a closed monoidal category in which every object is reflexive. The set of morphisms between two spaces is topologized by a topology finer (possibly) than LC convergence to provide the internal *hom*.

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The dual, moreover, has the strong topology.

One word about notation. For the most part we adhere to that of DVS. However there is one significant change. Owing to a lack of enough kinds of brackets and hieroglyphs, we take advantage of the fact that this paper concerns exclusively itself with the strong *hom* to use $(-,-)$ and $(-)^*$ to refer to the strong *hom* and strong dual, respectively. Similarly, a space V is called reflexive if it is isomorphic with the strong second dual, here denoted V^{**}.

1. Preliminaries.

The subject of this paper is the category $\mathfrak{V}$ as described in DVS, equipped with the strong internal *hom* functor defined there. Specifically, if U and V are spaces, we let (U,V) denote the set of continuous linear maps $U \to V$ topologized by uniform convergence on LC subspaces. A basic open subspace is

$$\{f \mid f(U_0) \subset V_0\}$$

where U_0 is an LC subspace in U and V_0 is an open subspace of V.

We let $V^* = (V,K)$.

PROPOSITION 1.1. *Let U be a fixed space. The functor $(U,-)$ commutes with projective limits and has an adjoint $-\otimes U$.*

PROOF. It certainly does at the underlying set level so only the topology is in question. Let

$$V = \prod_{\omega\in\Omega} V_\omega \quad \text{and} \quad \pi_\omega : V \to V_\omega$$

be the projection. A basic open set in (U,V) is $\{f \mid f(U_0) \subset V_0\}$ where U_0 is an LC subspace of U and V_0 an open subspace of V. We can suppose $V_0 = \prod W_\omega$ where W_ω is open in V_ω and is V_ω for all but a finite set Ω_0 of indices. Then

$$f(U_0) \subset V_0 \quad \text{iff} \quad \pi_\omega f(U_0) \subset W_\omega \quad \text{for } \omega\in\Omega_0 .$$

There is no restriction on the other coordinates of f. Then $\{f \mid f(U_0) \subset V_0\}$

corresponds to the set

$$\Big(\prod_{\omega\in\Omega_0}\{f\mid f(U_0)\subset W_\omega\}\Big)\times\Big(\prod_{\omega\notin\Omega_0}\{f:U\to V_\omega\}\Big).$$

The argument for equalizers is easy and is omitted. In fact, when V is a subspace of W, (U, V) has the subspace topology in (U, W).

Now the existence of the adjoint follows from the special adjoint theorem (cf. DVS, 1.2-1.4).

This *hom* is not symmetric and is not closed monoidal. A map from $U\otimes V$ to W can be easily seen to be a bilinear map $U\times V\to W$ which is, for each $u\in U$, continuous on V, and for each LC subspace $V_0\subset V$, an equicontinuous family on U. From this it is easy to see the assymmetry. To see that we don't even get a closed monoidal category, we observe that that would imply that the equivalences between maps

$$U\otimes V\to W \quad\text{and}\quad U\to(V, W)$$

arise from a natural isomorphism $(U\otimes V, W)\approx(U,(V, W))$ (see [2], II.3). Suppose X and Y are infinite sets, $U=K^X$ and $V=K^Y$. Then assuming that the above isomorphism held, we would have $(U\otimes V)^*\approx(U, V^*)$, which can be directly calculated to be $K^{X\times Y}$. Let W be the subspace of $K^{X\times Y}$, proper when X and Y are infinite, whose elements are those of the algebraic tensor product $K^X\otimes K^Y$. Then on purely algebraic grounds there is a map $K^X\to(K^Y, W)$ which is continuous when W is given the subspace topology. This clearly has no continuous extension $K^{X\times Y}\to W$.

There is, however, an alternative. To explain it we require a definition. A space V is called (linearly) *totally bounded* if for every open subspace U, there is a finite number of vectors $v_1,\dots,v_n$ which, together with U, span V. Equivalently, every discrete quotient is finite dimensional. The obvious analogy of this definition with the usual one is strengthened by the following proposition whose proof is quite easy and is omitted.

PROPOSITION 1.2. *The space V is LC iff it is complete and totally bounded.*

PROPOSITION 1.3. *Let U be totally bounded and V be LC. Then $U \otimes V$ is totally bounded.*

PROOF. Let W be discrete and $f: U \otimes V \to W$. Then there corresponds a $g: U \to (V, W)$ and the latter space is discrete. Hence the image is generated by the images of a finite number of elements, say $g(u_1), \dots, g(u_n)$. Each of these in turn defines a map $V \to W$ whose image is a finite dimensional subspace of W, and thus the whole image $f(U \otimes V)$ is a finite dimensional subspace of W.

2. ζ- and ζ^*-spaces.

We say that a space U is a *ζ-space* if every closed totally bounded subspace is LC (or, equivalently, complete). The full subcategory of ζ-spaces is denoted $\zeta\mathfrak{V}$.

PROPOSITION 2.1. *The space U is a ζ-space iff every map to U from a dense subspace of a LC space to U can be extended to the whole space.*

PROOF. Let $V_0 \to U$ be given where V_0 is a dense subspace of the LC space V. The image $U_0 \subset U$ is totally bounded and hence has an LC closure which we may as well suppose is U. Now U is LC, hence is a power of K, which means it is a complete uniform space and thus the map extends, since a continuous linear function is uniformly continuous.

The converse is trivial and so the proposition follows.

We say that U is a *ζ^*-space* provided U^* is a ζ-space. Since both discrete and LC spaces are ζ-spaces, they are each ζ^*-spaces.

PROPOSITION 2.2. *Let U be a ζ-space. Then U^* is a ζ^*-space; i.e., U^{**} is a ζ-space.*

PROOF. Let $V_0 \to V$ be a dense inclusion with V an LC space. If $V_0 \to U^{**}$ is given, we have, using the fact that U is a ζ-space, the commutative diagram

$$\begin{array}{ccc} V_0 & \longrightarrow & V \\ \downarrow & & \downarrow \\ U^{**} & \longrightarrow & U \end{array}$$

Double dualization gives us the required $V \approx V^{**} \to U^{**}$.

If U is a space, it has a uniform completion $U^{\sim}$, and we let ζU denote the intersection of all the ζ-subspaces of $U^{\sim}$ which contain U. Evidently, U is a dense subspace of ζU.

For any subspace $V \subset U^{\sim}$, let $\zeta_1 V$ be the union of the closures of the totally bounded subspaces of V. For an ordinal μ, let

$$\zeta_{\mu+1} V = \zeta_1(\zeta_\mu V)$$

and, for a limit ordinal μ, let

$$\zeta_\mu V = \cup \{ \zeta_\nu V \mid \nu < \mu \}.$$

Let $\zeta_\infty V$ be the union of all the $\zeta_\mu V$.

PROPOSITION 2.3. $\zeta U = \zeta_\infty U$.

PROOF. It is clear that $\zeta_\infty U$ is closed under the operation of ζ_1 and hence is a ζ-space containing U. Thus $\zeta U \subset \zeta_\infty U$, while the reverse inclusion is obvious.

PROPOSITION 2.4. *The construction* $U \mapsto \zeta U$ *is a functor which, together with the inclusion* $U \hookrightarrow \zeta U$, *determines a left adjoint to the inclusion of* $\zeta\mathfrak{V} \to \mathfrak{V}$.

PROOF. Let $f: U \to V$ be a map. Since it is uniformly continuous, there is induced a map $f^{\sim}: U^{\sim} \to V^{\sim}$. It is clearly sufficient to show that $f^{\sim}(\zeta U) \subset \zeta V$. But since the continuous image of a totally bounded space is totally bounded, we see that whenever W is a subspace of $U^{\sim}$ with $f(W) \subset \zeta V$, and $W_0 \subset W$ is totally bounded, $f(cl(W_0)) \subset \zeta V$ as well. From this it follows that $f(\zeta_1 W) \subset \zeta V$, and so we see by induction $f(\zeta W) \subset \zeta V$. Applying this to U, we see that $f(\zeta U) \subset \zeta V$.

PROPOSITION 2.5. *Let* U *be reflexive. Then so is* ζU.

PROOF. By 2.2, $(\zeta U)^{**}$ is a ζ-space, so that

$$U \longrightarrow U^{**} \longrightarrow (\zeta U)^{**}$$

can be extended to $\zeta U \to (\zeta U)^{**}$. The diagram

$$\begin{array}{ccccc} U & \longrightarrow & U^{**} & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ \zeta U & \longrightarrow & (\zeta U)^{**} & \longrightarrow & \zeta U \end{array}$$

commutes, the first square by construction, the second by naturality. Since U is dense in ζU and the top map is the identity, so is the bottom one. Thus ζU is reflexive.

PROPOSITION 2.6. *Let U be a reflexive ζ^*-space. Then so is ζU.*

PROOF. Let $V_0 \to V$ be a dense inclusion with V an LC space. Given a map

$$V_0 \longrightarrow (\zeta U)^* \longrightarrow U^*$$

this extends, since U^* is a ζ-space, to a map $V \to U^*$. This gives us

$$U \approx U^{**} \longrightarrow V^*$$

and since V^* is discrete, hence complete, this extends to $\zeta U \to V^*$ whose dual is a map

$$V \approx V^{**} \longrightarrow (\zeta U)^* .$$

The outer square and lower triangle of

$$\begin{array}{ccc} V_0 & \longrightarrow & V \\ \downarrow & \swarrow & \downarrow \\ (\zeta U)^* & \longrightarrow & U^* \end{array}$$

commute, and since the lower map is 1-1 and onto, so does the upper triangle.

If X is a topological space and X_1 and X_2 are subsets of X, say that X_1 *is closed in* X_2 if $X_1 \cap X_2$ is a closed subset of X_2. Equivalently there is a closed subset

$$X'_1 \subset X \quad \text{such that} \quad X_1 \cap X_2 = X'_1 \cap X_2 .$$

We use without proof the obvious assertion that X_1 closed in X_2 and X_3 closed in X_4 implies that $X_1 \cap X_2$ is closed in $X_3 \cap X_4$.

PROPOSITION 2.7. *Let $\{U_\omega\}$ be a family of discrete spaces and U be a subspace of ΠU_ω. Then U is a ζ-space iff for every choice of a collection* *of finite dimensional subspaces* $V_\omega \subset U_\omega$, *U is closed in* ΠV_ω.

PROOF. Suppose the latter condition is satisfied and U_0 is a closed totally bounded subspace of U. Then the image of

$$U_0 \longrightarrow U \longrightarrow \Pi U_\omega \longrightarrow U_\omega$$

is a totally bounded, hence finite dimensional subspace $V_\omega \subset U_\omega$. Evidently $U_0 \subset \Pi V_\omega$, and since U_0 is closed in U, it is closed in $U \cap \Pi V_\omega$, which is closed in ΠV_ω. Thus U_0 is LC. Conversely, if U is a ζ-space, then for any collection $\{ V_\omega \}$ of finite dimensional subspaces, $U \cap \Pi V_\omega$ is a closed totally bounded subspace of U and hence is LC, hence closed in ΠV_ω.

3. The internal *hom*.

If U and V are spaces, we recall that (U, V) denotes the set of continuous linear mappings $U \to V$ topologized by taking as a base of open subspaces $\{ f \mid f(U_0) \subset V_0 \}$ where U_0 is an LC subspace of U and V_0 an open subspace of V. An equivalent description is that (U, V) is topologized as a subspace of $\Pi(U_\psi, V/V_\omega)$ where U_ψ ranges over the LC subspaces of U and V_ω over the open subspaces of V. We may consider that V/V_ω range over the discrete quotients of V. Each factor is given the discrete topology. From that description and the duality between discrete and LC spaces, the following becomes a formal exercise.

PROPOSITION 3.1. *Let U and V be reflexive spaces. Then the equivalence between maps $U \to V$ and $V^* \to U^*$ underlies an isomorphism*

$$(U, V) \approx (V^*, U^*).$$

LEMMA 3.2. *Suppose U is a reflexive ζ^*-space and V a reflexive ζ-space. Then (U, V) is a ζ-space.*

PROOF. Let $\{ U_\psi \}$ and $\{ V_\omega \}$ range over the LC and open subspaces, respectively, of U and V. A finite dimensional subspace of $(U_\psi, V/V_\omega)$ is spanned by a finite number of maps, each of which has a finite dimensional range. Thus altogether it is contained in a subspace of the form

$$(U_\psi/U_{\psi\omega},\ V_{\psi\omega}/V_\omega)$$

where $U_{\psi\omega}$ is a cofinite dimensional subspace of U_ψ and $V_{\psi\omega}/V_\omega$ is a finite dimensional subspace of V/V_ω. To apply 2.7 it is sufficient to consider families of finite dimensional subspaces of the factors. So let us suppose that for all pairs ω, ψ of indices a cofinite dimensional quotient $U_\psi/U_{\psi\omega}$ and a finite dimensional subspace $V_{\psi\omega}/V_\omega$ have been chosen. Then for each ψ, V is closed in $\prod_\omega V_{\psi\omega}/V_\omega$ and so (U_ψ, V) is closed in

$$(U_\psi, \Pi V_{\psi\omega}/V_\omega) \approx \Pi(U_\psi, V_{\psi\omega}/V_\omega).$$

It follows that $\prod_\psi(U_\psi, V)$ is closed in $\prod_{\psi,\omega}(U_\psi, V_{\psi\omega}/V_\omega)$. Using 3.1, we have similarly that, for each ψ, $(U, V/V_\omega)$ is closed in $\prod_\psi(U_\psi/U_{\psi\omega}, V/V_\omega)$ and so $\prod_\omega(U, V/V_\omega)$ is closed in $\prod_{\psi,\omega}(U_\psi/U_{\psi\omega}, V/V_\omega)$. Thus

$$\prod_\psi(U_\psi, V)\cap\prod_\omega(U, V/V_\omega)$$

is closed in

$$\prod_{\psi,\omega}((U_\psi/U_{\psi\omega}, V/V_\omega)\cap(U_\psi, V_{\psi\omega}/V_\omega)).$$

A collection of maps $U_\psi \to V$ is the same as a map $\Sigma U_\psi \to V$, and similarly a collection of maps $U \to V/V_\omega$ is equivalent to one $U \to \Pi V/V_\omega$. Now a map in both $\prod_\psi(U_\psi, V)$ and $\prod_\omega(U, V/V_\omega)$ corresponds to a commutative square

$$\begin{array}{ccc} \Sigma U_\psi & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & \Pi V/V_\omega \end{array}$$

The top map is onto and the lower a subspace inclusion and hence there is a fill-in on the diagonal. Thus the intersection is exactly $(U, V)^{\cdot}$. Clearly any map $U_\psi/U_{\psi\omega} \to V_{\psi\omega}/V_\omega$ belongs to both

$$\Pi(U_\psi/U_{\psi\omega}, V/V_\omega) \ \text{ and } \ \Pi(U_\psi, V_{\psi\omega}/V_\omega)$$

and hence to their intersection. Thus (U, V) is closed in

$$\Pi(U_\psi/U_{\psi\omega},\ V_{\psi\omega}/V_\omega)$$

and is a ζ-space.

4. The category $\mathfrak{R}$.

We let $\mathfrak{R}$ denote the full subcategory of $\mathfrak{B}$ whose objects are the reflexive ζ-ζ^*-spaces.

PROPOSITION 4.1. *The functor* $U \mapsto \delta U = (\zeta(U^*))^*$ *is right adjoint to the inclusion* $\mathfrak{R} \to \zeta\mathfrak{B}$. *For any* U, $\delta U \to U$ *is 1-1 and onto.*

PROOF. The map $U^* \to \zeta U^*$ is a dense inclusion so that

$$(\zeta U^*)^* \to U^{**} \to U$$

are each 1-1 and onto and so their composite is. If V is in $\mathfrak{R}$ and $V \to U$ is a map, we get

$$U^* \to V^*, \quad \zeta U^* \to V^*, \quad V \approx V^{**} \to (\zeta U^*)^*.$$

Now U^* is a reflexive (DVS 4.4) ζ-ζ^*-space (2.2), and so is ζU^* (2.6), and hence its dual is reflexive as well. If V is in $\mathfrak{R}$ and $V \to U$, we get

$$U^* \to V^*, \quad \zeta U^* \to V^* \text{ and then } V \approx V^{**} \to \delta U.$$

The other direction comes from

$$U^* \to \zeta U^*, \quad \delta U \to U^{**} \to U.$$

We now define, for U, V in $\mathfrak{R}$, $[U, V] = \delta(U, V)$ (cf. 3.2). It consists of the continuous maps $U \to V$ with a topology (possibly) finer than that of uniform convergence on LC subspaces. Note, of course, that $U^* = [U, K]$ is unchanged.

If U, V are in $\mathfrak{R}$ and W_0 is a totally bounded subset of (U, V), its closure W is LC. Then $\delta W \approx W$ is an LC subspace of $[U, V]$ and contains the same W_0. Thus W_0 is totally bounded in $[U, V]$. The converse being clear, we see that (U, V) and $[U, V]$ have the same totally bounded subspaces.

LEMMA 4.2. *Suppose* U, V *in* $\mathfrak{R}$. *Any totally bounded subspace of* $[U, V]$ *is equicontinuous.*

PROOF. Let $W \subset (U, V)$ be totally bounded. Corresponding to $W \to (U, V)$ we have $W \to (V^*, U^*)$ (3.1), and thus $W \otimes V^* \to U^*$. If V_0 is an open subspace of V, its annihilator $ann\, V_0$ in V^* is LC. This follows from the reflexivity of V and the definition of the topology V^{**}. Then $W \otimes (ann\, V_0)$ is totally bounded (1.2) and hence so is its image in U^*. The closure of that image is an LC subspace of U^* which we can call $ann\, U_0$, with U_0 open in U (same reason as above). From this it is clear that the image of $W \otimes U_0$ is in V_0, which means that W is equicontinuous.

COROLLARY 4.3. *Let U, V, W be in $\mathfrak{R}$. There is a 1-1 correspondence between maps $U \to [V, W]$ and $V \to [U, W]$.*

PROOF. A map $U \to [V, W]$ gives $U \to (V, W)$ and $U \otimes V \to W$. To any LC subspace of U corresponds an equicontinuous family $V \to W$. Certainly any $v \in V$ gives a continuous map $U \to W$ and thus, by the discussion in Section 1, we get $V \to (U, W)$ and then $V \to [U, W]$.

PROPOSITION 4.4. *Let U and V be in $\mathfrak{R}$. Then $[U, V] \approx [V^*, U^*]$ by the natural map.*

PROOF. Apply δ to both sides in 3.1.

Now we define, for U, V in $\mathfrak{R}$, $U \owedge V = [U, V^*]^*$.

PROPOSITION 4.5. *Let U, V, W be in $\mathfrak{R}$. Then there is a 1-1 correspondence between maps $U \owedge V \to W$ and maps $U \to [V, W]$.*

PROOF. Each of the transformations below is a 1-1 correspondence

$$[U, V^*]^* \to W, \quad W^* \to [U, V^*], \quad U \to [W^*, V^*] \approx [V, W].$$

COROLLARY 4.6. *For any U, V in $\mathfrak{R}$, $U \owedge V \approx V \owedge U$.*

PROPOSITION 4.7. *Let U, V be LC spaces. Then $U \owedge V = \zeta(U \otimes V)$ and is an LC space.*

PROOF. We know it is totally bounded (1.2) so that $\zeta(U \otimes V)$ is LC. When W is in $\mathfrak{R}$, each of the transformations below is a 1-1 correspondence:

$$\zeta(U \otimes V) \to W, \quad U \otimes V \to W, \quad U \to (V, W), \quad U \to [V, W].$$

PROPOSITION 4.8. *Let U, V and W belong to $\mathfrak{R}$. The natural composition of maps $(V, W)\times(U, V)\to(U, W)$ arises from a map*

$$(V, W)\otimes(U, V)\to(U, W).$$

PROOF. If G is an LC subspace of (U, V), U_0 an LC subspace of U, and W_0 an open subspace of W, the closure of the image of the evaluation map

$$G\otimes U_0 \to (U, V)\otimes U \to V$$

is an LC subspace V_0 in V. Then the basic open set in (V, W),

$$\{f\colon V\to W \mid f(V_0)\subset W_0\},$$

is transformed by G into

$$\{h\colon U\to W \mid h(U_0)\subset W_0\}.$$

Thus G determines an equicontinuous family of maps $(V, W)\to(U, W)$ and so we have the indicated map.

PROPOSITION 4.9. *Let U, V and W belong to $\mathfrak{R}$. Then natural composition arises from a map*

$$[V, W]\circledS[U, V]\to[U, W].$$

PROOF. Let $F\subset[V, W]$ and $G\subset[U, V]$ be LC subspaces. From

$$F\otimes G\to(V, W)\otimes(U, V)\to(U, W)$$

and the fact that (U, W) is a ζ-space, we have

$$F\circledS G\approx\zeta(F\otimes G)\to(U, W)$$

and then, by adjointness,

$$F\circledS G\to[U, W].$$

This gives us $G\to[F,[U, W]]]$. Now G is an LC space and hence defines an equicontinuous family. Each $f\in F$ gives, by composition, a continuous function $f_0\text{-}\colon(U, V)\to(U, W)$ which extends by functoriality to a continuous function $[U, V]\to[U, W]$. Hence we have a bilinear map: $F\times[U, V]\to[U, W]$ which is, for all $f\in F$, continuous on $[U, V]$, and for all LC subspaces $G\subset[U, V]$, equicontinuous on F. There results a map

$F \otimes [U, V] \to [U, W]$ which gives $F \to ([U, V), [U, W])$ and, by adjointness, $F \to [[U, V], [U, W]]$. Then F determines an equicontinuous family, so that we have a bilinear map $[V, W] \times [U, V] \to [U, W]$ with the property that every LC subspace of $[V, W]$ determines an equicontinuous family on $[U, V]$. Repeating the argument used above, we see that every element of $[U, V]$ determines a continuous map on $[V, W]$. Hence we have

$$[U, V] \otimes [V, W] \to [U, W], \quad [U, V] \to ([V, W], [U, W]),$$

$$[U, V] \to [[V, W], [U, W]], \quad [V, W] \to [[U, V], [U, W]].$$

THEOREM 4.10. *The category* $\mathfrak{R}$, *equipped with* $-\otimes-$ *and* $[-,-]$, *is a closed monoidal category in which every object is reflexive.*

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