ACYCLIC MODELS AND TRIPLES

MICHAEL BARR AND JON BECK

1. Introduction

We shall prove two theorems on the "triple" cohomology of algebras [Beck 1967] using a method of acyclic models suggested by H. Appelgate. Specifically, we show that the triple cohomology coincides with slight modifications of the usual theories (the same modifications used in [Barr & Rinehart 1966] in the case of groups and associative algebras.) We also prove a direct sum theorem for the cohomology of the coproduct of algeebras, subject to a certain condition.

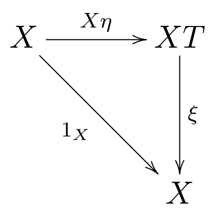
These theorems are proved by setting up cochain equivalences between standard cochain complexes for the theories involved. Unfortunately, algebra cohomology cannot in general be viewed as a derived functor, or equivalently, the standard complexes from which it comes need not be resolutions. Thus the usual techniques for comparing resolutions, using acyclicity, are inapplicable. However, it is usually possible to prove that the standard complexes become acyclic when applied to free algebras. This suggests using an acyclic models approach, with free algebras as models. Note that since the free functor from the underlying cateogry to the category of algebras has an adjoint, the main theorem on extension of maps and homotopies is simpler than normally appears in topology [Eilenberg & Mac/,Lane 1953].

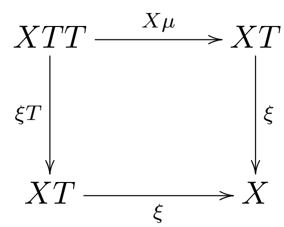
2. Triple cohomology

In this section we present a short discussion of the tripleable categories and cohomology. For detailed accounts, see [Beck 1967, Eilenberg & Moore 1965]. Let $\mathcal{A}$ be a category. $\mathbf{T} = (T, \eta, \mu)$ is a *triple* on $\mathcal{A}$ if $T : \mathcal{A} \longrightarrow \mathcal{A}$ is a functor, $\eta : 1_{\mathcal{A}} \longrightarrow T$ and $\mu : TT \longrightarrow T$ are morphisms of functors, called the *unit* and *multiplication* of $\mathbf{T}$ respectively, such that the diagrams

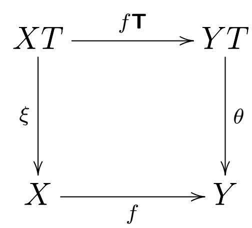
$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta T} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T \end{array}$$

commute. (X, ξ) is a $\mathbf{T}$-algebra if X is an object of $\mathcal{A}$ and $\xi : XT \longrightarrow X$ is a morphism of $\mathcal{A}$ such that





commute. ξ is called the $\mathbf{T}$-structure of the algebra. $f : (X, \xi) \longrightarrow (Y, \theta)$ is a morphism of $\mathbf{T}$-algebras if



commutes. The category of $\mathbf{T}$-algebras will be denoted by $\mathcal{A}^{\mathbf{T}}$. Categories of the form $\mathcal{A}^{\mathbf{T}}$ are called *tripleable.*

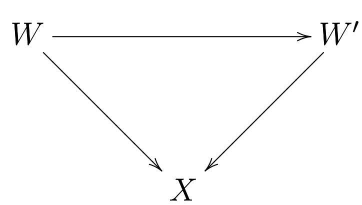
There is an adjoint pair of functors $F \dashv U$ where $F : \mathcal{A} \longrightarrow \mathcal{A}^{\mathbf{T}}$ is the *free* $\mathbf{T}$-algebra functor given by $XF = (XT, X\mu)$ and $U : \mathcal{A}^{\mathbf{T}} \longrightarrow \mathcal{A}$ is the forgetful functor $U(X, \xi) = X$. Clearly $T = FU$ and η and μ are derivable from the adjointness morphisms $1 \longrightarrow FU$ and $UF \longrightarrow 1$. Let $G = UF : \mathcal{A}^{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbf{T}}$. By adjointness G is a *cotriple* in $\mathcal{A}^{\mathbf{T}}$. Explicitly, $(X, \xi)G = (XT, X\mu)$, the counit $\epsilon : G \longrightarrow 1_{\mathcal{A}^{\mathbf{T}}}$ is

$$(X, \xi)\epsilon = \xi : (XT, X\mu) \longrightarrow (X, \xi)$$

and the comultiplication $\delta : G \longrightarrow GG$ is

$$(X, \xi)\delta = X\eta T : (XT, X\mu) \longrightarrow (XTT, XT\mu)$$

The cotriple $\mathbf{G} = (G, \epsilon, \delta)$ now gives rise to a cohomology theory in $\mathcal{A}^{\mathbf{T}}$. The theory will have coefficient in an X-module, for a $\mathbf{T}$-algebra X (omitting the $\mathbf{T}$-structure from the notation, for brevity). To define X-modules, let $(\mathcal{A}^{\mathbf{T}}, X)$ whose objects are $\mathbf{T}$-algebra morphisms $W \longrightarrow X$ and whose morphisms are commutative triangles



$(\mathcal{A}^{\mathbf{T}}, X)$ is called the category of $\mathbf{T}$-algebras and morphisms *over* X. $Y \longrightarrow X$ is an X-*module* if it is an abelian group object in the category $(\mathcal{A}^{\mathbf{T}}, X)$, that is if the functor

$\mathrm{Hom}_X(-, Y)$ which *a priori* has values in the category of sets, factors through the category of abelian groups (Hom_X stands for Hom in the category $(\mathcal{A}^{\mathsf{T}}, X)$). If $Y \longrightarrow X$ is an X-module, there is a T-algebra homomorphism $X \longrightarrow Y$, the *zero section*, such that $X \longrightarrow Y \longrightarrow X$ is the identity. Y will thus be a "split extension" of X with some abelian structure "fiberwise". In effect, we confuse modules with split extensions.

Now we define cohomology groups $H^n(W, Y)_X$ where $W \longrightarrow X$ is a T-algebra over X and $Y \longrightarrow X$ is an X-module. (The cohomology of X itself is retrievable as $H^n(X, Y)_X$, regarding $X \longrightarrow X$ as an algebra over X by the identity morphism.) The cotriple $\mathbf{G}$ acts on $(\mathcal{A}^{\mathsf{T}}, X)$ by the rule $(W \xrightarrow{p} X)G = WG \xrightarrow{pG.X\epsilon}$. Let G^{n+1} be the $(n+1)$-st iterate of G, and let $\epsilon_i = G^i \epsilon G^{n-i} : G^{n+1} \longrightarrow G^n$, $0 \le i \le n$. Then we get a simplicial object

$$\cdots \rightrightarrows WG^3 \rightrightarrows WG^2 \underset{\epsilon_1}{\overset{\epsilon_0}{\rightrightarrows}} WG \xrightarrow{\epsilon} W$$

in the category $(\mathcal{A}^{\mathsf{T}}, X)$. That is, each WG^{n+1} is an algebra over X by the above, and each face operator ϵ_i is a morphism over X. (Note that WG^{n+1} is in dimension n, and that degeneracy operators could also be defined using $\delta : G \longrightarrow G^2$.) If $Y \longrightarrow X$ is an X-module, we can form the cochain complex of abelian groups

$$0 \longrightarrow \mathrm{Hom}_X(WG, Y) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_X(WG^{n+1}, Y) \longrightarrow \cdots$$

with coboundary operators $d = \sum (-1)^i \mathrm{Hom}_X(W\epsilon_i)$. $H(W, Y)$ is the cohomology of this complex.

3. Acyclic models

Let $\mathcal{C}$ be a category and let K, L be standard cochain complexes $\mathcal{C}^* \longrightarrow \mathcal{Ab}$. This means that $K = \{K^n\}_{n \ge -1}$, there are morphisms $d : K^n \longrightarrow K^{n+1}$, $dd = 0$, and each K^n is an ordinary functor $\mathcal{C}^* \longrightarrow \mathcal{Ab}$; the same for L. In this paper we use only contravariant functors, but the dual is obvious.

Let $G : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor with a counit $\epsilon : G \longrightarrow 1_{\mathcal{C}}$. The standard cochain complex is *G-acyclic* if there is a functorial contracting homotopy s (of degree -1) in the "composite" complex GK. L is *G-representable*[1] if there are morphisms $\theta^n : GL^n \longrightarrow L^n$ such that $\epsilon L^n.\theta^n = 1$, for $n \ge 0$. (The θ's do not have to commute with coboundaries.)

3.1. Theorem. *Let K be G-acyclic and L be G-representable. Then any morphism of functos $f^{-1} : K^{-1} \longrightarrow L^{-1}$ can be extended to to a natural cochain transformation $f : K \longrightarrow L$. If $f, g : K \longrightarrow L$ and $f^{-1} = g^{-1}$, then there exists a natural cochain homotopy $\Phi : f \simeq g$.*

[1]this was poor terminology; representability is a fundamental concept in category theory and we now use the term *presentable*, which is even more descriptive since it recalls the notion of a presentation, which indeed it is.

Proof. f^n is constructed inductively as the composite

$$\begin{array}{ccccccc} K^n & \xrightarrow{\epsilon K^n} & GK^n & & GL^n & \xrightarrow{\theta^n} & L^n \\ & & \Big\downarrow{\scriptstyle s} & & \Big\uparrow{\scriptstyle Gd} & & \\ & & GK^{n-1} & \xrightarrow[Gf^{n-1}]{} & GL^{n-1} & & \end{array}$$

$\Phi^{-1} = \Phi^0 = 0$ and Φ^n is the upper minus the lower composite in the diagram

$$\begin{array}{ccccccc} K^n & \xrightarrow{\epsilon K^n} & GK^n & & & & \\ & & \Big\downarrow{\scriptstyle s} & & & & \\ & & GK^{n-1} & \xrightarrow{G(f^{n-1}-g^{n-1})} & GL^{n-1} & \xrightarrow{\theta^{n-1}} & L^{n-1} \\ & & & \searrow{\scriptstyle G\Phi^{n-1}} & \Big\downarrow{\scriptstyle Gd} & & \\ & & & & GL^{n-2} & & \end{array}$$

3.2. Corollary. *If $K,\ L$ are both G-acyclic and G-representable, and $K^{-1} \cong L^{-1}$, then $K \simeq L$.*

3.3. Corollary. *If $K,\ L$ are both G-acyclic and G-representable, $f : K \longrightarrow L$, and $f^{-1} : K^{-1} \cong L^{-1}$, then f is a cochain equivalence.*

3.4. Remark. In our applications, $\mathcal{C}$ is always a tripleable category, $\mathbf{G}$ is always the "free" cotriple on $\mathcal{C}$, and θ comes from the unit η of the triple. It should therefore be possible to sharpen 3.1 in the cases we are interested.

4. Cohomology of groups and algebras

Let $\mathcal{A}$ be the category of sets and $\mathcal{G}$ the category of groups. There is an adjoint pair of functors $F \dashv U$ where $F : \mathcal{A} \longrightarrow \mathcal{G}$ is the free group functor and $U : \mathcal{G} \longrightarrow \mathcal{A}$ is the forgetful functor. Let $T = FU : \mathcal{A} \longrightarrow \mathcal{A}$. By adjointness there are morphisms $\eta : 1_{\mathcal{A}} \longrightarrow T = FU$ and $\mu = F\epsilon U : TT = F(UF)U \longrightarrow FU = T$ which make $\mathbf{T} = (T, \eta, \mu)$ into a triple. If X is a set, there is a 1-1 correspondence between group laws on X and $\mathbf{T}$-structures $XT \longrightarrow X$. Hence the category of $\mathbf{T}$-algebras is isomorphic to the category groups $\mathcal{G}$.

If π is a group, let $Y \longrightarrow \pi$ be a π-module as defined in Section 2. Using the zero section $\pi \longrightarrow Y$, one shows that that $Y \longrightarrow \pi$ is isomorphic to the split extension of π by the ordinary π-module $M = \ker Y \longrightarrow \pi$. Thus $Y \cong M \times \pi$ as a set and the multiplication in Y is

$$(m_1, x_1)(m_2, x_2) = (m_1 + x_1 m_2, x_1 x_2)$$

in terms of the left π-operators on M. It follows that if $W \longrightarrow \pi$ is a group over π, there is a natural isomorphism $\mathrm{Hom}_\pi(W, Y) \cong \mathrm{Der}(W, M)$, where a *derivation* $f : W \longrightarrow M$ satisfies $(w_1 w_2)f = w_1 f + w_1(w_2 f)$. Here W acts on M via the given morphism $W \longrightarrow \pi$.

4.1. THEOREM. *There is a natural isomorphism*

$$H^n(W, Y)_\pi \longrightarrow \begin{cases} \mathrm{Der}(W, M), & n = 0 \\ H^{n+1}(W, M), & n > 0 \end{cases}$$

where $H^{n+1}(W, M)$ is the Eilenberg-Mac Lane cohomology group.

PROOF. Let $K : (\mathcal{G}, \pi)^* \longrightarrow \mathcal{Ab}$ be the cochain complex

$$WK^n = \mathrm{Hom}_\pi(WG^{n+1}, Y), \qquad n \geq -1$$

as in Section 2. Here $G : \mathcal{G} \longrightarrow \mathcal{G}$ is the cotriple $WG = WUF$, the free group generated by the underlying set of W, with the natural group epimorphism $\epsilon : WG \longrightarrow W$ as its counit, and if W is a group over π, so is WG. (Note that G^{n+1} is the *iterate* of the functor G, not its $(n+1)$-fold cartesian power.)

The form of the bar construction we use is $L : (\mathcal{G}, \pi)^* \longrightarrow \mathcal{Ab}$ where WL^n is the abelian group of functions from the cartesian power $W^{n+1} \longrightarrow M$, $n \geq 0$, and $WL^{-1} = \mathrm{Der}(W, M)$. The coboundary $d : WL^n \longrightarrow WL^{n+1}$ is

$$\begin{aligned}(w_0, \ldots, w_{n+1})(fd) &= w_0 p.(w_1, \ldots, w_{n+1})f \\ &\quad + \sum_{i=1}^{n+1} (w_0, \ldots, w_{n-1}w_n, \ldots, w_{n+1})f \\ &\quad + (-1)^{n+2}(w_0, \ldots, w_n)f\end{aligned}$$

where $n \geq 0$, p is the morphism $W \longrightarrow \pi$ and $d : WL^{-1} \longrightarrow WL^0$ is the obvious inclusion. Then $H(L)$ is the cohomology on the right in 4.1. In effect, we have just cut off the bottom complex given in [Eilenberg & Mac/,Lane 1947].

It follows from what was said leading up to 4.1 that $K^{-1} \cong L^{-1}$. We shall apply 3.2 to prove that there is a natural cochain equivalence $K \simeq L$. The functor with counit used to compare K and L will be the free group cotriple itself, acting in $(\mathcal{G}, \pi)$.

G-ACYCLICITY. A contracting homotopy $s : GK^n \longrightarrow GK^{n-1}$ is induced by $W\delta G^n : WGG^n \longrightarrow WGG^{n+1}$.

As to the homotopy in in GL, note that there is a natural cochain equivalence $L \simeq N$ where WN^n consists of those functions $W^{n+1} \longrightarrow M$ vanishing when any argument equals 1 ("normalized" cochains [Mac Lane 1963]). A natural contracting homotopy in GN has essentially been constructed in [Lyndon 1949]. Therefore there exists a natural contracting homotopy in GL. An explicit homotopy (probably different) is as follows. Let $f \in WGL^n$. Then $(g_0, \ldots, g_{n-1})(fs)$ is defined by induction on the length of the word $g_0 \in WG$. Spell the words in WG in letters (w), where $w \in W$. Then:

1. If $g_0 = (w)g$ where $g \in WG$, let

$$(g_0, \ldots, g_{n-1})(fs) = wp.(g, \ldots, g_{n-1})(fs) - ((w), g, \ldots, g_{n-1})f$$

2. If $g_0 = (w)^{-1}g$ where $g \in WG$, let

$$(g_0, \ldots, g_{n-1})(fs) = w^{-1}p.(g, \ldots, g_{n-1}) + w^{-1}p.((w), g_0 \ldots, g_{n-1})f$$

3. If $g_0 = 1$, let

$$(1, g_1, \ldots, g_{n-1})(fs) = (1, 1, g_1, \ldots, g_{n-1})f$$

Note that this homotopy is natural with respect to morphisms $W \longrightarrow W'$ in $(\mathcal{G}, \pi)$.

G-REPRESENTABILITY. $\theta^n : GK^n \longrightarrow K^n$ is also induced by

$$W\delta G^n : WG^{n+1} \longrightarrow WG^{n+2}, \qquad n \geq 0$$

$\theta^n : WL^n \longrightarrow L^n$ is given by

$$(w_0, \ldots, w_n)(f\theta) = ((w_0), \ldots, (w_n))f$$

This completes the proof of 4.1

As another application, the category of associative K-algebras with unit is tripleable over the category $\mathcal{A}$ of K-modules, using the tensor algebra as the triple. A similar argument shows:

4.2. THEOREM. *There is a natural isomorphism*

$$H^n(\Gamma, Y)_\Lambda = \begin{cases} \mathrm{Der}(\Gamma, M), & n = 0 \\ H^{n+1}(\Gamma, M), & n > 0 \end{cases}$$

where $\Gamma \longrightarrow \Lambda$ *is an algebra over* Λ*,* $Y \longrightarrow \Lambda$ *is a* Λ*-module, that is a split* K*-algebra extension by a kernel* M *with* $M^2 = 0$*,* Λ *operates on both sides of* M*, hence also* Γ*, and* $H^{n+1}(\Gamma, M)$ *is the Hochschild (*K*-relative) cohomology group.*

5. Cohomology of a coproduct

We return to the general situation outlined in Section 2, and assume:

1. $\mathcal{A}$ has pullbacks (which implies that $\mathcal{A}^\mathsf{T}$ has pullbacks).

2. $\mathcal{A}^\mathsf{T}$ has coproducts (denoted $X_1 * X_2$).

3. A natural morphism $u : X_1 * X_2 \longrightarrow X_1G * X_2G$ exists such that $u(X_1\epsilon * X_2\epsilon) = 1$.

The validity of these assumptions will be discussed below.

Let $Y \longrightarrow X_1 * X_2$ be an $X_1 * X_2$-module and let $i_j : X_j \longrightarrow X_1 * X_2$ be the natural morphisms in the coproduct, $j = 1, 2$. There are pullback diagrams

$$\begin{array}{ccc} Y_j & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ X_j & \longrightarrow & X_1 * X_2 \end{array}$$

One easily sees that Y_j is an X_j-module. Hence if $W_j \longrightarrow X_j$ is in $(\mathcal{A}^{\mathbf{T}}, X_j)$, there is a cohomology morphism, still denoted by i_j:

$$i_j : H(W_1 * W_2, Y)_{X_1 * X_2} \longrightarrow H(W_j, Y_j)_{X_j}$$

5.1. Theorem. $(i_1, i_2) : H(W_1 * W_2)_{X_1 * X_2} \longrightarrow H(W_1, Y_1)_{X_1} \oplus H(W_2, Y_2)_{X_2}$ *is an isomorphism.*

Proof. Define complexes $L, S : (\mathcal{A}^{\mathbf{T}}, X_1)^* \times (\mathcal{A}^{\mathbf{T}}, X_2)^* \longrightarrow \mathcal{Ab}$ by

$$(W_1, W_2)L^n = \mathrm{Hom}_{X_1 * X_2}((W_1 * W_2)G^{n+1}, Y)$$

$$(W_1 * W_2)S^n = \mathrm{Hom}_{X_1}(W_1 G^{n+1}, Y_1) \oplus \mathrm{Hom}_{X_2}(W_2 G^{n+1}, Y_2)$$

with coboundary operators as in the triple complex in Section 2. The inclusions into the coproduct induce a cochain morphism $(i_1, i_2) : L \longrightarrow S$, hence in cohomology, induce the morphism in the statement. It remains to show that $(i_1, i_2) : L \longrightarrow S$ is a cochain equivalence. First $(i_1, i_2)^{-1}$ is an isomorphism since

$$\begin{aligned} \mathrm{Hom}_{X_1 * X_2}(W_1 * W_2, Y) &\cong \mathrm{Hom}_{X_1 * X_2}(W_1, Y) \oplus \mathrm{Hom}_{X_1 * X_2}(W_2, Y) \\ &\cong \mathrm{Hom}_{X_1}(W_1, Y_1) \oplus \mathrm{Hom}_{X_2}(W_2, Y_2) \end{aligned}$$

To extend $(i_1, i_2)^{-1}$ to a homotopy inverse for (i_1, i_2) we apply 3.3 using the cotriple $\mathbf{G} \times \mathbf{G}$ which acts in the category $(\mathcal{A}^{\mathbf{T}}, X_1) \times (\mathcal{A}^{\mathbf{T}}, X_2)$ in the obvious way. L and S are acyclic on models by taking coproducts and sums of the contracting homotopy used for the triple complex in Section 4. S us also representable, in a similar manner. The representation morphism for L is the map

$$(W_1 G, W_2 G)L^n = \mathrm{Hom}((W_1 G * W_2 G)G^{n+1}, Y) \longrightarrow \mathrm{Hom}((W_1 * W_2)G^{n+1}, Y) = (W_1 * W_2)L^n$$

where we use condition 3 above and all the Hom's are in the category $(\mathcal{A}^{\mathbf{T}}, X_1 * X_2)$. This completes the proof of the theorem.

As to the assumptions used in proving 5.1, conditions 1 and 2 are routine and hold in all the usual algebraic categories. Assumption 3 is more delicate. Here are some cases in which it is known to hold:

1. $\mathscr{A}$ = sets, $\mathbf{T}$ = the free group triple, $\mathscr{A}^{\mathbf{T}}$ = the category of groups (Section 4). Thus the coproduct theorem holds for group cohomology (see also [Barr & Rinehart 1966] for a proof of this fact).

2. $\mathscr{A} = K$-modules, $\mathbf{T}$ = the tensor algebra triple, $\mathscr{A}^{\mathbf{T}}$ = the category of associative K-algebras with unit (Section 4), and the coproduct theorem holds for Hochschild cohomology.

3. $\mathscr{A} = K$-modules, $\mathbf{T}$ = the symmetric algebra cotriple, $\mathscr{A}^{\mathbf{T}}$ = the category of commutative K-algebras (associative with unit). The cohomology is probably Harrison's when K is a field.[2]

Assumption 3 fails in the case $\mathscr{A}$ = sets, XT = the commutative polynomial algebra generated by the elements of the set X. $\mathscr{A}^{\mathbf{T}}$ is again the category of commutative K-algebras but the cohomology is different. Take $K = \mathbf{Z}$, $W_1 = W_2 = X_1 = X_2 = \mathbf{Z}_2$ (integers mod 2), $Y = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \longrightarrow X_1 * X_2 = \mathbf{Z}_2 \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2$, by the projection, $\mathbf{Z}_2$ acting on itself as kernel in the usual way. Then in dimension 1, all three groups in 5.1 have the value $\mathbf{Z}_2$, which is a contradiction. It would be interesting to conditions on $\mathscr{A}$ and $\mathbf{T}$ guaranteeing the validity of Condition 3.

References

J. Beck (1967), Algebras and Cohomology. Dissertation, Columbia University. `http://www.tac.mta.ca/tac/reprints/articles/2/tr2abs.html`

S. Eilenberg and S. Mac Lane (1953) Acyclic models. Amer. J. Math. **75**, 189–199.

S. Eilenberg and S. Mac Lane (1947), Cohomology theory in abstract groups, I. Ann. Math. **48**, 51–78.

S. Eilenberg and J.C. Moore (1965), Adjoint functors and triples. Ill. J. Math. **9**, 381–398.

D.K. Harrison (1965), Commutative algebra and cohomology. Trans. Amer. Math. Soc. **104**, 191–204.

R.C. Lyndon (1949), New proof for a theorem of Eilenberg and Mac Lane. Ann. Math. **50**, 731–735.

S. Mac Lane (1963), Homology. Springer.

M. Barr and G.S. Rinehart (1966), Cohomology as the derived functor of derivations. Trans. Amer. Math. Soc. **122**, 416–426.

M. Barr (1968) Harrison homology, Hochschild homology and triples. J. Algebra **8**, 314–323.

[2]in characteristic 0 only, see [Barr 1968].