ABSOLUTE HOMOLOGY

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ABSTRACT. Call two maps, $f, g: C \longrightarrow C'$, of chain complexes absolutely homologous if for any additive functor F, the induced Ff and Fg are homologous (induce the same map on homology). It is known that the identity is absolutely homologous to 0 iff it is homotopic to 0 and tempting to conjecture that f and g are absolutely homologous iff they are homotopic. This conjecture is false, but there is an equational characterization of absolute homology. I also characterize left absolute and right absolute (in which F is quantified over left or right exact functors).

1. Introduction

An exactness property is generally called *absolute* if it is preserved by all functors (or all in a relevant class). For example, an arrow in a category is an absolute epimorphism if it is taken to a epimorphism by every functor. It is easy to characterize such arrows as split epics. And only one functor is actually needed, the one represented by its codomain. Probably the earliest non-trivial example was the characterization of absolute coequalizers. Paré [1969, 1971] characterized those diagrams

$$A \rightrightarrows B \longrightarrow C$$

that are taken to coequalizers by every functor, using a generalization of the notion of split coequalizer.

If (C, d) is a chain complex (or even just a differential object) in an abelian category, we say that it is *absolutely acyclic* if for every additive functor F to another abelian category, the complex (FC, Fd) is acyclic. Such an object can be characterized as being contractible. Here is a quick sketch of the proof (see [Barr, forthcoming] for details). Let $Z(C)$ denote the object of cycles—the kernel of d. Apply the functor represented by $Z(C)$ to conclude that the complex $(\mathrm{Hom}(Z(C), C), \mathrm{Hom}(Z(C), d)$ is an acyclic complex of abelian groups. The inclusion arrow $i: Z(C) \longrightarrow C$ is obviously a cycle and hence a boundary. This means that there is an arrow $z: Z(C) \longrightarrow C$ such that $d \circ z = i$. Since the image of d is contained in $Z(C)$, it makes sense to form the composite $z \circ d$ and then one can calculate that the image of $1 - z \circ d$ is also in $Z(C)$ and one can then form the composite $h = z \circ (1 - z \circ d)$ and calculate that $\mathrm{id} = d \circ h + h \circ d$, which means that (C, d) is contractible.

With this example in mind, it seemed reasonable to conjecture that any two absolutely homologous maps are homotopic. This turns out to be not quite true. For maps $f, g: C$

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$\longrightarrow C'$ to be homotopic requires that there be a morphism $h: C \longrightarrow C'$ such that $f - g = h \circ d + d \circ h$. What we actually find as the characterization is that there be two morphisms $h, k: C \longrightarrow C'$ such that $f - g = h \circ d + d \circ k$. This is still an equational condition, preserved by all additive functors, that implies that they induce the same map on homology, since on cycles, it says that $f - g = d \circ k$, which means that whenever $c \in C$ cycle, $(f - g)(c)$ is a boundary.

In the case of absolutely acyclic complexes, not all functors are needed; in fact, only a representable functor was used. For the more general case considered here, representable functors are not quite enough and what is used is the coequalizer of two representable functors, which is not left or right exact. It turns out that one can also characterize *left absolute* and *right absolute* homologous arrows, which remain homologous under the application of all left exact, resp. right exact, functors.

We have stated and proved the theorems for the ungraded case. The application to chain complexes and cochain complexes is easy; just replace the category of abelian groups by graded abelian groups and maps that have a degree, not necessarily 0.

1.1. NOTATION. For a differential object (C, d), we let $i: Z(C, d) \longrightarrow C$ denote the kernel of d and $q: C \longrightarrow B(C, d)$ denote the cokernel. We also let $j: B(C, d) \longrightarrow C$ be the inclusion of the image so that $d = j \circ q$. We will normally omit the d and write $Z(C)$ and $B(C)$.

2. Absolute homology

2.1. THEOREM. *Suppose $f, g: (C, d) \longrightarrow (C', d)$ are maps of differential objects. Then f and g are absolutely homologous if and only if there are morphisms $h, k: C \longrightarrow C'$ such that $f - g = h \circ d + d \circ k$.*

PROOF. It suffices to do this in the case that $g = 0$. So suppose that $f: (C, d) \longrightarrow (C', d)$ is a morphism such that form every functor $F: \mathscr{A} \longrightarrow \mathbf{Ab}$, the induced $H(Ff): H(F(C, d)) \longrightarrow H(F(C', d))$ is 0. We will use the functor F defined as the coequalizer of $\mathrm{Hom}(d, -)$ so that for an object A of $\mathscr{A}$,

$$\mathrm{Hom}(C, A) \xrightarrow{\mathrm{Hom}(d, A)} \mathrm{Hom}(C, A) \longrightarrow FA$$

is a coequalizer. Thus an element of FA is an equivalence class of morphisms $u: C \longrightarrow A$ with $u \equiv v$ if and only if there is a $w: C \longrightarrow A$ such that $u = v + w \circ d$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(C,C) & \xrightarrow{\mathrm{Hom}(d,C)} & \mathrm{Hom}(C,C) & \longrightarrow & FC & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \mathrm{Hom}(C,d)} & & \big\downarrow{\scriptstyle \mathrm{Hom}(C,d)} & & \big\downarrow{\scriptstyle Fd} & & \\
\mathrm{Hom}(C,C) & \xrightarrow[\mathrm{Hom}(d,C)]{} & \mathrm{Hom}(C,C) & \longrightarrow & FC & \longrightarrow & 0
\end{array}$$

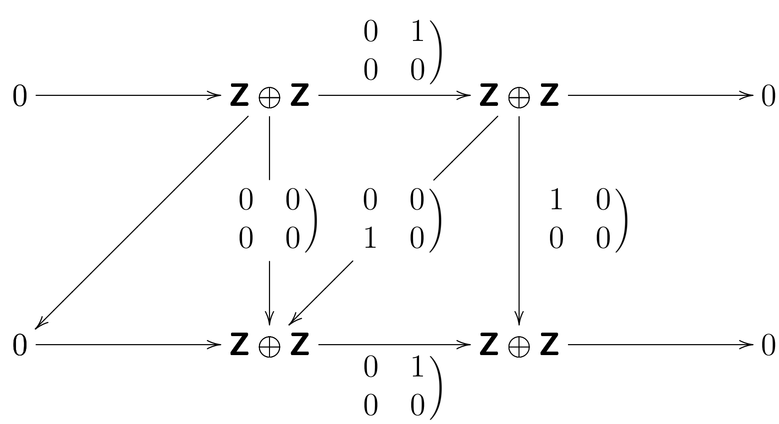
which means that for a morphism $u\colon C \longrightarrow C$, $Fd(u) = d \circ u$. This implies that if u commutes with d, then $Fd(u) = d \circ u = u \circ d \equiv 0$ and is thus a cycle in the differential abelian group (FC, Fd). In particular, the class containing $\mathrm{id}\colon C \longrightarrow C$ is a cycle. But then the class $Ff(\mathrm{id}) \in (FC', Fd)$ must be a boundary and this is just the class of f. This means that there has to be a morphism $h \in \mathrm{Hom}(C, C')$ such that $f \equiv d \circ h$, which in turn means there is a morphism $k \in \mathrm{Hom}(C, C')$ such that $f = d \circ h + k \circ d$. ■

2.2. Example. This is an example to show that absolute homology equivalence need not imply homotopy equivalence. Consider the following situation. Let $C = \mathbf{Z}_{16}$ and $C' = \mathbf{Z}_8$. The differential in each is multiplication by 4, which has square 0. Let $k\colon C \longrightarrow C'$ be the natural projection and let $f = d \circ k = k \circ d$. Then f is evidently absolutely null homotopic. On the other hand, for any $h\colon C \longrightarrow C'$, $d \circ h + h \circ d = 4h + 4h = 8h = 0$ in $\mathbf{Z}_8$, so f is not null homotopic.

2.3. Example. Here is an example of a morphism of chain complexes that is absolutely null without being homotopic to 0. Consider the chain complex

$$C_2 = 0 \longrightarrow C_1 = \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} C_0 = \mathbf{Z} \oplus \mathbf{Z} \longrightarrow 0$$

Let $h = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \colon C_0 \longrightarrow C_1$ and $f = d \circ h$. The picture is



Then $f = d \circ h$, but it is easy to see that there is no k for which $f = d \circ k + k \circ d$.

3. Left and right absolute homology equivalence

3.1. Theorem. *Suppose $f, g\colon (C,d) \longrightarrow (C',d)$ are maps of differential objects. Then f and g are left absolutely homologous if and only if there is a morphism $k\colon Z(C) \longrightarrow C'$ such that $(f-g)\circ i = d\circ k$. If, in addition, C' is injective, then f and g are absolutely homologous.*

Proof. It is sufficient to consider the case that $g = 0$. We let $F = \mathrm{Hom}(Z(C), -)$. Since $0 \longrightarrow Z(C) \xrightarrow{i} C \xrightarrow{d} C \longrightarrow 0$ is exact, so is

$$0 \longrightarrow \mathrm{Hom}(Z(C), Z(C)) \longrightarrow \mathrm{Hom}(Z(C), C) \xrightarrow{\mathrm{Hom}(Z(C),d)} \mathrm{Hom}(Z(C), C)$$

which means that $\mathrm{Hom}(Z(C), Z(C)) = Z(\mathrm{Hom}(Z(C), C))$. In particular, the identity arrow of $Z(C)$, whose image in $\mathrm{Hom}(Z(C), C)$ is i, is a cycle and hence its image $f\circ i$ in the differential abelian group $\mathrm{Hom}(Z(C), C')$ is a boundary. But this means that there is a $k\colon Z(C) \longrightarrow C'$ such that $d\circ k = f\circ i$ as claimed.

Now suppose that C' is injective. In that case, k can be extended to a map $\widehat{k}\colon C \longrightarrow C'$ such that $\widehat{k}\circ i = k$. It follows that $(f - d\circ\widehat{k})\circ i = f\circ i - d\circ\widehat{k}\circ i = f\circ i - d\circ k = 0$. Since $q\colon C \longrightarrow B(C)$ is the cokernel of i, there is a unique $h\colon B(C) \longrightarrow C'$ such that $h\circ q = f - d\circ\widehat{k}$. Another application of injectivity, implies the existence of $\widehat{h}\colon C \longrightarrow C'$ such that $\widehat{h}\circ j = h$. Then

$$\widehat{h}\circ d = \widehat{h}\circ j\circ q = h\circ q = f - d\circ\widehat{k}$$

from which we conclude that $f = \widehat{h}\circ d + d\circ\widehat{k}$. ∎

By replacing $\mathscr{A}$ by $\mathscr{A}^{\mathrm{op}}$, we can translate this theorem into one for right absolute homology.

3.2. Theorem. *Suppose $f, g\colon (C,d) \longrightarrow (C',d)$ are maps of differential objects. Then f and g are right absolutely homologous if and only if there is a morphism $h\colon C \longrightarrow C'/B(C')$ such that $q\circ(f-g) = k\circ d$. If, in addition, C is projective, then f and g are absolutely homologous.*

4. An example of a left absolute homology

The following theorem is suggested by Theorem 1.1 of [Bauer, 2001] although there are significant differences. But Bauer's complexes are functors into the category of free abelian groups. Although a subgroup of a free abelian group is free, such complexes are not projective in the functor category. Later, we will look at another result suggested by the same theorem.

4.1. Theorem. *Suppose that $\mathscr{A}$ is an abelian category in which subobjects of projectives are projective. Let (C,d) and (C',d') be differential objects of $\mathscr{A}$ with C projective. Then any homomorphism $u\colon H(C,d) \longrightarrow H(C',d')$ is induced by a chain homorphism $f\colon C \longrightarrow C'$.*

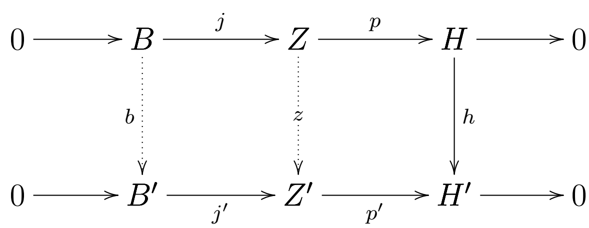
Proof. Let C, Z, B, H and C, Z', B', H' denote the differential objects and the objects of cycles, boundaries, and homology classes, resp. Let $p\colon Z \longrightarrow H$ and $p'\colon Z' \longrightarrow H'$ denote the projections from cycles to homology classes and let

$$B \xrightarrow{j} Z \xrightarrow{i} C \xrightarrow{q} B$$

and

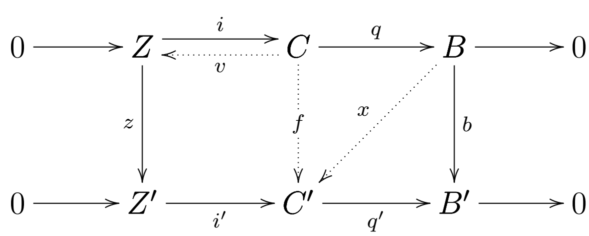
$$B' \xrightarrow{j'} Z' \xrightarrow{i'} C' \xrightarrow{q'} B'$$

denote, resp., the inclusion of boundaries into cycles, of cycles into chains and boundary map from chains to cycles. Thus the boundary operators are $d = i \circ j \circ q$ and $d' = i' \circ j' \circ q'$. The hypotheses imply that C, Z and B are projective. In the diagram



(∗)

the rows are exact. Then the projectivity of Z implies the existence of an arrow $z\colon Z \longrightarrow Z'$ that makes the right hand square commute. The exactness of the lower line implies the existence of $b\colon B \longrightarrow B'$ making the left hand square commute. Next consider the diagram with exact rows



(∗∗)

We make two uses of the projectivity of B. First we split the upper sequence and get a map $v\colon C \longrightarrow Z$ such that $v \circ i = \mathrm{id}$. Second we get a map $x\colon B \longrightarrow C'$ such that $q' \circ x = b$. Now let $f = i' \circ z \circ v + x \circ q\colon C \longrightarrow C'$. Then

$$q' \circ f = q' \circ i' \circ z \circ v + q' \circ x \circ q = b \circ q$$

while

$$f \circ i = i' \circ z \circ v \circ i + q' \circ x \circ q \circ i = i' \circ z$$

It then follows that

$$f \circ d = f \circ i \circ j \circ q = i' \circ z \circ j \circ q = i' \circ j' \circ b \circ q = i' \circ j' \circ q' \circ f = d' \circ f$$

so that f is a chain homomorphism. The commutativity of $(**)$ implies that $z = Z(f)$ and $b = B(f)$ and then the commutativity of $(*)$ implies that $h = H(f)$. ∎

What this theorem does not claim is that f is not unique up to homotopy. To show that this may fail, we give an example in which a chain map induces the 0 homomorphism in homology, but is not homotopic to 0. Consider,

4.2. Example.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\;2\;} & \mathbf{Z} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbf{Z} & \xrightarrow[\;0\;]{} & \mathbf{Z} & \longrightarrow & 0
\end{array}$$

In degree 0, the upper complex has $\mathbf{Z}_2$ homology, but the map is 0, while in degree 1, the upper complex has 0 homology, so the map on homology is also 0, while it is evident that the map is not homotopic to 0 since 1 is not a multiple of 2. The map is also not right absolutely null homologous as tensoring with $\mathbf{Z}_2$ will show. It is, however, left absolutely null homologous as follows from the next result.

4.3. Theorem. *Under the same hypotheses as in Theorem 4.1 any two extensions of u are left absolutely homologous.*

Proof. It suffices to consider the case that $u = 0$ and f is a chain map with $H(f) = 0$. In the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\;j\;} & Z & \xrightarrow{\;p\;} & H & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle b} & \overset{l}{\swarrow} & \Big\downarrow{\scriptstyle z} & & \Big\downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & B' & \xrightarrow[\;j'\;]{} & Z' & \xrightarrow[\;p'\;]{} & H' & \longrightarrow & 0
\end{array}$$

the fact that $u = 0$ implies the existence of $l \colon Z \longrightarrow B'$ making the diagram commute. The projectivity of Z lifts this to an arrow $k \colon Z \longrightarrow C'$ and Theorem 3.1 finishes the argument. ∎

References

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