p-ADIC L-FUNCTIONS FOR ORDINARY FAMILIES ON SYMPLECTIC GROUPS

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Abstract. We construct the p-adic standard L-functions for ordinary families of Hecke eigensystems of the symplectic group $\mathrm{Sp}(2n)_{/\mathbb{Q}}$ using the doubling method. We explain a clear and simple strategy of choosing the local sections for the Siegel Eisenstein series on the doubling group $\mathrm{Sp}(4n)_{/\mathbb{Q}}$, which guarantees the nonvanishing of local zeta integrals and allows us to p-adically interpolate the restrictions of the Siegel Eisenstein series to $\mathrm{Sp}(2n)_{/\mathbb{Q}} \times \mathrm{Sp}(2n)_{/\mathbb{Q}}$.

Contents

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1. Introduction

The goal of this article is to construct the p-adic standard L-functions for ordinary families of Hecke eigen-systems of symplectic groups. Let $G = \mathrm{Sp}(2n)_{/\mathbb{Q}}$ and $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Take S to be a finite set of places of $\mathbb{Q}$ containing the archimedean place and all the finite places where π_v is ramified. Suppose ξ is a primitive Dirichlet character unramified outside S. Consider the partial standard L-function $L^S(s,\pi\times\xi) = \prod_{v\notin S} L_v(s,\pi\times\xi)$ with the unramified local L-factor defined as

$$L_v(s,\pi\times\xi) = (1-\xi(q_v)q_v^{-s})^{-1}\prod_{i=1}^{n}(1-\xi(q_v)\alpha_{v,i}q_v^{-s})^{-1}(1-\xi(q_v)\alpha_{v,i}^{-1}q_v^{-s})^{-1},$$

where q_v is the cardinality of the residue field and $\alpha_{v,1}^{\pm1},\cdots,\alpha_{v,n}^{\pm1}$ are the Satake parameters of π_v. The Euler product converges absolutely for $\mathrm{Re}\,(s)\gg 0$ and has a meromorphic continuation to the whole complex plane with at most simple poles [PSR87, KR90b].

Assume $\pi_\infty \cong \mathcal{D}_{\underline{t}}$, the holomorphic discrete series of weight $\underline{t} = (t_1,\cdots,t_n)$ (so $t_1\geq\cdots\geq t_n\geq n+1$). The right half critical set of $L^S(s,\pi\times\xi)$ consists of points

$$s_0\in\mathbb{Z},\quad 1\leq s_0\leq t_n-n \text{ and } (-1)^{s_0+n}=\xi(-1).$$

At these critical points it is known that $L^S(s,\pi\times\xi)$ has no poles and the critical values divided by certain automorphic periods (depending on π and s_0, but independent of ξ) are algebraic numbers [Har81, Shi00, BS00].

Fix an odd prime p, an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. We study the p-adic interpolation (up to an explicit factor) of the critical L-values $L^S(s_0,\pi\times\xi)$, as the p-part of ξ varies among all finite order characters of $\mathbb{Z}_p^\times$, the point s_0 varies in the right half critical set, and moreover the Hecke eigen-system associated to π varies in an ordinary p-adic family.

First for the automorphic representation π, we define the modified Euler factor at p for p-adic interpolation, under the ordinarity assumption on π, i.e. there exist $(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)\in(\mathcal{O}_{\overline{\mathbb{Q}}_p}^\times)^n$ and $\varphi\in\pi$ such the $\mathbb{U}_p$-operator $U_{p,\underline{a}}$ acts on φ by $\prod_{j=1}^n\mathfrak{a}_j^{a_j}$ for all $\underline{a}=(a_1,\cdots,a_n)\in\mathbb{Z}^n$, $a_1\geq a_2\geq\cdots\geq a_n\geq 0$ (see §2.5 for the definition of $U_{p,\underline{a}}$, especially the normalization which depends on the $K_{G,\infty}$-type of the automorphic form it acts on). As shown in §5.5, for a fixed π, if such an $(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)\in(\mathcal{O}_{\overline{\mathbb{Q}}_p}^\times)^n$ exists, then it must be unique, and the corresponding φ must be an eigenvector for the action of $T_G(\mathbb{Z}_p)$, where T_G is the standard maximal torus of G. Furthermore, the ordinarity condition on π implies that the local factor π_p can be embedded into the principal series $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\theta_1,\cdots,\theta_n)$. Here B_G is the the standard Borel subgroup of G. The character θ_j of $\mathbb{Q}_p^\times$, $1\leq j\leq n$, is defined as $\theta_j(p)=\alpha_j=p^{-(t_j-j)}\mathfrak{a}_j$, and $\theta_j|_{\mathbb{Z}_p^\times}=\psi_j$ where $\underline{\psi}=(\psi_1,\cdots,\psi_n)$ is the character of $T_G(\mathbb{Z}_p)$ through which it acts on φ.

Let $N\geq 3$ be a positive integer prime to p and ϕ be a Dirichlet character whose conductor divides N. Let χ be a Dirichlet character whose conductor is a power of p. For π,ϕ,χ, define the modified Euler factor at p as

$$E_p(s,\pi\times\phi^{-1}\chi^{-1}) = \frac{\left(1-\chi^\circ(p)\cdot\phi(p)p^{s-1}\right)\prod_{j=1}^n\left(1-(\chi\psi_j)^\circ(p)\cdot\phi(p)\alpha_j^{-1}p^{s-1}\right)}{\left(1-\chi^\circ(p)\cdot\phi(p)^{-1}p^{-s}\right)\prod_{j=1}^n\left(1-(\chi\psi_j)^\circ(p)\cdot\phi(p)^{-1}\alpha_j p^{-s}\right)} \times\left(\phi(p)p^{s-1}\right)^{c_\chi}G(\chi)\prod_{j=1}^n\left(\phi(p)\alpha_j^{-1}p^{s-1}\right)^{c_{\chi\psi_j}}G(\chi\psi_j). \tag{1.0.1}$$

Here $G(\chi)$ is the Gauss sum of χ. The integer c_χ is defined such that the conductor of χ is p^{c_χ}, and χ° takes the value 0 at p, unless $c_\chi=0$ in which case $\chi^\circ(p)=1$. Similarly we define $G(\chi\psi_j)$, $c_{\chi\psi_j}$ and $(\chi\psi_j)^\circ$, $1\le j\le n$. The $E_p(s,\pi\times\phi^{-1}\chi^{-1})$ defined above agrees with Coates' definition in [Coa91, §6] of the modified Euler factor at p for the Weil–Deligne representation associated to π_p twisted by the character $\phi^{-1}\chi^{-1}$ (note that the definition does not depend on the monodromy operator).

Let F be a finite extension of $\mathbb{Q}_p$ containing all N-th roots of unity. Denote by T_n the standard maximal torus of $\mathrm{GL}(n)$, which can be identified with T_G by the diagonal emdedding of $\mathrm{GL}(n)$ into G. Denote by Γ_{T_n} the p-profinite subgroup of $T_n(\mathbb{Z}_p)$ and set $\Lambda_n=\mathcal{O}_F[[\Gamma_{T_n}]]$. The $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-algebra $\mathbb{T}^N_{\mathrm{ord}}$, consisting of unramified Hecke operators and $\mathbb{U}_p$-operators acting on Hida families of tame principal level N, is finite and torsion free over Λ_n [Hid02,Pil12b]. A point $x\in\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}})(\overline{\mathbb{Q}}_p)$ corresponds to an eigen-system of the unramified Hecke operators and $\mathbb{U}_p$-operators. If that eigen-system comes from an irreducible cuspidal automorphic representation $\pi\in\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$, then it completely determines the isomorphism class of π_v for all $v\nmid N$ and we write π_x^N for the isomorphism class of the $G(\mathbb{A}^N)$-representation $\bigotimes'_{v\nmid N}\pi_v$.

Given a point $(\kappa,\underline{\tau})$ inside $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$, we say it is arithmetic if it can be written as the product of an algebraic character with a finite order character, and we write its algebraic part (resp. finite part) as $\kappa_{\mathrm{alg}}=k$, $\underline{\tau}_{\mathrm{alg}}=\underline{t}=(t_1,\cdots,t_n)$ (resp. $\kappa_{\mathrm{f}}=\chi$, $\underline{\tau}_{\mathrm{f}}=\underline{\psi}=(\psi_1,\cdots,\psi_n)$). A point is called admissible if it is arithmetic with $t_1\ge\cdots\ge t_n\ge k\ge n+1$. Given a geometrically irreducible component $\mathcal{C}$ of $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}}\otimes_{\mathcal{O}_F}F)$ with function field $F_{\mathcal{C}}$, our main result is

Theorem 1.0.1. *For every Dirichlet character ϕ with conductor dividing N such that ϕ^2 is non-trivial, and a pair (β_1,β_2) of positive definite symmetric $n\times n$ matrices with rational entries, there exists a p-adic measure $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}\in\mathcal{M}eas(\mathbb{Z}_p^\times,\Lambda_n)\otimes_{\Lambda_n}F_{\mathcal{C}}$ with the following interpolation properties. Suppose that the weight projection map $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}})\to\mathrm{Spec}(\mathcal{O}_F[[T_n(\mathbb{Z}_p)]])$ is étale at $x\in\mathcal{C}(\overline{\mathbb{Q}}_p)$. Let $\underline{\tau}\in\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ be the projection of x into the weight space. If $(\kappa,\underline{\tau})$ is admissible, then the evaluation of $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}$ at κ,x is*

$$\left(\int_{\mathbb{Z}_p^\times}\kappa\,d\mu_{\mathcal{C},\phi,\beta_1,\beta_2}\right)(x)=\phi(-1)^n\mathrm{vol}\left(\widehat{\Gamma}(N)\right)\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}\cdot\frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}} \times\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty},v_{\underline{t}}^\vee,v_{\underline{t}})}{\langle v_{\underline{t}}^\vee,v_{\underline{t}}\rangle}\cdot\sum_{\varphi\in\mathfrak{s}_x}\frac{\mathfrak{c}(\varphi,\beta_1)\mathfrak{c}(eW(\varphi),\beta_2)}{\langle\varphi,\overline{\varphi}\rangle} \times E_p(k-n,\pi_x^N\times\phi^{-1}\chi^{-1})\cdot L^{Np\infty}(k-n,\pi_x^N\times\phi^{-1}\chi^{-1}), \tag{1.0.2}$$

if $\phi\chi(-1)=(-1)^k$, and otherwise the evaluation is 0. Here

- *For a positive integer m the Gamma function $\Gamma_m(s)$ is defined as $\pi^{\frac{m(m-1)}{4}}\prod_{j=0}^{m-1}\Gamma(s-\frac{j}{2})$.*
- *$Z_\infty(f_{\kappa,\underline{\tau},\infty},v_{\underline{t}}^\vee,v_{\underline{t}})$ is the archimedean zeta integral for the doubling method with $v_{\underline{t}}$ being the highest weight vector inside the lowest $K_{G,\infty}$-type of $\mathcal{D}_{\underline{t}}$. Our choice of the archimedean section $f_{\kappa,\underline{\tau},\infty}$ in §4.3 guarantees its nonvanishing. When $\phi\chi(-1)=(-1)^k$ the section $f_{\kappa,\underline{\tau},\infty}$ depends only on the algebraic part $(k,\underline{t})$ of $(\kappa,\underline{\tau})$.*

- *The finite set* $\mathfrak{s}_x = \{\varphi_1, \cdots, \varphi_d\}$ *consists of an orthogonal basis of the space spanned by cuspidal holomorphic forms on* $G(\mathbb{A})$ *of weight* $\underline{t}$ *and tame principal level* N *belonging to the Hecke eigenspace parametrized by* x*. In this article we use the bi-*$\mathbb{C}$*-linear Petersson inner product with respect to the Haar measure of* $G(\mathbb{A})$ *specified in* **Notation**. *By being orthogonal we mean the basis satisfies* $\langle \varphi_i, \overline{\varphi_j} \rangle = 0$ *if* $i \neq j$.
- $\mathfrak{c}(\cdot, \beta_i)$ *is the* β_i*-th Fourier coefficient for* $i = 1, 2$*. The measure depends on the choice of the indices* β_1, β_2*, and in general there is no canonical choice for them due to the lack of a canonical nonvanishing Fourier coefficient for Siegel modular eigenforms, which can be regarded as the analogue of the first Fourier coefficient in the case of modular forms.*
- *The operator* $W : \pi \to \pi$ *is defined as*

$$W(\varphi)(g) := \int_{N_G(\mathbb{Z}_p)} \overline{\varphi}^{\vartheta}(gu)\, du,$$

where the form φ^{ϑ} *is the* MVW *involution of* φ*, i.e. the conjugation of* φ *by* $\vartheta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$*, and* N_G *is the unipotent radical of* B_G*. Proposition 5.7.2 shows that the ordinary projection* $eW(\varphi)$ *is nonzero if* φ *is ordinary . The operator* W *can be viewed as an analogue of the operator sending a modular form* f *of level* $\Gamma_0(N_f)$ *to* $f^c | \begin{pmatrix} 0 & -1 \\ N_f & 0 \end{pmatrix}$.

Remark 1.0.2. The condition $\phi^2 \neq 1$ is used to make sure that the p-adic Dirichlet L-functions which appear in our construction have no poles. Without this condition we can pick a prime number ℓ coprime to p, and get a measure $\mu_{\mathcal{C},\ell,\phi,\beta_1,\beta_2} \in \mathcal{M}eas(\mathbb{Z}_p^\times, \Lambda_n) \otimes_{\Lambda_n} F_{\mathcal{C}}$ with almost the same interpolation properties as described above, with the only difference that we need to add the factor $1 - \chi(\ell)^{-1}\ell^{-k+n}$ on the RHS of (1.0.2).

When π is fixed with $\underline{t}$ being a scalar weight and $\psi_1 = \cdots = \psi_n$, the one-variable p-adic L-function is constructed in [BS00, CP04] with a weaker ordinarity condition only requiring the eigenvalue of the operator $U_{p,n}$ to be a p-adic unit. The computations there are done with the Siegel upper half space.

Our work in the construction of $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}$ can also be viewed as a first step towards the Iwasawa–Greenberg Main Conjecture for $\mathrm{Sp}(2n)$ generalizing [Urb06, SU14]. Our focus here is not only to show the existence of the measure $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}$ with the interpolation properties described above, but also to show that in the construction all section selections of the doubling method are completely natural, by illustrating how differential operators show up in p-adic applications of the doubling method, how representation theory at the archimedean place guides the selection of suitable differential operators, employing the ideas in [Har86, Har08], and how section selections at the place p and the archimedean place are related for p-adic interpolation purposes. The strategy of section selections here should generalize to many other cases where differential operators are involved.

It is well known that the doubling method reduces the study of analytic properties of L-functions, as well as, algebraicity and p-adic interpolation of special L-values to that of the Siegel Eisenstein series on the corresponding doubling group. Set $H = \mathrm{Sp}(4n)_{/\mathbb{Q}}$ and fix the (holomorphic) embedding $\iota : G \times G \hookrightarrow H$. Let $P_H \subset H$ be the doubling Siegel parabolic. Pick a factorizable section $f(s,\xi)$ from the normalized induction $I_{P_H}(s,\xi) = \mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})}(\xi|\cdot|^s \circ \det)$. Let $E(\cdot, f(s,\xi))$ be the Siegel Eisenstein series and $E^*(\cdot, f(s,\xi))$ be the normalization of $E(\cdot, f(s,\xi))$ by multiplying the product of Dirichlet L-functions $d^S(s,\xi)$ (see §3 for precise definitions). Given $\varphi_1, \varphi_2 \in \pi$ with factorizable images under $\pi \cong \bigotimes'_v \pi_v$ and assuming all data outside S are unramified, the doubling method formula [PSR87, Gar84, Shi00] reads

$$\left\langle E^*(\iota(\cdot,\cdot), f(s,\xi)), \overline{\varphi}_1 \otimes \varphi_2^{\vartheta} \right\rangle = L^S(s+\frac{1}{2}, \pi\times\xi)\cdot \prod_{v\in S} \frac{Z_v(f_v(s,\xi), \overline{\varphi}_{1,v}, \varphi_{2,v})}{\left\langle \overline{\varphi}_{1,v}, \varphi_{2,v}\right\rangle_v} \cdot \langle \overline{\varphi}_1, \varphi_2\rangle . \tag{1.0.3}$$

From (1.0.3) one sees that a key point in applying the doubling method to attain various results about the L-function is to select suitable local sections $f_v(s,\xi)$ for $v \in S$, such that one can get a good handle on both the resulting normalized Siegel Eisenstein series on the left hand side, and the local zeta integrals on the right hand side of the formula.

The proof of Theorem 1.0.1 consists of two main steps. The first step is to pick suitable local sections $f_{\kappa,\underline{\tau},v} \in I_{P_H,v}(k-\frac{2n+1}{2}, \phi^{-1}\chi^{-1})$ for all admissible points $(\kappa,\underline{\tau})$ inside $\mathrm{Hom}(\mathbb{Z}_p^\times \times T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$, and to compute the Fourier coefficients of the resulting Siegel Eisenstein series as well as the corresponding local zeta integrals. Away from $Np\infty$ we always set $f_{\kappa,\underline{\tau},v}$ to be the unramified section. The two major criteria for selecting $f_{\kappa,\underline{\tau},v}$ for $v \mid Np\infty$ are the nonvanishing and p-adic interpolation conditions, i.e.

(1) the local zeta integral $Z_v(f_{\kappa,\underline{\tau},v}, \overline{\varphi}_{1,v}, \varphi_{2,v})$ does not vanish identically for $\varphi_1, \varphi_2 \in \mathfrak{s}_x$ if the projection of the point $x \in \mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}})(\overline{\mathbb{Q}}_p)$ to the weight space is $\underline{\tau}$, and
(2) the resulting $E^*(\cdot, f_{\kappa,\underline{\tau}})\big|_{G\times G}$ after a further normalization is algebraic and its q-expansion admits p-adic interpolation.

For $v \mid N$, a very simple choice is the so-called "volume section" (see §4.2).

Regarding the selection for $v = p, \infty$, one observation is that if at p we consider sections supported on the "big cell" then, due to the p-adic interpolation condition on q-expansions, the archimedean section $f_{\kappa,\underline{\tau},\infty}$ almost determines the p-adic section $f_{\kappa,\underline{\tau},p}$ and vice versa. Our strategy is to make choices for the archimedean sections incorporating both representation theory results and p-adic considerations.

In §5, with all local sections selected, we interpolate the q-expansions of the restriction to $G\times G$ of the corresponding Siegel Eisenstein series to the p-adic measure $\mu_{\mathcal{E},q\text{-exp}}$ on $\mathbb{Z}_p^\times \times T(\mathbb{Z}_p)$ valued in $\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]$. It serves as the input for applying the machinery of Hida theory in the next step. The second part of that section is devoted to the calculation of local zeta integrals at p, where an important observation of Böcherer–Schmidt (see §5.3) helps simplify the computation. The calculation results for all local zeta integrals are summarized in Proposition 5.2.3, which give the interpolation properties of the p-adic L-function we will finally construct.

In the second step we apply Hida theory to produce, from the q-expansion-valued measure $\mu_{\mathcal{E},q\text{-exp}}$, a p-adic measure on $\mathbb{Z}_p^\times$ valued in cuspidal ordinary families of p-adic Siegel modular forms on $G\times G$. Combining it with a p-adic analogue of the Petersson inner product, constructed from the geometrically irreducible component $\mathcal{C}$ of $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}}\otimes_{\mathcal{O}_F} F)$, we get the measure in Theorem 1.0.1.

For unitary groups there are also works done towards the construction of p-adic L-functions [HLS06, EHLS16, Eis15, Eis14, Eis16, EW16] and Klingen Eisenstein families [Wan15]. The paper [EHLS16] where the construction is completed was not yet available at the time this paper was written. Their results are not used in our construction. The computations of the factors at p done in [Wan15] assume restrictive conditions on the conductors of the nebentypes. The general cases are treated in [EHLS16] with an innovative use of the Godement–Jacquet local functional equation. It is claimed in [Eis16] that the method for section selections there also works for symplectic groups. We expect the sections chosen by that method (although the expressions seem more complicated) to be no different from ours here because, as we have pointed out, the choice of archimedean sections imposes sections at p via p-adic interpolation considerations, and based on the ideas in [Har86], the choice of archimedean sections here is quite canonical as explained in the proof of Proposition 4.3.1. The nonvanishing of the archimedean zeta integral is not particularly discussed in [Eis16], but should follow from the arguments in [Har08].

Notation. We fix an odd prime p and a positive integer $N \geq 3$ prime to p. We also fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and an isomorphism between $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$.

For a Dirichlet character ξ we write ξ° to denote the primitive one associated to it. We denote by C_ξ the conductor of ξ° and by $G(\xi)$ the Gauss sum of ξ°. If the conductor C_ξ is a power of p we define the integer c_ξ such that $C_\xi = p^{c_\xi}$. The Dirichlet characters we consider in the following will be almost all primitive with only one exception. For a finite order character inside $\mathrm{Hom}(\mathbb{Z}_p^\times, \zeta_{p^\infty})$ when we regard it as a Dirichlet character we require it to take value 0 at p.

Let $\mathbf{L}_n$ be the free $\mathbb{Z}$-module of rank $2n$ spanned by the basis $e_1, \cdots, e_n, f_1, \cdots, f_n$. We will always use this basis to write related objects in matrix form. Equip $\mathbf{L}_n$ with the symplectic pairing given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then $e_1, \cdots, e_n$ (resp. $f_1, \cdots, f_n$) span a maximal isotropic subspace L_n (resp. L_n^*) and we have the polarization $\mathbf{L}_n = L_n \oplus L_n^*$. We use G to denote the reductive group $G(\mathbf{L}_n) = \mathrm{Sp}(2n)$ defined over $\mathbb{Z}$. In matrix form it is

$$\left\{ g \in \mathrm{GL}(2n) : {}^{\mathrm{t}}g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Let Q_G be the standard Siegel parabolic subgroup of G preserving L_n. We identify its Levi subgroup with $\mathrm{GL}(n)$ via the map $\mathbf{p} : Q_G \to \mathrm{GL}(n)$ sending $\begin{pmatrix} a & b \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix}$ to a. Denote by B_n the Borel subgroup of $\mathrm{GL}(n)$ of upper triangular matrices, and by N_n, T_n its unipotent radical and maximal torus respectively. We fix the isomorphism of $\mathbb{G}_m^n$ with T_n which sends $(a_1, \cdots, a_n)$ to $\mathrm{diag}(a_1, \cdots, a_n)$. The inverse image under $\mathbf{p}$ of B_n constitutes the standard Borel subgroup B_G of G with unipotent radical N_G and maximal torus T_G. The tori T_n and T_G are identified via the map $\mathbf{p}$.

Let $\mathfrak{g}$ (resp. $\mathfrak{q}_G$) be the Lie algebra of G (resp. Q_G). We use E_{ij} to denote the matrix with 1 in the (i,j) entry and 0 elsewhere, whose size will be clear from the context. Fix the following basis of $\mathfrak{g}$

$$\begin{aligned}
&\eta_{ij} = E_{ij} - E_{j+n,i+n}, && && 1 \leq i,j \leq n, \\
&\mu_{ii}^+ = E_{i,i+n}, && \mu_{ii}^- = E_{i+n,i}, && 1 \leq i \leq n, \\
&\mu_{ij}^+ = E_{i,j+n} + E_{j,i+n}, && \mu_{ij}^- = E_{i+n,j} + E_{j+n,i}, && 1 \leq i < j \leq n.
\end{aligned}$$

For a positive integer m and an algebra R, denote by $\mathrm{Sym}(m, R)$ the set of $m \times m$ symmetric matrices with entries in R.

Consider the connected Shimura datum (G, u) with

$$\begin{aligned}
u : U(1, \mathbb{R}) &\to G^{\mathrm{ad}}(\mathbb{R}) \\
e^{i\theta} &\mapsto \begin{pmatrix} \cos\theta \cdot I_n & \sin\theta \cdot I_n \\ -\sin\theta \cdot I_n & \cos\theta \cdot I_n \end{pmatrix}.
\end{aligned}$$

The group $G(\mathbb{R})$ acts on u by conjugation. The centralizer

$$K_{G,\infty} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + bi \in U(n, \mathbb{R}) \right\}$$

is a maximal compact subgroup of $G(\mathbb{R})$, and the conjugacy class of u is $G(\mathbb{R})/K_{G,\infty}$, which is isomorphic to the Siegel upper half space

$$\mathbb{H}_n = \left\{ z \in M_n(\mathbb{C}) : {}^{\mathrm{t}}z = z, \ \mathrm{Im}\, z > 0 \right\}.$$

The group $G(\mathbb{R})$ acts on $\mathbb{H}_n$ by $g\cdot z=(az+b)(cz+d)^{-1}$ for $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G(\mathbb{R})$, $z\in\mathbb{H}_n$, and we put $\mu(g,z)=cz+d$.

Fix the standard additive character $\mathbf{e}_{\mathbb{A}}=\bigotimes_v \mathbf{e}_v:\mathbb{Q}\backslash\mathbb{A}\to\mathbb{C}^\times$ with local component $\mathbf{e}_v$ defined as $\mathbf{e}_v(x)=\begin{cases} e^{-2\pi i\{x\}_v}, & v\neq\infty\\ e^{2\pi i x}, & v=\infty\end{cases}$ where $\{x\}_v$ is the fractional part of x.

For a finite place v we fix the Haar measure on $\mathbb{Q}_v$ (resp. $G(\mathbb{Q}_v)$) with $\mathbb{Z}_v$ (resp. $G(\mathbb{Z}_v)$) having volume 1. For the archimedean place we take the usual Lebesgue measure for $\mathbb{R}$. For the group $G(\mathbb{R})$ we take the product measure where the one on $K_{G,\infty}$ has total volume 1 and the one on the $\mathbb{H}_n$ is $\det(y)^{-n-1}\prod_{1\leq i\leq j\leq n} dx_{ij}\,dy_{ij}$. The Haar measures on $\mathbb{A}$ and $G(\mathbb{A})$ are obtained by taking products of the local ones. For the unipotent group $N_G(\mathbb{Q}_v)$ (resp. $U_G(\mathbb{Q}_v)$), we always take the Haar measure that gives the open compact subgroup $N_G(\mathbb{Z}_v)$ (resp. $U_G(\mathbb{Z}_v)$) volume 1 if v is finite, and the Haar measure $d_\infty u(x)=\prod_{1\leq i\leq j\leq n} dx_{ij}$ for the archimedean place, where $u(x)=\begin{pmatrix} I_n & x\\ 0 & I_n\end{pmatrix}$ for $x\in\mathrm{Sym}(n,\mathbb{R})$.

Similarly we have all the above definitions for $H=\mathrm{Sp}(4n)$.

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2. Nearly holomorphic Siegel modular forms

As a preparation for our following constructions we introduce the space of nearly holomorphic Siegel modular forms and the Maass–Shimura differential operators. With our choice of archimedean sections in §4.3 the Siegel Eisenstein series on H and their restrictions to $G\times G$ are in general not holomorphic but nearly holomorphic. The action of $\mathfrak{q}_G^+$ (see §2.4 for definition) will be applied for choosing the archimedean sections. We show how to translate the $\mathfrak{q}_G^+$-action on $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ to the Maass–Shimura differential operators (defined by Shimura) acting on smooth functions on the Siegel upper half space, and to the Gauss–Manin connections associated to the automorphic sheaves of nearly holomorphic forms.

Besides we define q-expansions, $\mathbb{U}_p$-operators and ordinary projections for nearly holomorphic forms. The $\mathbb{U}_p$-operators are defined both geometrically and adelically. One crucial aspect for the definition of the $\mathbb{U}_p$-operators is the normalization.

For more detailed treatment see [Liu15]. We formulate the theory for G and it is clear that everything applies to H and $G\times G$.

2.1. **Siegel modular variety and automorphic sheaves.** Let $\mathbf{G}=\mathrm{GSp}(2n)$ with the multiplier character $\nu:\mathbf{G}\to\mathbb{G}_m$ and $\mathbf{Q}$ be its standard Siegel parabolic subgroup consisting of matrices whose lower left $n\times n$ blocks are zero. Set $\Gamma=\Gamma_1(N,p^m)$ to be the congruence subgroup $\{\gamma\in\mathrm{Sp}(2n,\mathbb{Z}):\gamma\equiv I_{2n}\mod N,\ \ \gamma\mod p^m\in N_G(\mathbb{Z}/p^m\mathbb{Z})\}$. Denote by $Y_{\mathbf{G},\Gamma}$ the Siegel modular variety parametrizing principally polarized abelian schemes of dimension n with level Γ structure defined over $\mathbb{Q}$. Over it there is the universal abelian scheme $(\mathcal{A},\lambda,\psi_N,\mathrm{fil}^+_{p^m})$ where λ is a principal polarization, ψ_N is an isomorphism $\mathbf{L}_n\otimes\mathbb{Z}/N\mathbb{Z}\xrightarrow{\sim}\mathcal{A}[N]$ respecting the Weil pairing up to similitude, and $\mathrm{fil}^+_{p^m}$ is a full flag with trivialization of graded pieces of an isotropic free $\mathbb{Z}/p^m\mathbb{Z}$-submodule of $\mathcal{A}[p^m]$ of rank n.

Take $X_{\mathbf{G},\Gamma}$ to be a smooth toroidal compactification of $Y_{\mathbf{G},\Gamma}$ with boundary C (see [FC90,Lan12]). The universal abelian scheme $\mathcal{A}$ extends to a semi-abelian scheme $\mathbf{p}: \mathcal{G} \to X_{\mathbf{G},\Gamma}$ with a canonical section $\mathbf{e}$. Let $\omega(\mathcal{G}/X_{\mathbf{G},\Gamma}) = \mathbf{e}^*\Omega^1_{\mathcal{G}/X_{\mathbf{G},\Gamma}}$ be the sheaf of invariant differentials, which is locally free of rank n. The sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma}) = R^1\mathbf{p}_*(\Omega^\bullet_{\mathcal{A}/Y_{\mathbf{G},\Gamma}})$ has a canonical extension $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})^{\mathrm{can}}$, which is a locally free sheaf over $X_{\mathbf{G},\Gamma}$ of rank $2n$ equipped with the Hodge filtration

(2.1.1) $$0 \longrightarrow \omega(\mathcal{G}/X_{\mathbf{G},\Gamma}) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})^{\mathrm{can}} \longrightarrow \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X_{\mathbf{G},\Gamma}) \to 0,$$

and a symplectic pairing such that $\omega(\mathcal{G}/X_{\mathbf{G},\Gamma})$ is maximally isotropic. The Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})$ also extends to an integrable connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})^{\mathrm{can}}$ with log poles along the boundary

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})^{\mathrm{can}} \otimes_{\mathcal{O}_{X_{\mathbf{G},\Gamma}}} \Omega^1_{X_{\mathbf{G},\Gamma}}(\log C).$$

There is a standard way to attach an automorphic sheaf to an object in $\mathrm{Rep}_{\mathbb{Q}}\,\mathbf{Q}$, the category of algebraic representations of $\mathbf{Q}$ over $\mathbb{Q}$-vector spaces. Over $X_{\mathbf{G},\Gamma}$ we have the right $\mathbf{Q}$-torsor

$$T^\times_{\mathcal{H}} = \underline{\mathrm{Isom}}_{X_{\mathbf{G},\Gamma}}\left(\mathbf{L}_n \otimes_{\mathbb{Z}} \mathcal{O}_{X_{\mathbf{G},\Gamma}}, \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{\mathbf{G},\Gamma})^{\mathrm{can}}\right),$$

where the isomorphisms are required to respect the Hodge filtration and the symplectic pairing up to similitude. Using the contracted product one defines the functor

$$\begin{aligned} \mathcal{E} : \mathrm{Rep}_{\mathbb{Q}}\,\mathbf{Q} &\longrightarrow \mathrm{QCoh}(X_{\mathbf{G},\Gamma}) \\ V &\longmapsto T^\times_{\mathcal{H}} \times^{\mathbf{Q}} V \end{aligned}$$

sending a $\mathbf{Q}$-representation to a locally free sheaf over $X_{\mathbf{G},\Gamma}$. We will also write $\mathcal{E}(V)$ as $\mathcal{V}$. For a prime number ℓ with $(\ell, Np) = 1$ and $\gamma_\ell \in \mathbf{G}(\mathbb{Z}_\ell)\backslash\,\mathbf{G}(\mathbb{Q}_\ell)/\,\mathbf{G}(\mathbb{Z}_\ell)$ the Hecke action of T_{γ_ℓ} on $H^0(X_{\mathbf{G},\Gamma}, \mathcal{V})$ can be defined in the standard way using algebraic correspondence. We call such a $\mathcal{V}$, together with the Hecke actions on its global sections, an automorphic sheaf over $X_{\mathbf{G},\Gamma}$. The multiplier character ν is a character of $\mathbf{Q}$ and so gives an invertible automorphic sheaf $\mathcal{E}(\nu)$. As a coherent sheaf, $\mathcal{E}(\nu)$ is isomorphic to the structure sheaf but the Hecke actions differ by a Tate twist. We use $\mathcal{V}(i)$ to denote $\mathcal{V} \otimes \mathcal{E}(\nu)^{\otimes i}$.

Let $X_{G,\Gamma}$ be a connected component of the base change of $X_{\mathbf{G},\Gamma}$ to $\mathbb{Q}(\zeta_N)$. Here we do all the constructions over $X_{\mathbf{G},\Gamma}$. For applications later we restrict everything to $X_{G,\Gamma}$. What we need to be careful about is the Hecke operators. For $v \nmid N$, over $X_{\mathbf{G},\Gamma}$ we consider operators corresponding to elements inside $\mathbf{G}(\mathbb{Q}_v)$, while over the connected component $X_{G,\Gamma}$ we only consider those inside $G(\mathbb{Q}_v)$.

2.2. **Nearly holomorphic forms and differential operators.** If we want to consider automorphic sheaves further endowed with an integrable connection, the right objects to consider are $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-modules. A $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-module V is an object in $\mathrm{Rep}_{\mathbb{Q}}\,\mathbf{Q}$ with an extra action of $\mathrm{Lie}\,\mathbf{G}$ such that its restriction to $\mathrm{Lie}\,\mathbf{Q}$ agrees with the one induced from the action of $\mathbf{Q}$, and the compatibility condition

$$g \cdot X \cdot g^{-1} \cdot v = \big(Ad(g)X\big) \cdot v$$

holds for all $v \in V$, $X \in \mathrm{Lie}\,\mathbf{G}$ and $g \in \mathbf{Q}$. Denote by $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$ the category of $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-modules.

Suppose V is a $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-module. It follows from [Liu15, Proposition 2.2.3] that there is the Gauss–Manin connection for the locally free sheaf $\mathcal{E}(V)$,

$$\nabla : \mathcal{E}(V) \longrightarrow \mathcal{E}(V) \otimes_{\mathcal{O}_{X_{\mathbf{G},\Gamma}}} \Omega^1_{X_{\mathbf{G},\Gamma}}(\log C)$$

which induces Hecke equivariant maps on global sections. Its construction uses the Gauss–Manin connection (2.1.1) and the $\mathrm{Lie}\,\mathbf{G}$-module structure of V.

Now let (σ, W_σ) be a finite dimensional algebraic representation of $\mathrm{GL}(n)$. We define the $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-module V_σ as follows. Let $\underline{Y} = (Y_{ij})_{1\le i,j\le n}$ the symmetric $n \times n$ matrix with the (i,j) and (j,i) entries being the indeterminate $Y_{ij} = Y_{ji}$. As a $\mathbb{Q}$-vector space $V_\sigma = W_\sigma[\underline{Y}]$, the space of polynomials in Y_{ij} with coefficients in W_σ. The action of $\mathbf{Q}$ is defined as

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot P(\underline{Y}) = a \cdot P(a^{-1}b + a^{-1}\,\underline{Y} d)$$

for $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbf{Q}$ and $P(\underline{Y}) \in V_\sigma$. The element μ^-_{ij} acts on $P(\underline{Y})$ by

$$(\mu^-_{ij} \cdot P)(\underline{Y}) = \sum_{1\le k\le n} (Y_{ki}\eta_{kj} + Y_{kj}\eta_{ki}) \cdot P(\underline{Y}) - \sum_{1\le k\le l\le n} (Y_{ki}Y_{jl} + Y_{kj}Y_{il}) \frac{\partial}{\partial Y_{kl}} P(\underline{Y}) \;\; i \neq j,$$

$$(\mu^-_{ii} \cdot P)(\underline{Y}) = \sum_{1\le k\le n} Y_{ki}\eta_{ki} \cdot P(\underline{Y}) - \sum_{1\le k\le l\le n} Y_{ki}Y_{il} \frac{\partial}{\partial Y_{kl}} P(\underline{Y}).$$

It is easy to check that the above formulas define a $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-module structure on V_σ.

As a $\mathbf{Q}$-representation, V_σ admits an increasing filtration $V^r_\sigma := W_\sigma[\underline{Y}]_{\le r}$, $r \ge 0$, where the subscript $\le r$ means polynomials in $Y_{ij}, 1 \le i, j \le n$ with total degree less or equal to r. We have $\mathfrak{g} \cdot V^r_\sigma \subset V^{r+1}_\sigma$.

The locally free sheaf over $X_{\mathbf{G},\Gamma}$ of (nonholomorphy) degree r nearly holomorphic forms valued in W_σ is defined to be $\mathcal{V}^r_\sigma = \mathcal{E}(V^r_\sigma)$. The connection on $\mathcal{V}_\sigma$ restricts to

$$\nabla_\sigma : \mathcal{V}^r_\sigma \longrightarrow \mathcal{V}^{r+1}_\sigma \otimes_{\mathcal{O}_{X_{\mathbf{G},\Gamma}}} \Omega^1_{X_{\mathbf{G},\Gamma}}(\log C).$$

Let τ_n be the symmetric square of the standard representation of $\mathrm{GL}(n)$. Combining the connection with the Kodaira–Spencer isomorphism we get the differential operator

$$D_\sigma : \mathcal{V}^r_\sigma \xrightarrow{\nabla_\sigma} \mathcal{V}^{r+1}_\sigma \otimes_{X_{\mathbf{G},\Gamma}} \Omega^1_{X_{\mathbf{G},\Gamma}}(\log C) \xrightarrow{\mathrm{KS}} \mathcal{V}^{r+1}_{\sigma\otimes\tau_n}(-1) \longrightarrow \mathcal{V}^{r+1}_{\sigma\otimes\tau_n}.$$

The map on global sections induced by D_σ fails to be Hecke equivariant by a Tate twist because of the last morphism above. By iteration one can define the differential operator $D^e_\sigma : \mathcal{V}^r_\sigma \to \mathcal{V}^{r+e}_{\sigma\otimes\mathrm{Sym}^e\,\tau_n}$.

Given a dominant weight $\underline{t} = (t_1, \cdots, t_n) \in X(T_n)_+$ with respect to B_n, set $\underline{t}' = (-t_n, \cdots, -t_1)$ and define

$$W_{\underline{t}} := \left\{ f : \mathrm{GL}(n)/N_n \to \mathbb{A}^1 : f(gx) = \underline{t}'(x) f(g) \text{ for all } x \in T_n \right\}$$

with $g \in \mathrm{GL}(n)$ acting by left inverse translation. Then $W_{\underline{t}}$ is an irreducible finite dimensional representation of $\mathrm{GL}(n)$ with highest weight $\underline{t}$. Evaluation at I_n gives a canonical element in its dual representation and we denote it by $\mathfrak{e}_{\mathrm{can}}$. From $W_{\underline{t}}$ one constructs the $(\mathrm{Lie}\,\mathbf{G}, \mathbf{Q})$-module $V_{\underline{t}}$ and its sub-$\mathbf{Q}$-representations $V^r_{\underline{t}}$.

Definition 2.2.1. *The automorphic sheaf over $X_{\mathbf{G},\Gamma}$ of weight $\underline{t}$, (non-holomorphy) degree r nearly holomorphic forms is defined to be $\mathcal{V}^r_{\underline{t}} = \mathcal{E}(V^r_{\underline{t}})$.*

Put $\omega_{\underline{t}} = \mathcal{V}^0_{\underline{t}}$. It is the sheaf of holomorphic Siegel modular forms of weight $\underline{t}$.

Denote by τ^*_n the dual representation of τ_n. The natural $\mathbf{Q}$-representation morphism $V^r_\sigma \to V^r_\sigma/V^0_\sigma \hookrightarrow V^{r-1}_{\sigma\otimes\tau^*_n}(-1)$ induces an $\mathcal{O}_{X_{\mathbf{G},\Gamma}}$-linear operator

$$E_\sigma : \mathcal{V}^r_\sigma \longrightarrow \mathcal{V}^{r-1}_{\sigma\otimes\tau^*_n}(-1) \longrightarrow \mathcal{V}^{r-1}_{\sigma\otimes\tau^*_n}, \tag{2.2.1}$$

whose induced map on global sections fails to commute with Hecke actions by a Tate twist. Given a σ-valued nearly holomorphic Siegel modular form, it is holomorphic if and only if it is annihilated by the operator E_σ.

2.3. **Equivalence to Shimura's theory on Siegel upper half space.** Let us recall Shimura's definition of nearly holomorphic forms and Maass–Shimura differential operators [Shi00]. Let $C^\infty_\sigma(\mathbb{H}_n,\Gamma)$ be the $\mathbb{C}$-vector space of smooth functions $f:\mathbb{H}_n\to W_\sigma(\mathbb{C})$ satisfying the transformation property $f(\gamma\cdot z)=\sigma\big(\mu(\gamma,z)\big)f(z)$. Shimura defines the space $N^r_\sigma(\mathbb{H}_n,\Gamma)$ of σ-valued, degree r and level Γ nearly holomorphic Siegel modular forms to be the subspace of $C^\infty_\sigma(\mathbb{H}_n,\Gamma)$ consisting of those functions that can be written as a polynomial in the entries of $(\operatorname{Im} z)^{-1}$ of degree less or equal to r with coefficients being holomorphic maps from $\mathbb{H}_n$ to $W_\sigma(\mathbb{C})$ (if $n=1$ a growth condition at ∞ is also needed).

The Maass–Shimura differential operator is defined as

$$\begin{aligned} D_{\sigma,\mathbb{H}_n}: C^\infty_\sigma(\mathbb{H}_n,\Gamma) &\longrightarrow C^\infty_{\sigma\otimes\tau_n}(\mathbb{H}_n,\Gamma)\\ f&\longmapsto \sigma(\operatorname{Im} z)^{-1}d_z\big(\sigma(\operatorname{Im} z)f\big),\end{aligned}$$

where d_z stands for $\sum_{1\le i\le j\le n} dz_{ij}\cdot\frac{\partial}{\partial z_{ij}}$. It restricts to $D_{\sigma,\mathbb{H}_n}: N^r_\sigma(\mathbb{H}_n,\Gamma)\to N^{r+1}_{\sigma\otimes\tau_n}(\mathbb{H}_n,\Gamma)$.

The base change of $\mathcal{A}\to Y_{G,\Gamma}$ to the field of complex numbers is isomorphic to $\Gamma\backslash\mathbb{C}^n\times\mathbb{H}_n/\mathbb{Z}^{2n}\to\Gamma\backslash\mathbb{H}_n$. Here $(m_1,m_2)\in\mathbb{Z}^{2n}$ and $\gamma\in\Gamma$ act on $(w,z)\in\mathbb{C}^n\times\mathbb{H}_n$ by

$$\begin{aligned}(w,z)\cdot(m_1,m_2)&=(w+m_1z+m_2,z),\\ \gamma\cdot(w,z)&=(w\mu(\gamma,z)^{-1},\gamma\cdot z).\end{aligned}$$

Over $\mathbb{H}_n$ there is the principally polarized abelian scheme $A_{\mathbb{H}_n}=\mathbb{C}^n\times\mathbb{H}_n/\mathbb{Z}^{2n}$ with a canonical basis $dw_1,\cdots,dw_n$ for the sheaf of invariant differentials $\omega(A_{\mathbb{H}_n}/\mathbb{H}_n)$. The Kodaira–Spencer isomorphism identifies dw_idw_j with $2\pi i\cdot dz_{ij}$. As in [Liu15, §2.5] a canonical test object can be constructed from $A_{\mathbb{H}_n}$. Using the modular interpretation à la Katz for the global sections of automorphic sheaves, the evaluation of the sheaf-theoretically defined nearly holomorphic forms at that test object defines the map

$$H^0(X_{G,\Gamma},\mathcal{V}^r_\sigma)\otimes_{\mathbb{Q}(\zeta_N)}\mathbb{C}\to N^r_\sigma(\mathbb{H}_n,\Gamma). \tag{2.3.1}$$

We summarize the results there in the proposition below.

Proposition 2.3.1. *The map* (2.3.1) *is an isomorphism and the diagram below commutes.*

$$\begin{array}{ccc} H^0(X_{G,\Gamma},\mathcal{V}^r_\sigma)\otimes_{\mathbb{Q}(\zeta_N)}\mathbb{C} & \xrightarrow{\ \sim\ } & N^r_\rho(\mathbb{H}_n,\Gamma)\\ \big\downarrow D_\sigma & & \big\downarrow D_{\sigma,\mathbb{H}_n}\\ H^0(X_{G,\Gamma},\mathcal{V}^{r+1}_{\sigma\otimes\tau_n})\otimes_{\mathbb{Q}(\zeta_N)}\mathbb{C} & \xrightarrow{\ \sim\ } & N^{r+1}_{\rho\otimes\tau_n}(\mathbb{H}_n,\Gamma)\end{array}$$

2.4. **Equivalence to the action of $\mathfrak{q}^+_G$.** Let $C^\infty(\Gamma\backslash G(\mathbb{R}))$ be the $\mathbb{C}$-vector space of smooth functions on $G(\mathbb{R})$ that are invariant under the left translation by Γ. Let W^*_σ be the dual representation of $W_\sigma(\mathbb{C})$ and $\langle,\rangle: W_\sigma\times W^*_\sigma\to\mathbb{A}^1$ be the canonical pairing. For each $w^*\in W^*_\sigma(\mathbb{C})$ there is the embedding

$$\varphi_G(\cdot,w^*): C^\infty_\sigma(\mathbb{H}_n,\Gamma)\to C^\infty(\Gamma\backslash G(\mathbb{R})), \tag{2.4.1}$$

defined as

$$\varphi_G(f,w^*)(g)=\Big\langle \sigma\left(\mu(g,i)\right)^{-1}f(g\cdot i),\, w^*\Big\rangle$$

for $f\in C^\infty_\sigma(\mathbb{H}_n,\Gamma)$ and $g\in G(\mathbb{R})$. The maximal compact subgroup $K_{G,\infty}$ acts on $W_\sigma(\mathbb{C})$, $W^*_\sigma(\mathbb{C})$ via the isomorphism $K_{G,\infty}\cong U(n,\mathbb{R})\subset \mathrm{GL}(n,\mathbb{C})$. One can check that for $k\in K_{G,\infty}$ we have

$$\varphi_G(f,w^*)(gk)=\varphi_G(f,{}^{\mathrm{t}}k^{-1}\cdot w^*)(g).$$

Therefore if we put $V_f=\{\varphi_G(f,w^*): w^*\in W^*_\sigma(\mathbb{C})\}$ then it is a subspace of $C^\infty(\Gamma\backslash G(\mathbb{R}))$ closed under the action of $K_{G,\infty}$ and is isomorphic to $W_\sigma(\mathbb{C})$ as a $K_{G,\infty}$-representation.

The torus $\mathbb{C}^\times$ acts on $G(\mathbb{R})$ by

$$(x+iy)\cdot g = \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix} g \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix}^{-1},$$

inducing an action of $\mathbb{C}^\times$ on $\mathfrak{g}_\mathbb{C}$. Let $\mathfrak{g}_\mathbb{C}^{a,b}$ be the subspace of $\mathfrak{g}_\mathbb{C}$ on which $z\in\mathbb{C}^\times$ acts by the scalar $z^{-a}\overline{z}^{-b}$. Then $\mathfrak{g}_\mathbb{C}$ decomposes as $\mathfrak{g}_\mathbb{C}^{-1,1}\oplus\mathfrak{g}_\mathbb{C}^{0,0}\oplus\mathfrak{g}_\mathbb{C}^{1,-1}$. We have $\mathfrak{g}_\mathbb{C}^{0,0}=\mathfrak{k}_{G,\mathbb{C}}$, the complexified Lie algebra of $K_{G,\infty}$. Set $\mathfrak{q}_G^+=\mathfrak{g}_\mathbb{C}^{-1,1}$ and $\mathfrak{q}_G^-=\mathfrak{g}_\mathbb{C}^{1,-1}$. The aim of this section is to show that the $\mathfrak{q}_G^+$-action on $C^\infty(\Gamma, G(\mathbb{R}))$ translates to the Maass–Shimura differential operators on $C^\infty_\sigma(\mathbb{H}_n,\Gamma)$ under the embedding (2.4.1). This is explained in [Shi00, A.8] but we include a proof here for the convenience of our later application.

Fix a basis $\underline{X}=(X_{ij})_{1\le i,j\le n}$, $X_{ij}=X_{ji}$ of τ_n with $a\in\mathrm{GL}(n)$ acting on it by ${}^{\mathrm{t}}a\,\underline{X}a$. We will assume that under the trivialization of $\omega(A_{\mathbb{H}_n}/\mathbb{H}_n)$ by the basis $dw_1,\cdots,dw_n$, the element X_{ij} corresponds to $dw_idw_j=2\pi i\cdot dz_{ij}$. Denote by $\underline{X}^*=(X^*_{ij})_{1\le i,j\le n}$, $\underline{X}^*_{ij}=\underline{X}^*_{ji}$ the basis of τ_n^* dual to $\underline{X}$.

Let $\mathfrak{c}=\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & iI_n \\ iI_n & I_n\end{pmatrix}$ and $\widehat{\mu}^+_{ij}=\mathfrak{c}\mu_{ij}\mathfrak{c}^{-1}$. Then $\widehat{\mu}^+_{ij}$, $1\le i\le j\le n$, span $\mathfrak{q}_G^+$.

Proposition 2.4.1. *The diagram*

$$\begin{array}{ccc} C^\infty_\sigma(\mathbb{H}_n,\Gamma) & \xrightarrow{\varphi_G(\cdot,w^*)} & C^\infty(\Gamma\backslash G(\mathbb{R})) \\ \Big\downarrow{\scriptstyle 4\pi i\cdot D_{\sigma,\mathbb{H}_n}} & & \Big\downarrow{\scriptstyle \widehat{\mu}^+_{ij}} \\ C^\infty_{\sigma\otimes\tau_n}(\mathbb{H}_n,\Gamma) & \xrightarrow{\varphi_G(\cdot,w^*\otimes X^*_{ij})} & C^\infty(\Gamma\backslash G(\mathbb{R})) \end{array}$$

commutes.

Proof. We need to show the identity

$$4\pi i\cdot\varphi_G(D_{\sigma,\mathbb{H}_n}f,w^*\otimes X^*_{ij})(g)=\widehat{\mu}^+_{ij}\varphi_G(f,w^*)(g) \tag{2.4.2}$$

for all $f\in C^\infty_\sigma(\mathbb{H}_n,\Gamma)$, $g\in G(\mathbb{R})$ and $1\le i\le j\le n$. Notice that for all $k\in K_{G,\infty}$ we have

$$\varphi_G(D_{\sigma,\mathbb{H}_n}f,w^*\otimes X^*_{ij})(gk)=\varphi_G(D_{\sigma,\mathbb{H}_n}f,{}^{\mathrm{t}}k^{-1}\cdot(w^*\otimes X^*_{ij}))(g),$$

and

$$\widehat{\mu}^+_{ij}\varphi_G(f,w^*)(gk)=\big(Ad(k)\widehat{\mu}^+_{ij}\big)\varphi_G(f,{}^{\mathrm{t}}k^{-1}\cdot w^*)(g).$$

Thus it is enough to show (2.4.2) for $g\in Q_G(\mathbb{R})$.

Write elements in $Q_G(\mathbb{R})$ as $g=\begin{pmatrix} a & x\,{}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}$ with $a\in\mathrm{GL}(n,\mathbb{R})$ and x an $n\times n$ symmetric matrix with real coefficients. Put $y=a\,{}^{\mathrm{t}}a$ which is positive definite symmetric. Then $z=x+iy$ belongs to $\mathbb{H}_n$, and by definition

$$\begin{aligned}\varphi_G(D_{\sigma,\mathbb{H}_n}f,w^*\otimes\mu(g,i)^{-1}\cdot X^*_{ij})(g) &= \big\langle\sigma\otimes\tau_n\left(\mu(g,i)^{-1}\right)\sigma(y)^{-1}d_z\big(\sigma(y)f(z)\big),w^*\otimes\mu(g,i)^{-1}\cdot X^*_{ij}\big\rangle \\ &= \Big\langle\sigma\left(\mu(g,i)\right)^{-1}\sigma(y)^{-1}d_z\big(\sigma(y)f(z)\big),w^*\otimes X^*_{ij}\Big\rangle \\ &= \frac{1}{2\pi i}\left\langle\sigma(a)^{-1}\frac{\partial}{\partial z_{ij}}\big(\sigma(y)f(z)\big),w^*\right\rangle.\end{aligned}$$

Given $\alpha\in\mathrm{GL}(n,\mathbb{C})$ we define $\alpha\cdot\widehat{\mu}^+_{ij}$ to be $\mathfrak{c}\begin{pmatrix}{}^{\mathrm{t}}\alpha^{-1} & 0\\ 0 & \alpha\end{pmatrix}\mu^+_{ij}\begin{pmatrix}{}^{\mathrm{t}}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\mathfrak{c}^{-1}$. It is easy to see that under this definition if $\alpha\cdot\widehat{\mu}^+_{ij}=\sum_{1\le i\le j\le n}c_{ij}\widehat{\mu}^+_{ij}$ with $c_{ij}\in\mathbb{C}$, then $\alpha\cdot X^*_{ij}=\sum_{1\le i\le j\le n}c_{ij}X^*_{ij}$.

Let ε_{ij}, $1 \le i \le j \le n$ be variables and we write ε to mean the $n \times n$ symmetric matrix whose (i,j) and (j,i) entries are ε_{ij}. Then we have

$$\mu(g,i)^{-1} \cdot \sum_{1 \le i \le j \le n} \varepsilon_{ij} \widehat{\mu}_{ij}^{+} = -\frac{i}{2} \begin{pmatrix} a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} & 0 \\ 0 & -a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} \\ a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} & 0 \end{pmatrix}.$$

Now we compute

$$\begin{aligned}
&\big(\mu(g,i)^{-1} \cdot \widehat{\mu}_{ij}^{+}\big)\varphi_G(f,w^*)(g) \\
= &-\frac{i}{2}\frac{\partial}{\partial \varepsilon_{ij}} \varphi_G(f,w^*)\left(\begin{pmatrix} a & x {}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix} \exp \begin{pmatrix} a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} & 0 \\ 0 & -a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} \end{pmatrix} \right)\Bigg|_{\varepsilon=0} \\
&+ \frac{1}{2}\frac{\partial}{\partial \varepsilon_{ij}} \varphi_G(f,w^*)\left(\begin{pmatrix} a & x {}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} \\ a^{-1}\varepsilon {}^{\mathrm{t}}a^{-1} & 0 \end{pmatrix} \right)\Bigg|_{\varepsilon=0} \\
= &-\frac{i}{2}\frac{\partial}{\partial \varepsilon_{ij}} \left\langle \sigma(a)^{-1}\sigma(y+\varepsilon) f(z+2i\varepsilon), w^* \right\rangle\Big|_{\varepsilon=0} + \frac{1}{2}\frac{\partial}{\partial \varepsilon_{ij}} \left\langle \sigma(a)^{-1}\sigma(y-i\varepsilon) f(z+2\varepsilon), w^* \right\rangle\Big|_{\varepsilon=0} \\
= &-\frac{i}{2}\left\langle \sigma(a)^{-1}\frac{\partial}{\partial y'_{ij}}\sigma(y') f(z), w^* \right\rangle\Bigg|_{y'=y} - i \left\langle \sigma(a)^{-1}\sigma(y)\frac{\partial}{\partial y_{ij}} f(z), w^* \right\rangle \\
&-\frac{i}{2}\left\langle \sigma(a)^{-1}\frac{\partial}{\partial y'_{ij}}\sigma(y') f(z), w^* \right\rangle\Bigg|_{y'=y} + \left\langle \sigma(a)^{-1}\sigma(y)\frac{\partial}{\partial x_{ij}} f(z), w^* \right\rangle \\
= &\; 2\left\langle \sigma(a)^{-1}\left(\frac{1}{2}\frac{\partial}{\partial x'_{ij}} - \frac{i}{2}\frac{\partial}{\partial y'_{ij}}\right)\sigma(y') f(z), w^* \right\rangle\Bigg|_{z'=z} + 2\left\langle \sigma(a)^{-1}\sigma(y)\left(\frac{1}{2}\frac{\partial}{\partial x_{ij}} - \frac{i}{2}\frac{\partial}{\partial y_{ij}}\right) f(z), w^* \right\rangle \\
= &\; 2\left\langle \sigma(a)^{-1}\frac{\partial}{\partial z_{ij}}\big(\sigma(y) f(z)\big), w^* \right\rangle.
\end{aligned}$$

Therefore for a given $g \in Q(\mathbb{R})$ we have the identity

$$4\pi i \cdot \varphi_G(D_{\sigma,\mathbb{H}_n} f, w^* \otimes \mu(g,i)^{-1} \cdot X_{ij}^*)(g) = \big(\mu(g,i)^{-1} \cdot \widehat{\mu}_{ij}^{+}\big)\varphi_G(f,w^*)(g)$$

for all $1 \le i \le j \le n$, from which (2.4.2) follows. □

Remark 2.4.2. A similar computation as above shows that for the action of $\mathfrak{q}_G^-$ on $C^\infty(\Gamma\backslash G(\mathbb{R}))$ we have

$$\Big(\widehat{\mu}_{ij}^{-}\varphi_G(f,w^*)\Big)_{1 \le i,j \le n} = \mu(g,i)^{-1}\left(\varphi_G\Big(\frac{\partial f}{\partial \bar{z}_{ij}}, w^*\Big)\right)_{1 \le i,j \le n} {}^{\mathrm{t}}\mu(g,i)^{-1}.$$

Up to scalars the action of $\mathfrak{q}_G^-$ on $C^\infty(\Gamma\backslash G(\mathbb{R}))$ corresponds to the operator

$$\begin{aligned} E_{\sigma,\mathbb{H}_n} : C_\sigma^\infty(\mathbb{H}_n,\Gamma) &\longrightarrow C_{\sigma\otimes\tau_n^*}^\infty(\mathbb{H}_n,\Gamma) \\ f &\longmapsto d_{\bar{z}} f, \end{aligned}$$

which translates to the operator E_σ defined as (2.2.1) by the map (2.3.1).

Let $\widehat{\Gamma}$ be the completion of Γ inside $G(\mathbb{A}_{\mathrm{f}})$. The strong approximation implies that

$$G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma} \cong \Gamma\backslash G(\mathbb{R}).$$

Let $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})$ be the space of automorphic forms on $G(\mathbb{A})$ that are invariant under the right translation of $\widehat{\Gamma}$. For $\underline{t} \in X(T_n)_+$ we use $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$ to denote its $\underline{t}$-isotypic part as a $K_{G,\infty}$-representation. The composition of (2.3.1) with (2.4.1) gives the map

(2.4.3) $$\varphi_G(\cdot,\mathfrak{e}_{\mathrm{can}}) : H^0(X_{G,\Gamma},\mathcal{V}_{\underline{t}}^r)\otimes_{\mathbb{Q}(\zeta_N)}\mathbb{C} \longrightarrow N_{\underline{t}}^r(\mathbb{H}_n,\Gamma) \overset{\varphi_G(\cdot,\mathfrak{e}_{\mathrm{can}})}{\longrightarrow} \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}.$$

In §4.3 we use operators in $\mathfrak{q}_H^+$ to construct the archimedean sections. The corresponding adelic Eisenstein series are obtained from the scalar weight holomorphic Eisenstein series by applying the action of $\mathfrak{q}_H^+$. Propositions 2.3.1, 2.4.1 make it clear how to translate the adelic picture to the geometric picture.

2.5. **$\mathbb{U}_p$-operators.** For each $\underline{a} \in \mathbb{Z}^n$ we define $\Delta\underline{a} := (a_1 - a_2, \cdots, a_{n-1} - a_n, a_n)$ and $p^{\underline{a}} := \mathrm{diag}(p^{a_1}, \cdots, p^{a_n}, p^{-a_1}, \cdots, p^{-a_n}) \in G(\mathbb{Q})$. Denote by C_n^+ be the subset of $\mathbb{Z}^n$ consisting of $\underline{a}$ such that $\Delta\underline{a} \geq 0$. We construct operators $U_{p,\underline{a}}$ for all $\underline{a} \in C_n^+$ acting on $H^0(X_{G,\Gamma}, \mathcal{V}_{\underline{t}}^r)$ and $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$, such that the map (2.4.3) is $U_{p,\underline{a}}$-equivariant. All such operators will be called $\mathbb{U}_p$-operators. What we need to be careful about is the normalization. In §6.2.1 we show that nearly holomorphic forms embed into the space of p-adic Siegel modular forms. The normalization should make the $\mathbb{U}_p$-operators here compatible with those defined for the space of p-adic Siegel modular forms, for which there is a canonical optimal normalization that preserves its natural integral structure.

First we look at the geometric picture. Set

$$\gamma_{p,i} = \begin{pmatrix} I_i & 0 & 0 & 0 \\ 0 & pI_{n-i} & 0 & 0 \\ 0 & 0 & p^2 I_i & 0 \\ 0 & 0 & 0 & pI_{n-i} \end{pmatrix} \quad 1 \leq i \leq n-1, \tag{2.5.1}$$

$$\gamma_{p,n} = \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \qquad \text{and} \qquad \gamma_{p,0} = pI_{2n}.$$

We associate to $\gamma_{p,i}$ an operator $U_{\gamma_{p,i}}$ acting on $H^0(X_{\mathbf{G},\Gamma}, \mathcal{V}_{\underline{t}}^r)$ for each $0 \leq i \leq n$, and define $U_{p,\underline{a}}$ as $U_{\gamma_{p,0}}^{-a_1} U_{\gamma_{p,n}}^{2a_n} \prod_{i=1}^{n-1} U_{\gamma_{p,i}}^{a_i - a_{i+1}}$. It will be clear that $U_{p,\underline{a}}$ induces an endomorphism of $H^0(X_{G,\Gamma}, \mathcal{V}_{\underline{t}}^r)$.

We make $\gamma_{p,0}$ act invertibly on $Y_{\mathbf{G},\Gamma}$ by sending the quadruple $(A, \lambda, \psi_N, \mathrm{fil}_{p^m}^+)$ to $(A, \lambda, \psi_N \circ p, \mathrm{fil}_{p^m}^+)$. The canonical isomorphism between $\mathcal{H}_{\mathrm{dR}}^1(A/Y_{\mathbf{G},\Gamma})$ and $\gamma_{p,0}^* \mathcal{H}_{\mathrm{dR}}^1(A/Y_{\mathbf{G},\Gamma})$ gives an isomorphism between $\mathcal{V}_{\underline{t}}^r$ and $\gamma_{p,0}^* \mathcal{V}_{\underline{t}}^r$. The operator $U_{\gamma_{p,0}}$ is defined to be the composition

$$H^0(Y_{\mathbf{G},\Gamma}, \mathcal{V}_{\underline{t}}^r) \xrightarrow{\gamma_{p,0}^*} H^0(Y_{\mathbf{G},\Gamma}, \gamma_{p,0}^* \mathcal{V}_{\underline{t}}^r) \longrightarrow H^0(Y_{\mathbf{G},\Gamma}, \mathcal{V}_{\underline{t}}^r),$$

and its action is easily seen to be invertible.

For $\gamma_{p,i}$, $1 \leq i \leq n$, consider the moduli scheme C_i parametrizing the quintuple $(A, \lambda, \psi_N, \mathrm{fil}_{p^m}^+, L)$ with L being a Lagrangian subgroup of $A[p^2]$ (resp. $A[p]$) if $1 \leq i \leq n-1$ (resp. $i = n$) satisfying $L[p] \oplus p^{m-1} \mathrm{fil}_{p^m,i}^+ = A[p]$. There are two projections $p_1, p_2 : C_i \to Y_{\mathbf{G},\Gamma}$. The projection p_1 is simply forgetting L. Let $\pi : A \to A/L$ be the universal isogeny. The projection p_2 sends $(A, \lambda, \psi_N, \mathrm{fil}_{p^m}^+, L)$ to $(A/L, \lambda', \pi \circ \psi_N, \mathrm{fil}_{p^m}^{+\prime})$. Here the polarization is defined by $\pi^* \lambda' = p^2 \lambda$ (resp. $\pi^* \lambda' = p\lambda$) for $1 \leq i \leq n-1$ (resp. $i = n$). If $x_1, \cdots, x_n \in A[p^m]$ represents $\mathrm{fil}_{p^m}^+$ with $p^{m-1} x_j \in L$ for $i+1 \leq j \leq n$, we put $\mathrm{fil}_{p^m}^{+\prime}$ to be the filtration represented by $\pi(x_1), \cdots, \pi(x_i), \pi(p^{-1} x_{i+1}), \cdots, \pi(p^{-1} x_n)$. Since p is inverted the pullback gives the morphism $\pi^* : p_2^* T_{\mathcal{H}}^\times \to p_1^* T_{\mathcal{H}}^\times$ which induces $\pi^* : p_2^* \mathcal{V}_{\underline{t}}^r \to p_1^* \mathcal{V}_{\underline{t}}^r$. The operator $U_{\gamma_{p,i}}$ is the composition

$$H^0(Y_{\mathbf{G},\Gamma}, \mathcal{V}_{\underline{t}}^r) \xrightarrow{p_2^*} H^0(C_i, p_2^* \mathcal{V}_{\underline{t}}^r) \xrightarrow{\pi^*} H^0(C_i, p_1^* \mathcal{V}_{\underline{t}}^r) \xrightarrow{p^{-u_i} \mathrm{Tr}\, p_1} H^0(Y_{\mathbf{G},\Gamma}, \mathcal{V}_{\underline{t}}^r),$$

where the normalization factor u_i is defined as

$$u_i = \begin{cases} i(n+1) + (t_{i+1} + \cdots + t_n) & \text{if } 1 \leq i \leq n-1, \\ n(n+1)/2 & \text{if } i = n. \end{cases}$$

The normalization factor is picked in order to make the definition compatible with the $\mathbb{U}_p$-operators defined for (integral) p-adic Siegel modular forms. Considerations in two aspects make contributions. The part involving $\underline{t}$ appears because when working with p-adic Siegel modular forms the action of $p^{\underline{a}}$ on $W_{\underline{t}}$ needs to be renormalized to preserve integrality optimally. The part independent of $\underline{t}$ is the pure inseparability degree of the map p_1 restricted to the ordinary locus. If more generally we consider the operator $U_{\gamma_{p,i}}$ on the space $H^0(X_{\mathbf{G},\Gamma}, \mathcal{V}^r_{\underline{t}\otimes\mathrm{Sym}^e\,\tau_n^*})$, one can check that the optimal normalization that makes all eigenvalues p-adically integral is p^{-u_i+2e} if $1 \leq i \leq n-1$ and p^{-u_n} if $i = n$.

Adelically for $\underline{a} \in C_n^+$ we define the operator $U_{p,\underline{a}}$ acting on $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{\underline{t}}$ as

$$U_{p,\underline{a}} := p^{\langle \underline{t}+2\rho_{G,c},\, \underline{a}\rangle} \int_{N_G(\mathbb{Z}_p)} R_p(up^{\underline{a}})\, du, \tag{2.5.2}$$

where $R_p(g)$ is the right translation by $g \in G(\mathbb{Q}_p)$, and the measure of $N_G(\mathbb{Z}_p)$ is its Haar measure with total volume 1. We use $\rho_{G,c}$ (resp. ρ_G, $\rho_{G,nc}$) to denote the half sum of positive compact (resp. positive, positive noncompact) roots of $\mathfrak{g}$ with respect to B_G. If $K_p \subset G(\mathbb{Z}_p)$ is an open compact subgroup containing $N_G(\mathbb{Z}_p)$, then as an action on $\pi_p^{K_p}$, the above defined $U_{p,\underline{a}}$ equals, up to scalar, the usual Hecke operator associated to the characteristic function of the compact open subset $K_p p^{\underline{a}} K_p$ of $G(\mathbb{Q}_p)$. Set $N_G(\underline{a})$ to be the set of representatives of the quotient $N_G(\mathbb{Z}_p)/p^{\underline{a}} N_G(\mathbb{Z}_p) p^{-\underline{a}}$. Then the action of $U_{p,\underline{a}}$ on $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$ can also be written as be

$$U_{p,\underline{a}} = p^{\langle \underline{t}-2\rho_{G,nc},\, \underline{a}\rangle} \sum_{u \in N_G(\underline{a})} R_p(up^{\underline{a}}). \tag{2.5.3}$$

It is easy to check that with the above definitions of the operator $U_{p,\underline{a}}$ on $H^0(X_{G,\Gamma}, \mathcal{V}^r_{\underline{t}})$ and $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$, the map (2.4.3) is $\mathbb{U}_p$-equivariant.

Remark 2.5.1. Note that although up to a scalar one may think of the adelic operator $U_{p,\underline{a}}$ as defined locally at the place p, the correct normalization for studying p-adic properties of these operators essentially depends on the $K_{G,\infty}$-type. This illustrates a common phenomenon in the study of p-adic automorphic forms that the place p and the archimedean place are closely related.

Proposition 2.5.2. *Given a weight $\underline{t}$ nearly holomorphic form $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{\underline{t}}$ (for example an automorphic form of $K_{G,\infty}$-type $\underline{t}$ inside a cuspidal automorphic representation whose archimedean component is a holomorphic discrete series), let $\mathbb{U}_p(\varphi)$ be the finite dimensional $\mathbb{C}$-vector space (viewed also as a $\overline{\mathbb{Q}}_p$-vector space by our fixed isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$) spanned by $U_{p,\underline{a}}\varphi$, $\underline{a} \in C_n^+$. Then by our normalization for the $\mathbb{U}_p$-operators, for each $U_{p,\underline{a}}$ all of its eigenvalues on $\mathbb{U}_p(\varphi)$ are p-adic integers.*

Proof. This results follows from two facts. One is that the space of nearly holomorphic forms can be embedded, $\mathbb{U}_p$-equivariantly, into the space of p-adic forms (see §6.2.1). The other is that the natural p-adic integral structure of the space of p-adic forms are preserved by the $\mathbb{U}_p$-operators (the normalization of $\mathbb{U}_p$-operators is optimal for preserving the integral structure) [Hid04, §8.3] [Liu15, §2.9.5]. ∎

For $1 \leq j \leq n$, we define the operator $U_{p,j}$ to be the one that corresponds to the element $\mathrm{diag}(pI_j, I_{n-j}, p^{-1}I_j, I_{n-j})$ inside $G(\mathbb{Q})$, and $U_p = \prod_{j=1}^n U_{p,j}$. Equivalently we can define $U_p = U_{p,\rho_G}$, the operator associated to $\rho_G = (n, n-1, \cdots, 1) \in C_n^+$. The above proposition tells us that, for a nearly holomorphic form φ, the limit

$$\lim_{r\to\infty} U_p^{r!}\varphi, \tag{2.5.4}$$

with respect to the usual p-adic topology of the finite dimensional $\overline{\mathbb{Q}}_p$-vector spaces $\mathbb{U}_p(\varphi)$, is well defined. We denote by $e\varphi$ this limit, which is called the ordinary projection of φ, because it is the projection of φ to the direct sum of the generalized eigenspaces of the $\mathbb{U}_p$-operators associated to eigenvalues that are all p-adic units. Although in the uniform definition (2.5.4) a limit with respect to the p-adic topology is involved, in each specific cases the ordinary projector is a $\mathbb{C}$-linear endomorphism of a finite dimensional vector space that can be written as a polynomial of U_p.

The following proposition proved in [Liu15, Corollary 3.10.3] will be used later. It shows that ordinary nearly holomorphic forms must be holomorphic.

Proposition 2.5.3. *As maps from $H^0(X_{G,\Gamma}, \mathcal{V}^r_{\underline{t}})$ to $H^0(X_{G,\Gamma}, \mathcal{V}^{r-1}_{\underline{t}\otimes\tau_n^*})$, we have*

$$E_\sigma U_p = p^2 \cdot U_p E_\sigma. \tag{2.5.5}$$

2.6. **q-expansions of nearly holomorphic forms.** We have fixed the rank $2n$ lattice $\mathbf{L}_n = L_n \oplus L_n^*$ with a symplectic pairing where L_n, L_n^* are both maximal isotropic and are dual to each other. Let S_{L_n} be the symmetric quotient of $L_n \times L_n$ and $S_{L_n,\geq 0}$ be the intersection of S_{L_n} with the cone dual to the cone inside $S^*_{L_n} \otimes_{\mathbb{Z}} \mathbb{R}$ consisting of semi-positive definite forms. Take a basis $s_1, \cdots, s_{n(n+1)/2}$ of S_{L_n} lying inside $S_{L_n,\geq 0}$, and set $\mathbb{Z}((S_{L_n,\geq 0})) = \mathbb{Z}[[S_{L_n,\geq 0}]][1/s_1 s_2 \cdots s_{n(n+1)/2}]$. For $\beta \in S_{L_n,\geq 0}$, the corresponding element in $\mathbb{Z}[[S_{L_n,\geq 0}]]$ is sometimes written as q^β.

The natural map $L_n \to S_{L_n} \otimes L_n^*$ defines a period group $L_n \subset L_n^* \otimes \mathbb{G}_{m/\mathbb{Z}((S_{L_n,\geq 0}))}$, principally polarized by the duality between L_n and L_n^*. Mumford's construction [FC90] gives an abelian variety $A_{/\mathbb{Z}((S_{L_n,\geq 0}))}$ with a canonical polarization λ_{can} and a canonical basis $\omega_{\mathrm{can}} = (\omega_{1,\mathrm{can}}, \cdots, \omega_{n,\mathrm{can}})$ of $\omega(A/\mathbb{Z}((S_{L_n,\geq 0})))$. From the exact sequence

$$0 \to L_n^* \otimes \prod_l \varprojlim_m \mu_{l^m} \to \prod_l T_l(A) \to L_n \otimes \widehat{\mathbb{Z}} \to 0$$

one can define the level structure $\psi_{N,\mathrm{can}}$ and $\mathrm{fil}^+_{p^m,\mathrm{can}}$ after base changing to $\mathbb{Z}((N^{-1}S_{L_n,\geq 0}))[\zeta_N, 1/Np]$. Let $D_{ij} \in \mathrm{Der}(\mathbb{Z}((S_{L_n,\geq 0})), \mathbb{Z}((S_{L_n,\geq 0})))$ be the element dual to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $\delta_{i,\mathrm{can}} = \nabla(D_{ii})\omega_{i,\mathrm{can}}$. For $\beta \in S_{L_n,\geq 0}$ we have $D_{ij}(q^\beta) = (2 - \delta_{ij})\beta_{ij} q^\beta$ with $\delta_{ij} = 0$ if $i \neq j$, and 1 if $i = j$. Then $\delta_{\mathrm{can}} = (\delta_{1,\mathrm{can}}, \cdots, \delta_{n,\mathrm{can}})$ together with ω_{can} forms a basis of $\mathcal{H}^1_{\mathrm{dR}}(A/\mathbb{Z}((S_{L_n,\geq 0})))$ respecting both the Hodge filtration and the symplectic pairing.

Let F be a number field containing $\mathbb{Q}(\zeta_N)$. Evaluating a nearly holomorphic form f inside $H^0(X_{G,\Gamma}, \mathcal{V}^r_\sigma) \otimes_{\mathbb{Q}(\zeta_N)} F$ at the test object $(A_{/\mathbb{Z}((N^{-1}S_{L_n,\geq 0}))[\zeta_N,1/Np]}, \lambda_{\mathrm{can}}, \psi_{N,\mathrm{can}}, \mathrm{fil}^+_{p^m,\mathrm{can}}, \omega_{\mathrm{can}}, \delta_{\mathrm{can}})$ defines its polynomial q-expansion which we denote by $f(q, \underline{Y})$. It lies inside $\mathbb{Z}[[N^{-1}S_{L_n,\geq 0}]] \otimes_{\mathbb{Z}} W_\sigma(F)[\underline{Y}]_{\leq r}$. For a dominant weight $\underline{t}$, applying $\mathfrak{e}_{\mathrm{can}}$ to the polynomial q-expansion of a weight $\underline{t}$ nearly holomorphic form and putting $\underline{Y} = 0$ gives the (p-adic) q-expansion map

$$\varepsilon_{q,p\text{-adic}} : H^0(X_{G,\Gamma}, \mathcal{V}^r_\sigma) \otimes_{\mathbb{Q}(\zeta_N)} F \to \mathbb{Z}[[N^{-1}S_{L_n,\geq 0}]] \otimes_{\mathbb{Z}} F. \tag{2.6.1}$$

This q-expansion map is injective and can be used to give an integral structure on the space of nearly holomorphic forms. We call it p-adic because it agrees with the one obtained by viewing nearly holomorphic forms as p-adic forms and applying the q-expansion map for p-adic forms to them (see §6.2.1).

Let $\mathrm{Sym}(n,\mathbb{Z})^*$ be the subset of $\mathrm{Sym}(n,\mathbb{Q})$ consisting of elements α such that $\mathrm{Tr}\,\alpha a \in \mathbb{Z}$ for all $a \in \mathrm{Sym}(n,\mathbb{Z})$. Using our fixed basis of $\mathbf{L}_n$, we identify $S_{L_n,\geq 0}$ with $\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}$, and $\mathbb{Z}[[N^{-1}S_{L_n,\geq 0}]]$ with $\mathbb{Z}[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]]$.

We record here [Liu15, Proposition 2.6.1] the formulas of differential operators in terms of polynomial q-expansions. Recall that we have fixed a basis $\underline{X}$ of the $\mathrm{GL}(n)$-representation τ_n.

Proposition 2.6.1. *Let $f \in H^0(X_{G,\Gamma}, \mathcal{V}_\sigma^r) \otimes_{\mathbb{Q}(\zeta_N)} F$ be a nearly holomorphic form with polynomial q-expansion $f(q, \underline{Y})$. Then*

$$(D_\sigma f)(q, \underline{Y}) = \sum_{1 \le i \le j \le n} \Big(D_{ij} f(q, \underline{Y}) + \mu_{ij}^- \cdot f(q, \underline{Y}) \Big) \otimes X_{ij}.$$

3. Siegel Eisenstein series and their Fourier coefficients

Let k be an integer larger or equal to $n+1$ and ξ be a primitive Dirichlet character with conductor dividing Np^∞ such that the parity condition $\xi(-1) = (-1)^k$ holds. We record here some computation results of Shimura [Shi82, ShE97] on the Fourier coefficients of certain holomorphic Siegel Eisenstein series of weight k, and put the formulas into a form that is ready for p-adic interpolation.

3.1. **Siegel Eisenstein series on H.** Take a primitive Dirichlet character ξ whose conductor divides Np^∞. For a complex number s we denote by $\xi_s = \xi|\cdot|^s \circ \det$ the character of $Q_H(\mathbb{A})$ sending $\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix}$ to $\xi(\det A)|\det A|^s$. Let $I_{Q_H}(s, \xi) = \mathrm{Ind}_{Q_H(\mathbb{A})}^{H(\mathbb{A})} \xi_s$ be the normalized induction consisting of smooth functions f on $H(\mathbb{A})$ that satisfy $f(qh) = \xi_s(q)\delta_{Q_H}^{1/2}(q) f(h)$ for all $h \in H(\mathbb{A})$ and $q \in Q_H(\mathbb{A})$. Here the modulus character δ_{Q_H} takes value $|\det A|^{\frac{2n+1}{2}}$ at $\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix}$. Similarly we define the local degenerate principal series $I_{Q_H,v}(s, \xi)$ for all places of $\mathbb{Q}$.

Given a section $f(s, \xi) \in I_{Q_H}(s, \xi)$, its associated Siegel Eisenstein series is defined as

$$E(h, f(s,\xi)) = \sum_{\gamma \in Q_H(\mathbb{Q}) \backslash H(\mathbb{Q})} f(s, \xi)(\gamma h).$$

The sum is absolutely convergent for $\mathrm{Re}\,(s)$ sufficiently large and admits a meromorphic continuation.

We have already fixed an additive character $\mathbf{e}_\mathbb{A}$ of $\mathbb{Q}\backslash\mathbb{A}$ and a Haar measure on $\mathbb{A}$. If $x \in \mathrm{Sym}(2n, \mathbb{A})$ set $u(x)$ to be the element $\begin{pmatrix} I_{2n} & x \\ 0 & I_{2n} \end{pmatrix}$ of the unipotent radical $U_H(\mathbb{A}) \subset Q_H(\mathbb{A})$. For $\boldsymbol{\beta} \in \mathrm{Sym}(2n, \mathbb{Q})$ the $\boldsymbol{\beta}$-th Fourier coefficient for $E\big(\cdot, f(s, \xi)\big)$ is defined as

$$E_{\boldsymbol{\beta}}(h, f(s,\xi)) := \int_{\mathrm{Sym}(2n,\mathbb{Q}) \backslash \mathrm{Sym}(2n,\mathbb{A})} E(u(x)h, f(s,\xi)) \mathbf{e}_\mathbb{A}(-\mathrm{Tr}\,\boldsymbol{\beta} x)\, dx.$$

If $\det(\boldsymbol{\beta}) \neq 0$ and $f(s, \xi) = \otimes_v f_v(s, \xi)$ is factorizable, then

$$E_{\boldsymbol{\beta}}(h, f(s,\xi)) = \prod_v W_{\boldsymbol{\beta},v}(h, f(s,\xi)) \tag{3.1.1}$$

with

$$W_{\boldsymbol{\beta},v}(h, f_v(s,\xi)) = \int_{\mathrm{Sym}(2n,\mathbb{Q}_v)} f_v(s,\xi)(w_H u(\varsigma) h) \mathbf{e}_v(-\mathrm{Tr}\,\boldsymbol{\beta}\varsigma)\, d_v\varsigma$$

where $w_H = \begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix}$.

Let S_f be the set of finite places of $\mathbb{Q}$ dividing Np and S be the union of S_f with $\{\infty\}$. In the following, for $v \notin S$ we always take $f_v(s, \xi)$ to be the unique section $f_v^{\mathrm{ur}}(s,\xi) \in I_{Q_H,v}(s, \xi)$ that takes value 1 on $H(\mathbb{Z}_v)$ (the uniqueness is due to the Iwasawa decomposition $H(\mathbb{Q}_v) = Q_H(\mathbb{Q}_v)H(\mathbb{Z}_v)$). For $v \in S_\mathrm{f}$ the section $f_v(s,\xi)$ we will consider is supported on the so-called "big cell" inside $H(\mathbb{Q}_v)$, i.e. $Q_H(\mathbb{Q}_v) w_H U_H(\mathbb{Q}_v)$. An element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H(\mathbb{Q}_v)$ belongs to the "big cell" if and only if

$\det C \neq 0$. Given $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{H}_{2n}$ we put $h_{\boldsymbol{z}} = 1_{\mathrm{f}} \cdot \begin{pmatrix} \sqrt{\boldsymbol{y}} & \boldsymbol{x}\sqrt{\boldsymbol{y}}^{-1} \\ 0 & \sqrt{\boldsymbol{y}}^{-1} \end{pmatrix}_{\infty}$. With $h = h_{\boldsymbol{z}}$ and at least one local section supported on the "big cell", (3.1.1) holds for all $\boldsymbol{\beta}$. Next we compute formulas for $W_{\boldsymbol{\beta},v}(h, f_v(s,\xi))$ place by place.

3.2. **The ramified places.** Let α_v be a compactly supported smooth function on $\mathrm{Sym}(2n, \mathbb{Q}_v)$. We define the section $f_v^{\alpha_v}(s,\xi) \in I_{Q_H,v}(s,\xi)$ as

$$(3.2.1) \qquad f_v^{\alpha_v}(s,\xi)\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{cases} \xi^{-1}(\det C)|\det C|^{-(s+\frac{2n+1}{2})}\alpha_v(C^{-1}D) & \text{if } \det C \neq 0, \\ 0 & \text{if } \det C = 0. \end{cases}$$

An easy computation shows that

$$(3.2.2) \qquad W_{\boldsymbol{\beta},v}(1_v, f_v^{\alpha_v}(s,\xi)) = \int_{\mathrm{Sym}(2n,\mathbb{Q}_v)} \alpha_v(\varsigma)\mathbf{e}_v(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\, d_v\varsigma = \widehat{\alpha}_v(\boldsymbol{\beta}).$$

Since the Fourier transform is an isomorphism on the space of compactly supported smooth functions on $\mathrm{Sym}(2n, \mathbb{Q}_v)$, the above formula gives us adequate flexibility in arranging, for our purpose of p-adic interpolation, the contribution of ramified places to the Fourier coefficients of the Siegel Eisenstein series. Later when choosing sections at p we will first decide what $\widehat{\alpha}_p$ should be and then get the corresponding $f_p^{\alpha_p}(s,\xi)$. Notice also that for such "big cell" sections, $W_{\boldsymbol{\beta},v}(1_v, f_v^{\alpha_v}(s,\xi))$ is independent of s and ξ.

In the following we always require the $\widehat{\alpha}_p$ to be supported on the following compact set

$$(3.2.3) \qquad \left\{ b = \begin{pmatrix} b_1 & b_0 \\ {}^{t}b_0 & b_2 \end{pmatrix} \in \mathrm{Sym}(2n, \mathbb{Z}_p) : b_1 \equiv 0 \mod p^2, \quad b_0 \in \mathrm{GL}(n, \mathbb{Z}_p) \right\}.$$

In particular under this requirement the Fourier coefficient $E_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f(s,\xi))$ vanishes for all degenerate $\boldsymbol{\beta}$.

3.3. **The unramified places.** For $v \notin S$ we record here Shimura's calculation of $W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{ur}}(s,\xi))$ in the case when $\boldsymbol{\beta}$ is nondegenerate. Let val_v be the valuation of $\mathbb{Q}_v$ taking value 1 at the uniformizer and q_v be the cardinality of the residue field. Denote by $\mathrm{Sym}(2n, \mathbb{Z}_v)^*$ the set of symmetric matrices $\eta \in \mathrm{Sym}(2n, \mathbb{Q}_v)$ such that $\operatorname{Tr}\eta\varsigma \in \mathbb{Z}_v$ for all $\varsigma \in \mathrm{Sym}(2n, \mathbb{Z}_v)$. Define

$$d_v(s,\xi) := L_v(s + \frac{2n+1}{2}, \xi) \prod_{j=1}^{n} L_v(2s + 2n + 1 - 2j, \xi^2).$$

With all data unramified at v we have

Theorem 3.3.1 ([ShE97, Theorem 13.6, Proposition 14.9])**.** *The Fourier coefficient $W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{ur}}(s,\xi))$ vanishes unless $\boldsymbol{\beta}$ lies inside the intersection of $\mathrm{Sym}(2n, \mathbb{Q})$ with $\mathrm{Sym}(2n, \mathbb{Z}_v)^*$. When it is nonvanishing, we have*

$$(3.3.1) \qquad W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{ur}}(s,\xi)) = d_v(s,\xi)^{-1} L_v(s + \frac{1}{2}, \xi\lambda_{\boldsymbol{\beta}}) \cdot g_{\boldsymbol{\beta},v}\left(\xi(q_v) q_v^{-(s+\frac{2n+1}{2})}\right).$$

Here $\lambda_{\boldsymbol{\beta}}(q_v) := \left(\frac{(-1)^n \det(2\boldsymbol{\beta})}{q_v}\right)$ and $g_{\boldsymbol{\beta},v}(t)$ is a polynomial with coefficients in $\mathbb{Z}$ whose constant term is 1 and degree is at most $4n \cdot \mathrm{val}_v\left(\det(2\boldsymbol{\beta})\right)$. In particular $g_{\boldsymbol{\beta},v}(t) = 1$ if $\det(2\boldsymbol{\beta}) \in \mathbb{Z}_v^\times$.

What is relevant to us is the evaluation of $E(\cdot, f(s,\xi))$ at $s_0 = k - \frac{2n+1}{2}$ with $\xi(-1) = (-1)^k$ and $k \geq n+1$. In that case we have the parity $(-1)^{k-n} = \xi\lambda_{\boldsymbol{\beta}}(-1)$ so the special value $L(s_0 + \frac{1}{2}, \xi\lambda_{\boldsymbol{\beta}}) = L(k-n, \xi\lambda_{\boldsymbol{\beta}})$ belongs to the set of interpolation points of the p-adic Dirichlet L-function.

3.4. **The archimedean place.** For an integer $k \geq n+1$ satisfying $\xi(-1) = (-1)^k$ we consider the canonical section $f_\infty^k(s,\xi) \in I_{Q_H,\infty}(s,\xi)$ defined as

$$f_\infty^k(s,\xi)(h) = j(h,i)^{-k}|j(h,i)|^{k-(s+\frac{2n+1}{2})}$$

where $j(h,i) = \det(\mu(h,i)) = \det(Ci+D)$ for $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. It gives rise to a Siegel Eisenstein series of scalar weight k. Then

$$\begin{aligned} &W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}}, f_\infty^k(s,\xi)) \\ &= \int_{\mathrm{Sym}(2n,\mathbb{R})} \det\left(\sqrt{\boldsymbol{y}}i + (\boldsymbol{x}+\varsigma)\sqrt{\boldsymbol{y}}^{-1}\right)^{-k} \left|\det\left(\sqrt{\boldsymbol{y}}i + (\boldsymbol{x}+\varsigma)\sqrt{\boldsymbol{y}}^{-1}\right)\right|^{k-(s+\frac{2n+1}{2})} \mathbf{e}_\infty(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\,d\varsigma \\ &= \mathbf{e}_\infty(\operatorname{Tr}\boldsymbol{\beta x})(\det \boldsymbol{y})^{\frac{1}{2}(s+\frac{2n+1}{2})}\xi_{2n}\left(\boldsymbol{y},\boldsymbol{\beta}; \frac{1}{2}(s+\frac{2n+1}{2})+\frac{k}{2}, \frac{1}{2}(s+\frac{2n+1}{2})-\frac{k}{2}\right), \end{aligned}$$

where for $h_1, h_2 \in \mathrm{Sym}(2n,\mathbb{R})$ and $s_1, s_2 \in \mathbb{C}$ the function ξ_{2n} is defined as

$$\xi_{2n}(h_1,h_2;s_1,s_2) := \int_{\mathrm{Sym}(2n,\mathbb{R})} \det(\varsigma+ih_1)^{-s_1}\det(\varsigma-ih_1)^{-s_2}\mathbf{e}_\infty(\operatorname{Tr} h_2\varsigma)d\varsigma.$$

The function $\xi_{2n}(h_1,h_2;s_1,s_2)$ is studied by Shimura in full generality [Shi82]. Before stating the result we define the Gamma function

$$\Gamma_m(s) := \pi^{\frac{m(m-1)}{4}} \prod_{j=0}^{m-1} \Gamma(s-\frac{j}{2}).$$

Theorem 3.4.1 (Theorem 4.2, *loc. cit*)**.** *Let r_+ (resp. r_-) be the number of positive (resp. negative) eigenvalues of $\boldsymbol{\beta}$ and $r = 2n - r_+ - r_-$. Set $\delta_+(\boldsymbol{\beta y})$ (resp. $\delta_-(\boldsymbol{\beta y})$) to be the product of all positive eigenvalues (resp. absolute values of negative eigenvalues) of $\boldsymbol{\beta y}$.*

$$\begin{aligned} \xi_{2n}(\boldsymbol{y},\boldsymbol{\beta};s_1,s_2) =& 2^{2n+\frac{r_+r_-}{2}+2s_1(r_+-n)+2s_2(r_--n)+\frac{r(2n+1)}{2}} e^{\pi i\cdot n(s_1-s_2)}\pi^{r_+s_1+r_-s_2-\frac{r_+r_-}{2}+\frac{r(r+1)}{2}} \\ &\times (\det \boldsymbol{y})^{\frac{2n+1}{2}-(s_1+s_2)}\delta_+(\boldsymbol{\beta y})^{s_1-\frac{2n+1}{2}+\frac{r_-}{4}}\delta_-(\boldsymbol{\beta y})^{s_2-\frac{2n+1}{2}+\frac{r_+}{4}} \\ &\times \frac{\Gamma_r(s_1+s_2-\frac{2n+1}{2})}{\Gamma_{2n-r_-}(s_1)\Gamma_{2n-r_+}(s_2)}\omega(2\pi\boldsymbol{y},\boldsymbol{\beta};s_1,s_2) \end{aligned}$$

Here $\omega(2\pi\boldsymbol{y},\boldsymbol{\beta};s_1,s_2)$ is a holomorphic function in s_1, s_2, and if $\boldsymbol{\beta}$ is strictly positive definite

$$\omega(2\pi\boldsymbol{y},\boldsymbol{\beta};s_1,0) = 2^{-n(2n+1)}\mathbf{e}_\infty(i\operatorname{Tr}\boldsymbol{\beta y}).$$

The value $W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}}, f_\infty^k(s,\xi))$ we are interested in is at $s_0 = k - \frac{2n+1}{2}$, which corresponds to the evaluation of $\xi_{2n}(\boldsymbol{y},\boldsymbol{\beta};s_1,s_2)$ at $s_1 = k, s_2 = 0$. Look at the term $\frac{\Gamma_r(s_1+s_2-\frac{2n+1}{2})}{\Gamma_{2n-r_-}(s_1)\Gamma_{2n-r_+}(s_2)}$. By our requirement on $\widehat{\alpha}_p$ only nondegenerate $\boldsymbol{\beta}$'s need to be considered, for which $r = 0$ and the numerator is 1. Meanwhile the function $\Gamma_{2n-r_+}(s_2)$ in the denominator has a pole at $s_2 = 0$ unless $r_+ = 2n$. Hence for nondegenerate $\boldsymbol{\beta}$ the value $W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}}, f_\infty^k(k-\frac{2n+1}{2},\xi))$ is nonvanishing only if $\boldsymbol{\beta}$ is strictly positive definite. For those $\boldsymbol{\beta}$'s we have

$$W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}}, f_\infty^k(k-\frac{2n+1}{2},\xi)) = (-1)^{nk}\frac{2^{2n}}{\Gamma_{2n}(k)}\pi^{2nk}(\det 2\boldsymbol{\beta})^{k-\frac{2n+1}{2}}(\det \boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\operatorname{Tr}\boldsymbol{\beta z}).$$

3.5. **Summary.** Let $d^S(s,\xi) = \prod_{v\notin S} d_v(s,\xi)$ and we normalize the Siegel Eisenstein series as

$$E^*(h, f(s,\xi)) = d^S(s,\xi)E(h, f(s,\xi)).$$

Let $\mathrm{Sym}(2n,\mathbb{Q})_{>0}$ be the subset of $\mathrm{Sym}(2n,\mathbb{Q})$ consisting of (strictly) positive definite elements, and $\Sigma_{p,+}$ be the subset of $\mathrm{Sym}(2n,\mathbb{Q})_{>0}$ consisting of elements that belong to both the set (3.2.3) and $\mathrm{Sym}(2n,\mathbb{Z}_v)^*$ for all $v\notin S$. Use $\alpha_{S_{\mathrm{f}}}$ to denote the collection of the Schwartz functions α_v on $\mathrm{Sym}(2n,\mathbb{Q}_v)$ for $v\in S_{\mathrm{f}}$ with $\widehat{\alpha}_p$ always assumed to be supported on the set (3.2.3). Put

$$(3.5.1)\quad f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\xi) = \bigotimes_{v\notin S} f_v^{\mathrm{ur}}(k-\frac{2n+1}{2},\xi)\otimes\bigotimes_{v\in S_{\mathrm{f}}} f_v^{\alpha_v}(k-\frac{2n+1}{2},\xi)\otimes f_\infty^k(k-\frac{2n+1}{2},\xi),$$

which is a section inside $I_{Q_H}(k-\frac{2n+1}{2},\xi)$.

Combining results from the previous three sections we know that the normalized Siegel Eisenstein series $E^*(\cdot, f^{k,\alpha_{S_{\mathrm{f}}}})$ on $H(\mathbb{A})$ is holomorphic of weight k with Fourier coefficients supported on $\Sigma_{p,+}$. Put $g^S_{\boldsymbol{\beta}}(k,\xi^{-1}) = \prod_{v\notin S} g_{\boldsymbol{\beta},v}(\xi(q_v)q_v^{-k})$. For $\boldsymbol{\beta}\in\Sigma_{p,+}$ there is the formula

(3.5.2)
$$\begin{aligned}&E^*_{\boldsymbol{\beta}}\left(h_{\boldsymbol{z}}, f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\xi)\right)\\ &=(-1)^{nk}\frac{2^{2n}}{\Gamma_{2n}(k)}\pi^{2nk}L^S(k-n,\lambda_{\boldsymbol{\beta}}\xi)g^S_{\boldsymbol{\beta}}(k,\xi^{-1})\prod_{v\in S_{\mathrm{f}}}\widehat{\alpha}_v(\boldsymbol{\beta})(\det 2\boldsymbol{\beta})^{k-\frac{2n+1}{2}}(\det\boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\,\boldsymbol{\beta}\boldsymbol{z}).\end{aligned}$$

Implementing the q-expansion principle, with suitable $\alpha_{S_{\mathrm{f}}}$, one can deduce the algebraicity of $E^*(\cdot, f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\xi))$, i.e. up to an explicit normalization factor it lies inside the image under the map (2.4.3) of algebraic global sections.

We modify (3.5.2) into a form that is more convenient for later p-adic interpolation. Under our parity condition on k and ξ, the functional equation for Dirichlet L-functions indicates

$$(3.5.3)\qquad L(k-n,\lambda_{\boldsymbol{\beta}}\xi) = \frac{(2\pi i)^{k-n}}{2\Gamma(k-n)C_{\lambda_{\boldsymbol{\beta}}\xi}^{k-n-1}G(\lambda_{\boldsymbol{\beta}}^{-1}\xi^{-1})}L(1-k+n,\lambda_{\boldsymbol{\beta}}^{-1}\xi^{-1}).$$

Now write ξ as the product $\phi^{-1}\chi^{\circ -1}$ of two primitive characters, where the conductor of ϕ (resp. χ°) divides N (is a power of p). We write χ to mean the character associated to χ° taking value 0 at p. When there is no need to emphasize the primitivity of χ° we also simply write χ. Set $\phi_{\boldsymbol{\beta}} = \lambda_{\boldsymbol{\beta}}^{-1}\phi$ whose conductor is prime to p. Using the relation $G(\phi_{\boldsymbol{\beta}}\chi) = \phi_{\boldsymbol{\beta}}(C_\chi)\chi(C_{\phi_{\boldsymbol{\beta}}})G(\phi_{\boldsymbol{\beta}})G(\chi)$ and (3.5.3) we get from (3.5.2)

$$\begin{aligned}&E^*_{\boldsymbol{\beta}}\left(h_{\boldsymbol{z}}, f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\right)\\ &=\frac{2^{k+n-1}(\pi i)^{2nk+k-n}}{\Gamma(k-n)\Gamma_{2n}(k)\phi(C_\chi)C_\chi^{k-n-1}G(\chi)}\cdot\frac{\lambda_{\boldsymbol{\beta}}(C_\chi)L_p(1-k+n,\phi_{\boldsymbol{\beta}}\chi^\circ)}{L_p(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{\circ-1})}\\ &\quad\times\frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})}\cdot\chi^{-1}(C_{\phi_{\boldsymbol{\beta}}})C_{\phi_{\boldsymbol{\beta}}}^{-k+n+1}\cdot L_N(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1}\cdot L^p(1-k+n,\phi_{\boldsymbol{\beta}}\chi)\\ &\quad\times g^S_{\boldsymbol{\beta}}(k,\phi\chi)\cdot\prod_{v\in S_{\mathrm{f}}}\widehat{\alpha}_v(\boldsymbol{\beta})\det(2\boldsymbol{\beta})^{k-n-1}\cdot(\det\boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\,\boldsymbol{\beta}\boldsymbol{z}).\end{aligned}$$

For readers who are familiar with p-adic interpolation, it is noticeable that the above formula has been grouped into factors each of which is ready for p-adic interpolation with respect to k and χ, with the possible exception of the term $\frac{L_p(1-k+n,\phi_{\boldsymbol{\beta}}\chi^\circ)}{\lambda_{\boldsymbol{\beta}}(C_\chi)L_p(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{\circ-1})}$, especially the term $\lambda_{\boldsymbol{\beta}}(C_\chi)$. This term depends both on k,χ and $\boldsymbol{\beta}$ and in general does not admit p-adic interpolation. However by our requirement on $\widehat{\alpha}_p$, it suffices to consider only $\boldsymbol{\beta}=\begin{pmatrix}\beta_1 & \beta_0\\ {}^{\mathrm{t}}\beta_0 & \beta_2\end{pmatrix}$ that lies inside $\Sigma_{p,+}$. For such a $\boldsymbol{\beta}$ it is easy to see that $\det\boldsymbol{\beta}$ is a p-adic integer and $\det\boldsymbol{\beta}\equiv(-1)^n(\det\beta_0)^2 \mod p$. Thus $\lambda_{\boldsymbol{\beta}}(p)=1$. Let c_χ be the integer such that $C_\chi=p^{c_\chi}$. Define

$$A_{n,\phi,k,\chi}:=\frac{2^{k+n-1}(\pi i)^{2nk+k-n}}{\Gamma(k-n)\Gamma_{2n}(k)}\cdot\frac{L_p(1-k+n,\phi\chi^\circ)}{L_p(k-n,\phi^{-1}\chi^{\circ-1})}\left(\left(\phi(p)p^{k-n-1}\right)^{c_\chi}G(\chi)\right)^{-1}. \tag{3.5.4}$$

Proposition 3.5.1. *For $\boldsymbol{\beta}\in\Sigma_{p,+}$ we have*

$$\begin{aligned}&E^*_{\boldsymbol{\beta}}\left(h_{\boldsymbol{z}},f^{k,\alpha_{S_\mathrm{f}}}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\right)\\ &=A_{n,\phi,k,\chi}\cdot\frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})}\cdot\chi^{-1}(C_{\phi_{\boldsymbol{\beta}}})C_{\phi_{\boldsymbol{\beta}}}^{-k+n+1}\cdot L_N(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1}\cdot L^p(1-k+n,\phi_{\boldsymbol{\beta}}\chi)\\ &\quad\times g^S_{\boldsymbol{\beta}}(k,\phi\chi)\cdot\prod_{v\in S_\mathrm{f}}\widehat{\alpha}_v(\boldsymbol{\beta})\det(2\boldsymbol{\beta})^{k-n-1}\cdot(\det\boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\operatorname{Tr}\boldsymbol{\beta}\boldsymbol{z}).\end{aligned}$$

One can observe that on the RHS of the equality, the term $A_{n,\phi,k,\chi}$ is independent of $\boldsymbol{\beta}$ and other terms admit p-adic interpolations with respect to k,χ for suitably chosen α_{S_f} (c.f. §5.2).

4. Sections away from p and their local zeta integrals

Let $(\kappa,\underline{\tau})$ be an arithmetic point of $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$, i.e. it can be written as the product of an algebraic character $(\kappa_{\mathrm{alg}},\underline{\tau}_{\mathrm{alg}})$ and a finite order character $(\kappa_\mathrm{f},\underline{\tau}_\mathrm{f})$. We write $\kappa_{\mathrm{alg}}=k$, $\underline{\tau}_{\mathrm{alg}}=\underline{t}=(t_1,\cdots,t_n)$ with $k,t_1,\cdots,t_n$ being integers, and $\kappa_\mathrm{f}=\chi$, $\underline{\tau}_\mathrm{f}=\underline{\psi}=(\psi_1,\cdots,\psi_n)$ with $\chi,\psi_1,\cdots,\psi_n$ being characters of $\mathbb{Z}_p^\times$ of finite order. We call an arithmetic point $(\kappa,\underline{\tau})$ admissible if $t_1\geq\cdots\geq t_n\geq k\geq n+1$.

From now on we fix a primitive Dirichlet character ϕ whose conductor divides N, and we will sometimes omit N and ϕ from some notation that actually depends on them. Proposition 3.5.1 basically gives us a one-variable family of Siegel Eisenstein series on H where the variable is κ. What we want is an $(n+1)$-variable cuspidal family on $G\times G$, whose members are the restrictions to $G\times G$ of Siegel Eisenstein series on H, and its pairing with an n-variable family on $G\times G$ will give the desired $(n+1)$-variable p-adic L-function. Constructing this $(n+1)$-variable family boils down to selecting sections $f_{\kappa,\underline{\tau}}$ inside $I_{Q_H}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})$ for each admissible $(\kappa,\underline{\tau})$. It is no surprise that for all $v\notin S$ we set $f_{\kappa,\underline{\tau},v}$ to be the unramified section $f_v^{\mathrm{ur}}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})$. For $v\in S_\mathrm{f}$ we consider the "big cell" sections. Thus what we need to select is the collection of Schwartz functions $\alpha_{\kappa,\underline{\tau},S_\mathrm{f}}$ and the archimedean section $f_{\kappa,\underline{\tau},\infty}$.

In this section we make the choices for $\alpha_{\kappa,\underline{\tau},N}$ and $f_{\kappa,\underline{\tau},\infty}$. With our choices we compute the local zeta integrals for the doubling method for $v\mid N$, and show the nonvanishing of the archimedean zeta integral. In the next section we treat the place p. Based on the two criteria in the introduction, i.e. nonvanishing local zeta integrals and p-adically interpolatable q-expansions, all choices are completely natural.

4.1. **Doubling method for symplectic groups.** Let us first briefly recall the formulas of the doubling method. We have fixed the rank $2n$ free $\mathbb{Z}$-module $\mathbf{L}_n$ with a symplectic pairing and $G=G(\mathbf{L}_n)$. Let $\mathbf{V}_n=V_n\oplus V_n^*$ be the polarized symplectic space over $\mathbb{Q}$ with basis $e_1,\cdots,e_n,f_1,\cdots,f_n$ obtained from $\mathbf{L}_n$ by tensoring with $\mathbb{Q}$. Take another copy of $\mathbf{V}_n$ with basis $e'_1,\cdots,e'_n,f'_1,\cdots,f'_n$, and put $\mathbf{V}_{2n}=\mathbf{V}_n\oplus\mathbf{V}_n$ with the induced symplectic pairing. Elements in $H=G(\mathbf{V}_{2n})$ will be written in matrix form with respect to the basis $e_1,\cdots,e_n,e'_1,\cdots,e'_n,f_1,\cdots,f_n,f'_1,\cdots,f'_n$. Then there is the (holomorphic) embedding ι of $G\times G$ into H given by

$$\iota: G\times G\hookrightarrow H$$
$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\times\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\longmapsto\begin{pmatrix} a & 0 & b & 0\\ 0 & a' & 0 & b'\\ c & 0 & d & 0\\ 0 & c' & 0 & d'\end{pmatrix}.$$

Fix the map ϑ from $\mathbf{V}_n$ into itself whose matrix is $\begin{pmatrix} 0 & I_n\\ I_n & 0\end{pmatrix}$ with respect to our fixed basis. It does not preserve the symplectic pairing but has the similitude -1. Let $\mathbf{V}_{2n}^d=\{(v,\vartheta(v)) : v\in\mathbf{V}_n\}$ and $\mathbf{V}_{2n,d}=\{(v,-\vartheta(v)) : v\in\mathbf{V}_n\}$ which are both maximal isotropic subspaces of $\mathbf{V}_{2n}$. The doubling Siegel parabolic P_H is defined to be the stabilizer of $\mathbf{V}_{2n}^d$. The standard Siegel parabolic Q_H is the stabilizer of the maximal isotropic subspace $V_n\oplus V_n$ and we have

$$P_H=\mathcal{S}Q_H\mathcal{S}^{-1}\quad\text{ with }\quad\mathcal{S}=\begin{pmatrix} I_n & 0 & 0 & 0\\ 0 & I_n & 0 & 0\\ 0 & I_n & I_n & 0\\ I_n & 0 & 0 & I_n\end{pmatrix}.$$

For each section $f(s,\xi)\in I_{Q_H}(s,\xi)$ we set

$$f^d(s,\xi)(h)=f(s,\xi)(\mathcal{S}^{-1}h) \tag{4.1.1}$$

for $h\in H(\mathbb{A})$. Then $f^d(s,\xi)$ lies inside $I_{P_H}(s,\xi)$ and $E(\cdot,f(s,\xi))=E(\cdot,f^d(s,\xi))$. For an element $g\in G$ we define g^ϑ to be $\vartheta g\vartheta\in G$. This conjugation by ϑ is called the MVW involution. The MVW involution of an irreducible smooth representation of $G(\mathbb{Q}_v)$ is isomorphic to its contragredient [MVW87, p. 91].

Given an irreducible cuspidal automorphic representation $\pi\subset\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$ and its complex conjugation $\overline{\pi}\subset\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$, which is isomorphic to the contragredient of π, we fix isomorphisms $\pi\cong\bigotimes_v'\pi_v$ and $\overline{\pi}\cong\bigotimes_v'\tilde{\pi}_v$ such that for factorizable $\varphi_1,\varphi_2\in\pi$ with images $\bigotimes_v\varphi_{1,v}\in\bigotimes_v'\pi_v$ and $\bigotimes_v\overline{\varphi}_{2,v}\in\bigotimes_v'\tilde{\pi}_v$, we have

$$\langle\varphi_1,\overline{\varphi}_2\rangle=\prod_v\left\langle\varphi_{1,v},\overline{\varphi}_{2,v}\right\rangle_v,$$

where the pairing on the left hand side is the bi-$\mathbb{C}$-linear Petersson inner product with respect to our fixed Haar measure on $G(\mathbb{A})$ and the pairing on the right hand side is the natural pairing between π_v and its contragredient $\tilde{\pi}_v$.

For $\varphi\in\pi$ we define its MVW involution φ^ϑ by $\varphi^\vartheta(g)=\varphi(g^\vartheta)$, and we know that φ^ϑ lies inside $\overline{\pi}$ due to the multiplicity one theorem [Art13].

For a local section $f_v(s,\xi)\in I_{Q_H,v}(s,\xi)$ we define the operator

$$T_{f_v(s,\xi)}:\pi\longrightarrow\pi$$
$$\varphi\longmapsto\left(T_{f_v(s,\xi)}\varphi\right)(g)=\int_{G(\mathbb{Q}_v)}f_v^d(s,\xi)(\iota(g_v',1))\varphi(gg_v')d_vg_v'.$$

Certainly in order for $T_{f_v(s,\xi)}$ to be well defined we must address convergence issues. The absolute convergence can be proved for $s\in\mathbb{C}$ with $\mathrm{Re}\,(s)$ sufficiently large. In our applications a meromorphic continuation always exists and we use it to define $T_{f_v(s,\xi)}$ for general $s\in\mathbb{C}$. In fact when $v\mid N$ or when $v=p$ and $\chi\psi_1,\cdots,\chi\psi_n$ are all nontrivial, by our choices the function $f^d_{\kappa,\underline{\tau},v}(\iota(\cdot,1))$ on $G(\mathbb{Q}_v)$ is compactly supported. When $v=\infty$ the absolute convergence follows from the fact that

π_∞ is a discrete series as discussed in [Li90]. The only place we need to be careful with the convergence issue is the computation in §5.7, i.e. the local zeta integral at p with some of $\chi\psi_1, \cdots, \chi\psi_n$ being trivial.

The doubling local zeta integral is defined as (purely locally)

$$Z_v(f_v(s,\xi),\cdot,\cdot) : \pi_v \times \tilde{\pi}_v \longrightarrow \mathbb{C}$$

$$(v_1, \tilde{v}_2) \longmapsto Z_v(f_v(s,\xi), v_1, \tilde{v}_2) = \int_{G(\mathbb{Q}_v)} f_v^d(s,\xi)(\iota(g_v,1)) \left\langle \pi_v(g_v)v_1, \tilde{v}_2 \right\rangle_v d_v g_v. \tag{4.1.2}$$

As a pairing between $I_{Q_H,v}(s,\xi)$ and $\pi_v \times \tilde{\pi}_v$, the doubling local zeta integral has the equivariance property that for $(g_1, g_2) \in G(\mathbb{Q}_v) \times G(\mathbb{Q}_v)$,

$$Z_v\left(R_v(\iota(g_1, g_2^\vartheta))f(s,\xi), \pi_v(g_1)v_1, \tilde{\pi}_v(g_2)\tilde{v}_2\right) = Z_v(f_v(s,\xi), v_1, \tilde{v}_2). \tag{4.1.3}$$

Remark 4.1.1. The standard notation for the zeta integral should be written as $Z_v(f_v^d(s,\xi), v_1, \tilde{v}_2)$. In our construction we always use $f_v(s,\xi)$ for computing the Fourier coefficients of $E^*(\cdot, f_v(s,\xi)) = E^*(\cdot, f_v^d(s,\xi))$ while the zeta integral is always computed with $f_v^d(s,\xi)$. The notation in (4.1.2) is more convenient for us here, and should cause no confusion.

Theorem 4.1.2 ([PSR87, Gar84, Shi00])**.** *Suppose $f(s,\xi) = \bigotimes_{s\notin S} f_v^{\mathrm{ur}}(s,\xi) \otimes \bigotimes_{v\in S} f_v(s,\xi)$ is a section inside to $I_{Q_H}(s,\xi)$. If $\varphi \in \pi^{K_G^S}$ with $K_G^S = \prod_{v\notin S} G(\mathbb{Z}_v)$, then*

$$\left\langle E^*\left(\iota(\cdot, g), f(s,\xi)\right), \overline{\varphi}\right\rangle = L^S(s+\frac{1}{2}, \pi\times\xi)\cdot\left(\prod_{v\in S} T_{f_v(s,\xi)}\overline{\varphi}\right)(g^\vartheta).$$

Equivalently for all factorizable $\varphi_1, \varphi_2 \in \pi^{K_G^S}$,

$$\left\langle E^*(\cdot, f(s,\xi))|_{G\times G}, \overline{\varphi}_1 \otimes \varphi_2^\vartheta \right\rangle = L^S(s+\frac{1}{2}, \pi\times\xi)\cdot\prod_{v\in S}\frac{Z_v(f_v(s,\xi), \overline{\varphi}_{1,v}, \varphi_{2,v})}{\left\langle \overline{\varphi}_{1,v}, \varphi_{2,v}\right\rangle_v}\left\langle \overline{\varphi}_1, \varphi_2\right\rangle.$$

Remark 4.1.3. Our formulation of the doubling method aligns with those of [Gar84, Shi00] where if the Siegel Eisenstein series on H is holomorphic its restriction to $G\times G$ is still holomorphic on both factors, because the embedding $\iota : G\times G \hookrightarrow H$ corresponds to the holomorphic embedding of the Siegel upper half spaces $\mathbb{H}_n \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{2n}$ sending (z_1, z_2) to $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. However it differs from the standard formulation in the study of the doubling method from the point of view of theta correspondence, where the embedding is equivalent to ι with a conjugation by ϑ on the second factor. The translation from the standard formulation to ours here depends on the choice of the map ϑ from $\mathbf{V}_n$ to itself with similitude -1.

4.2. **The "volume sections" at places dividing** N**.** For a place $v \mid N$ we pick a very simple so-called "volume section" that gives simple Fourier coefficients and easily computed local zeta integrals. Moreover it makes the restriction of the resulting Siegel Eisenstein series to $G\times G$ cuspidal when the archimedean section is taken to be f_∞^k. The cuspidality fact is crucial for us to apply Hida theory on G.

Define the Schwartz function $\alpha_v^{\mathrm{vol}} : \mathrm{Sym}(2n, \mathbb{Q}_v) \to \mathbb{C}$ to be the characteristic function of the compact open subset $-\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + N\,\mathrm{Sym}(2n, \mathbb{Z}_v)$ of $\mathrm{Sym}(2n, \mathbb{Q}_v)$. The "volume section" inside $I_{Q_H,v}(s,\xi)$ is defined as $f_v^{\mathrm{vol}}(s,\xi) = f_v^{\alpha_v^{\mathrm{vol}}}(s,\xi)$. It gives the Fourier coefficient

$$W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{vol}}(s,\xi)) = \widehat{\alpha}_v^{\mathrm{vol}}(\boldsymbol{\beta}) = |N|_v^{n(2n+1)}\mathbf{e}_v(2\,\mathrm{Tr}\,\beta_0)\cdot\mathbb{1}_{N^{-1}\,\mathrm{Sym}(2n,\mathbb{Z}_v)^*}(\boldsymbol{\beta})$$

for $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_0 \\ {}^{\mathrm{t}}\beta_0 & \beta_2 \end{pmatrix}$, where $\mathbb{1}_{N^{-1}\operatorname{Sym}(2n,\mathbb{Z}_v)^*}$ is the characteristic function of the set $N^{-1}\operatorname{Sym}(2n,\mathbb{Z}_v)^*$. The "volume section" $f^{\mathrm{vol}}_{\kappa,\underline{\tau},v}$ is independent of $\underline{\tau}$ and its corresponding Fourier coefficient is a p-adic integer independent of both κ and $\underline{\tau}$.

Next we compute the local zeta integral. Let $\Gamma(N)_v$ be the open compact subgroup of $G(\mathbb{Q}_v)$ consisting of elements in $G(\mathbb{Z}_v)$ whose reduction modulo N is 1.

Proposition 4.2.1. *Suppose $\varphi \in \pi$ is invariant under right translation by $\Gamma(N)_v$. Then*

$$T_{f^{\mathrm{vol}}_v(s,\xi)}\varphi = \xi_v(-1)^n \mathrm{vol}(\Gamma(N)_v) \cdot \varphi.$$

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q}_v)$ we have

$$\mathcal{S}^{-1}\iota(g,1) = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & -I_n & I_n & 0 \\ -I_n & 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_n & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_n & 0 & 0 \\ c & -I_n & d & 0 \\ -a & 0 & -b & I_n \end{pmatrix}. \tag{4.2.1}$$

It belongs to the support of $f^{\mathrm{vol}}_v(s,\xi)$ if and only if $\det\begin{pmatrix} c & -I_n \\ -a & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} c & -I_n \\ -a & 0 \end{pmatrix}^{-1}\begin{pmatrix} d & 0 \\ -b & I_n \end{pmatrix} = \begin{pmatrix} a^{-1}b & -a^{-1} \\ -d+ca^{-1}b & -ca^{-1} \end{pmatrix}$ belongs to $-\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + N\operatorname{Sym}(2n,\mathbb{Z}_v)$. Therefore

$$f^{d,\mathrm{vol}}_v(s,\xi)(\iota(g,1)) = \begin{cases} \xi_v(-1)^n & \text{if } g \in \Gamma(N)_v, \\ 0 & \text{otherwise,} \end{cases}$$

and the proposition follows. $\square$

For an admissible $(\kappa,\underline{\tau})$ we set $f_{\kappa,\underline{\tau},v} = f^{\mathrm{vol}}_{\kappa,\underline{\tau},v} = f^{\mathrm{vol}}_v(k-\frac{2n+1}{2}, \phi^{-1}\chi^{\circ-1})$ and use $f^{\mathrm{vol}}_{\kappa,\underline{\tau},N}$ to denote the product of local sections $\bigotimes_{v|N} f^{\mathrm{vol}}_{\kappa,\underline{\tau},v}$. We also put $\widehat{\alpha}^{\mathrm{vol}}_N = \prod_{v|N}\widehat{\alpha}^{\mathrm{vol}}_v$.

Before moving to the archimedean place, we record here the following theorem due to Garrett concerning the cuspidality of the restriction to $G \times G$ of the Siegel Eisenstein series.

Theorem 4.2.2 ([Gar92, p. 465-473])**.** *Let $f(s,\xi)$ be a factorizable section inside $I_{Q_H}(s,\xi)$ with $f_v(s,\xi) = f^{\mathrm{vol}}_v(s,\xi)$ for some finite place v and $f_\infty(s,\xi) = f^k_\infty(s,\xi)$, $k > 2n+1$. Then the evaluation at $s = k - \frac{2n+1}{2}$ of the restriction of the Siegel Eisenstein series $E(\cdot, f(s,\xi))|_{G\times G}$ is a cuspidal holomorphic Siegel modular form of scalar weight k on $G \times G$.*

4.3. **The archimedean sections.** We select a section $f_{\kappa,\underline{\tau},\infty}$ from $I_{Q_H,\infty}(k-\frac{2n+1}{2}, \phi^{-1}\chi^{\circ-1})$ for each admissible $(\kappa,\underline{\tau}) = (k\cdot\chi, \underline{t}\cdot\underline{\psi})$ with κ satisfying the parity condition $\phi\chi(-1) = (-1)^k$. Denote by $\mathcal{D}_{\underline{t}}$ the holomorphic discrete series $(\mathfrak{g}, K_{G,\infty})$-module whose lowest $K_{G,\infty}$-type is of highest weight $\underline{t}$, and by $\mathcal{D}_{\underline{t}}(\underline{t})$ the lowest $K_{G,\infty}$-type inside $\mathcal{D}_{\underline{t}}$. Let $\widetilde{\mathcal{D}_{\underline{t}}}$ be the contragredient of $\mathcal{D}_{\underline{t}}$ and $\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})$ be its highest $K_{G,\infty}$-type.

In our application of the doubling method formula, the cuspidal automorphic forms φ on $G(\mathbb{A})$ we consider are those coming from global sections of the automorphic sheaf $\omega_{\underline{t}} = \mathcal{V}^0_{\underline{t}}$ over Shimura varieties of certain level through the map (2.4.3). Thus the archimedean factor π_∞ is a holomorphic discrete series and φ_∞ lies inside its lowest $K_{G,\infty}$-type. The nonvanishing condition we put on $f_{\kappa,\underline{\tau},\infty}$ is that for all such φ, the $(-\underline{t})$-isotypic part of $T_{f_{\kappa,\underline{\tau},\infty}}\overline{\varphi}$ is nontrivial, or equivalently the map $Z_\infty(f_{\kappa,\underline{\tau},\infty},\cdot,\cdot) : \widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t}) \to \mathbb{C}$ is nonzero.

For the case $t_1 = \cdots = t_n = k$ the very canonical choice for the archimedean section is f^k_∞. The corresponding local zeta integral is computed in [Shi95] and the results clearly imply the nonvanishing. From Proposition 3.5.1 one sees that f^k_∞ also satisfies the condition that after dividing an explicit scalar, its Fourier coefficients are all algebraic.

In order for $E^*(\cdot, f_{\kappa,\underline{\tau}})$ to be algebraic it is natural to consider sections obtained by applying operators constructed from $\mathfrak{q}^+_H$ to f^k_∞, because then our discussion in §2 shows that the resulting Siegel Eisenstein series can be obtained by applying the (geometrically defined) differential operators to $E^*(\cdot, f^{k,\alpha_{S_{\mathrm{f}}}})$, and the differential operators have an algebraic structure as well as formulas on q-expansions.

Recall that we have fixed a basis $\widehat{\mu}^+_{ij}$, $1 \le i \le j \le 2n$ for the Lie algebra $\mathfrak{q}^+_H$. Putting $\widehat{\mu}^+_{ij} = \widehat{\mu}^+_{ji}$ for $i > j$, we let $\widehat{\mu}^+_H$ be the symmetric $2n \times 2n$ matrix whose (i,j) entry is $\widehat{\mu}^+_{ij}$. Write $\widehat{\mu}^+_H = \begin{pmatrix} \widehat{\mu}^+_1 & \widehat{\mu}^+_0 \\ {}^{\mathrm{t}}\widehat{\mu}^+_0 & \widehat{\mu}^+_2 \end{pmatrix}$ in $n \times n$ blocks.

Inspired by [Har86], we define the following archimedean section

$$f_{\kappa,\underline{\tau},\infty} = \prod_{l=1}^{n} \det{}_l \left(\frac{1}{4\pi i} \widehat{\mu}^+_0 \right)^{t_l - t_{l+1}} \cdot f^k_\infty, \tag{4.3.1}$$

where we put $t_{n+1} = k$ and for a matrix A we use $\det_l(A)$ to denote the determinant of its upper left $l \times l$ minor. The rest of this section is devoted to proving the following proposition stating that this $f_{\kappa,\underline{\tau},\infty}$ satisfies the nonvanishing condition. The strategy for making this selection will manifest in the proof.

Proposition 4.3.1. *With $f_{\kappa,\underline{\tau},\infty}$ defined as in (4.3.1), the map*

$$Z_\infty(f_{\kappa,\underline{\tau},\infty}, \cdot, \cdot) : \widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t}) \longrightarrow \mathbb{C} \tag{4.3.2}$$

is nonzero. Let $v_{\underline{t}} \in \mathcal{D}_{\underline{t}}(\underline{t})$ be a nonzero vector of highest weight and $v^\vee_{\underline{t}} \in \widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})$ be its dual vector. Then the number $\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v^\vee_{\underline{t}}, v_{\underline{t}})}{\langle v^\vee_{\underline{t}}, v_{\underline{t}} \rangle}$ is nonzero

Proof. Let $U(\mathfrak{h}_\mathbb{C}) \cdot f^{d,k}_\infty$ be the sub-$(\mathfrak{h}_\mathbb{R}, K_{H,\infty})$-module of $I_{P_H,\infty}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{\circ-1})$ generated by $f^{d,k}_\infty$. As explained above due to the algebraicity consideration we want to pick our $f^d_{\kappa,\underline{\tau},\infty}$ from $U(\mathfrak{h}_\mathbb{C}) \cdot f^{d,k}_\infty$. Regarding $U(\mathfrak{h}_\mathbb{C}) \cdot f^{d,k}_\infty$ as a representation of the compact group $K_{G,\infty} \times K_{G,\infty}$, we prove that in the decomposition of $U(\mathfrak{h}_\mathbb{C}) \cdot f^{d,k}_\infty|_{K_{G,\infty} \times K_{G,\infty}}$, there is a unique piece $\sigma_{k,\underline{t}}$ which pairs nontrivially with $\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t})$ under the zeta integral. Then we finish the proof by showing that $f^d_{\kappa,\underline{\tau},\infty}$ has a nonzero projection into $\sigma_{k,\underline{t}}$.

We start by introducing several unitarizable irreducible $(\mathfrak{h}_\mathbb{R}, K_{H,\infty})$-modules whose $K_{H,\infty}$-finite parts are isomorphic to $U(\mathfrak{h}_\mathbb{C}) \cdot f^{d,k}_\infty$ or its contragradient when the parameters are within the range relevant to us here. Let (σ, W_σ) be a finite dimensional algebraic representation of $\mathrm{GL}(2n)$. Then $W_\sigma(\mathbb{C})$ is a $K_{H,\infty}$-representation. Define the $H(\mathbb{R})$-representation

$$\mathcal{O}(H(\mathbb{R}), K_{H,\infty}, \sigma) = \left\{ \begin{matrix} \text{analytic functions } f : H(\mathbb{R}) \to W_\sigma(\mathbb{C}) \text{ that are annihilated} \\ \text{by the action of } \mathfrak{q}^-_H \text{ on the right, and } f(hk) = \sigma^{-1}(k) f(g) \\ \text{for all } k \in K_{H,\infty}, h \in H(\mathbb{R}) \end{matrix} \right\}$$

with $H(\mathbb{R})$ acting by left inverse translation. Let $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ be the $(\mathfrak{h}_\mathbb{R}, K_{H,\infty})$-module which is the subspace of $\mathcal{O}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ spanned by $K_{H,\infty}$-finite vectors. Let $\mathcal{O}(\mathbb{H}_{2n}, \sigma)$ be the space of W_σ-valued holomorphic functions on $\mathbb{H}_{2n}$ with $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \in H(\mathbb{R})$ acting on $f \in \mathcal{O}(\mathbb{H}_{2n}, \sigma)$ via

$$(h \cdot f)(\boldsymbol{z}) = \sigma\left({}^{\mathrm{t}}(C\boldsymbol{z} + D)\right) f\left((A\boldsymbol{z} + B)(C\boldsymbol{z} + D)^{-1}\right).$$

It is easily seen that $\mathcal{O}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ is isomorphic to $\mathcal{O}(\mathbb{H}_{2n}, \sigma)$ (c.f. Remark 2.4.2). One can also check that the $\mathfrak{h}_{\mathbb{C}}$-module $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ is isomorphic to the base change to $\mathbb{C}$ of the $\mathfrak{h}$-module V_σ defined in §2.2, and that the formulas there show that it has a unique highest $K_{H,\infty}$-type σ which is contained inside every sub-representation.

Let $W_{2k,0}$ be the real vector space of dimension $2k$ with a positive definite symmetric pairing and $O(2k,0)$ be the associated orthogonal group. The action of $O(2k,0) \times H(\mathbb{R})$ on the Schrödinger model $\mathcal{S}(W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}})$, the space of Schwartz functions on $W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}}$, of its Weil representation with respect to the polarization $\mathbf{V}_{2n} = \mathbf{V}_{2n}^d \oplus \mathbf{V}_{2n,d}$ is given by

$$
\begin{aligned}
&(\omega(a)\mathfrak{s})(x) = \mathfrak{s}({}^{\mathrm{t}}ax), && a \in O(2k,0),\\
&(\omega(m)\mathfrak{s})(x) = \det\left(m|_{V^d_{2n,\mathbb{R}}}\right)^k \mathfrak{s}({}^{\mathrm{t}}mx), && m \in P(\mathbf{V}^d_{2n,\mathbb{R}}) \cap P(\mathbf{V}_{2n,d,\mathbb{R}}),\\
&(\omega(u)\mathfrak{s})(x) = \mathbf{e}_\infty(\langle -u(x), x\rangle/2)\mathfrak{s}(x), && u \in N(\mathbf{V}^d_{2n,\mathbb{R}}),\\
&(\omega(w)\mathfrak{s})(x) = i^{2nk}\int_{W_{2k,0}\otimes\mathbf{V}_{2n,d,\mathbb{R}}} \mathbf{e}_\infty(\langle y, wx\rangle)\mathfrak{s}(y)dy.
\end{aligned}
$$

Here for an isotropic subspace V, $P(V)$ is the stabilizer of V and $N(V)$ is the unipotent radical of $P(V)$. The element w in $H(\mathbb{R})$ is the one sending $(v, \vartheta(v))$ to $(v, -\vartheta(v))$ and $(v, -\vartheta(v))$ to $-(v, \vartheta(v))$ for $v \in \mathbf{V}_n$.

Let $\Theta_{2k,0}(0) = \mathcal{S}(W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}})^{O(2k,0)}$ be the theta lift of the trivial representation from $O(2k,0)$ to $H(\mathbb{R})$. The morphism

$$
\begin{aligned}
\Phi : \mathcal{S}(W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}}) &\longrightarrow I_{P_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)\\
\mathfrak{s} &\longmapsto \Phi(\mathfrak{s})(g) := (\omega(g)\mathfrak{s})(0)
\end{aligned}
$$

embeds $\Theta_{2k,0}(0)$ into the degenerate principal series [KR90a, Theorem 3]. We denote by $R^d_{2k,0}$ the image of $\Theta_{2k,0}(0)$ inside $I_{P_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)$ and $R_{2k,0}$ be the sub-$H(\mathbb{R})$-representation of $I_{Q_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)$, which corresponds to $R^d_{2k,0}$ via (4.1.1).

The representation $\Theta_{2k,0}(0)$ is unitary and embeds into $\mathcal{O}(\mathbb{H}_{2n}, -k)$ through the map

$$
\mathfrak{s} \longmapsto \int_{W_{2k,0}\otimes\mathbf{V}_{2n,d,\mathbb{R}}} \mathbf{e}_\infty(\mathrm{Tr}\, x\,{}^{\mathrm{t}}x\boldsymbol{z})\mathfrak{s}(x)dx.
$$

Therefore $\Theta_{2k,0}(0)$ is irreducible. If $k \geq n$ (we have always assumed $k \geq n+1$) the image is dense [KV78, p. 3]. It follows that $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, -k)$ is irreducible, so isomorphic to the Verma module $U(\mathfrak{h}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{H,\mathbb{C}}\oplus\mathfrak{q}^+_H)} \det^{-k}$ of highest weight $-k$.

We use the superscript MVW to denote the MVW-involution, i.e. conjugation by ϑ, of the above defined representations. In our case, thanks to the irreducibility, the MVW-involution is isomorphic to the contragredient representation. By using $-W_{2k,0}$ we define $\Theta_{0,2k}(0)$ and $R_{0,2k} \subset I_{P_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)$. It is easily seen that $\Theta_{0,2k}(0) \cong \Theta_{2k,0}(0)^{\mathrm{MVW}}$. The $K_{H,\infty}$-finite part of $R_{0,2k}$ will be denoted as $R^{\mathrm{f}}_{0,2k}$.

The degenerate principal $I_{Q_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)$ is $K_{H,\infty}$-multiplicity free [Gui80]. Both $U(\mathfrak{h}_{\mathbb{C}}) \cdot f^{d,k}_\infty$ and $R^{\mathrm{f}}_{0,2k}$ are irreducible $\mathfrak{h}_{\mathbb{C}}$-submodules of the degenerate principal series and contain the $K_{H,\infty}$-type of scalar weight k. Hence they must be equal to each other, and we are reduced to studying the $\mathfrak{h}_{\mathbb{C}}$-module $R^{\mathrm{f}}_{0,2k}$, which by the above discussion is isomorphic to $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, -k)^{\mathrm{MVW}}$ and the Verma module $M_k = U(\mathfrak{h}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{H,\mathbb{C}}\oplus\mathfrak{q}^-_H)} \det^k$ of lowest weight k. Regarding its decomposition as a $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-module there is the following theorem.

Theorem 4.3.2 ([JV79, Proposition 2.2, Corollary 2.3])**.** *If $k \geq n+1$, then*

$$\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, -k)|_{\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}} = \bigoplus_{r=0}^{\infty} \mathcal{O}^{\mathrm{f}}\left(G(\mathbb{R}) \times G(\mathbb{R}), K_{G,\infty} \times K_{G,\infty}, \det^{-k} \otimes \mathrm{Sym}^r(\mathfrak{q}_H^-/\mathfrak{q}_G^- \times \mathfrak{q}_G^-)\right).$$

Applying the decomposition results on algebraic $\mathrm{GL}(n)$-representations [Shi84, Theorem 2.A], we have

$$\mathrm{Sym}^r(\mathfrak{q}_H^+/\mathfrak{q}_G^+ \times \mathfrak{q}_G^+) \cong \bigoplus_{\substack{a_1 \geq \cdots \geq a_n \geq 0 \\ |\underline{a}| = r}} W_{\underline{a}}(\mathbb{C}) \boxtimes W_{\underline{a}}(\mathbb{C})$$

as $K_{G,\infty} \times K_{G,\infty}$-representations, where $|\underline{a}| = a_1 + \cdots + a_n$. Let $\underline{a}' = (-a_n, \cdots, -a_1)$. When $t_n \geq n+1$ the $(\mathfrak{g}_{\mathbb{R}}, K_{G,\infty})$-module $\mathcal{O}^{\mathrm{f}}(\mathbb{H}_n, \underline{a}')^{\mathrm{MVW}}$ gives the holomorphic discrete series $\mathcal{D}_{\underline{a}}$ of lowest $K_{G,\infty}$-type $\underline{a}$. Since we have always assumed $k \geq n+1$ we obtain the the multiplicity free decomposition

$$R_{0,2k}|_{\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}} \cong \bigoplus_{a_1 \geq \cdots \geq a_n \geq k} \mathcal{D}_{\underline{a}} \boxtimes \mathcal{D}_{\underline{a}}. \tag{4.3.3}$$

Let $\sigma_{k,\underline{t}}$ be the unique $K_{G,\infty} \times K_{G,\infty}$-sub-representation of $R_{0,2k}$ that corresponds to $\mathcal{D}_{\underline{t}}(\underline{t}) \boxtimes \mathcal{D}_{\underline{t}}(\underline{t})$ under the above isomorphism. Now due to the equivariance property (4.1.3) it is clear that the zeta integral pairing

$$Z_\infty : R_{0,2k} \times \left(\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t})\right) \longrightarrow \mathbb{C} \tag{4.3.4}$$

factors through $\sigma_{k,\underline{t}}$.

Lemma 4.3.3. *The pairing* (4.3.4) *is nontrivial.*

Proof. Since the representation of $G(\mathbb{R})$ we are considering is discrete series, the arguments in [Li90] demonstrates the equivalence between the nontriviality of (4.3.4) and the nonvanishing of the theta lift of $\mathcal{D}_{\underline{t}}$ from $G(\mathbb{R})$ to $O(0, 2k)$. The nonvanishing of this theta lift is easily seen from [KV78, Theorem (6.13)] or from (4.3.3) plus the doubling seesaw. ☐

Thus a section inside $R_{0,2k}$ pairs nontrivially with $\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t})$ by the zeta integral if and only if its projection to $\sigma_{k,\underline{t}}$ is nontrivial. Once we know that the projection of $f^d_{\kappa,\underline{\tau},\infty}$ to $\sigma_{k,\underline{t}}$ is nonzero, we can deduce the nonvanishing of the map (4.3.2), as well as that of the number $\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v^\vee_{\underline{t}}, v_{\underline{t}})}{\langle v^\vee_{\underline{t}}, v_{\underline{t}}\rangle}$ in the statement of Proposition 4.3.1 since by [Shi84, Theorem 2.A] and the definition of $f^d_{\kappa,\underline{\tau},\infty}$, its projection to $\sigma_{k,\underline{t}}$ is the highest weight vector on both factors. Therefore the last step is to prove the following lemma.

Lemma 4.3.4. *The section $f^d_{\kappa,\underline{\tau},\infty}$ projects nontrivially onto $\sigma_{k,\underline{t}}$.*

Proof. Let v_k be the lowest weight vector of the Verma module $M_k = U(\mathfrak{h}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{H,\mathbb{C}} \oplus \mathfrak{q}_H^-)} \det^k$. Under the isomorphism between $R_{0,2k}$ and M_k the section $f^{d,k}_\infty$ corresponds to v_k. Therefore by the definition of $f^d_{\kappa,\underline{\tau},\infty}$, what we need to show is that $\prod_{l=1}^n \det_l \left(\widehat{\mu}_0^+\right)^{t_l - t_{l+1}} \cdot v_k$ has a nontrivial projection onto the lowest $\mathfrak{k}_{G,\mathbb{C}} \times \mathfrak{k}_{G,\mathbb{C}}$-type of the $\mathcal{D}_{\underline{t}} \boxtimes \mathcal{D}_{\underline{t}}$-isotypic component of $M_k|_{\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}}$. The universal enveloping algebra $U(\mathfrak{h}_{\mathbb{C}})$ comes with a natural grading $\bigcup_{r\geq 0} U_r(\mathfrak{h}_{\mathbb{C}})$, where $U_r(\mathfrak{h}_{\mathbb{C}})$ is spanned by elements that can be written as a product of no more than r vectors in $\mathfrak{h}_{\mathbb{C}}$. Viewing M_k as a $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ module, it has the natural filtration $\bigcup_{r\geq 0} M_{k,r}$, with $M_{k,r}$ being the module generated by v_k under the action of $U(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}})$ and $U_r(\mathfrak{h}_{\mathbb{C}})$. Let $\mathfrak{q}_i^+$ be the Lie subalgebra of the abelian Lie algebra of $\mathfrak{q}_H^+$ spanned by entries of $\widehat{\mu}_i^+$, for $i=0,1,2$. Consider the morphism of $\mathbb{C}$-vector spaces

$$\begin{aligned} U(\mathfrak{q}_1^+\oplus\mathfrak{q}_2^+)\otimes_{\mathbb{C}} U_r(\mathfrak{q}_0^+) &\longrightarrow M_{k,r} \\ \alpha\otimes\beta &\longmapsto \alpha\beta\cdot v_k. \end{aligned} \tag{4.3.5}$$

It is injective by the PBW theorem, and the image contains $U_r(\mathfrak{h}_{\mathbb{C}})\cdot v_k$. From the relation $[[(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\times(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-),\mathfrak{q}_0^+],\mathfrak{q}_0^+]\subset\mathfrak{q}_H^+$ we see that $\big((\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\times(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\big)\cdot U_r(\mathfrak{q}_0^+)\cdot v_k$ is contained in $U(\mathfrak{q}_1^+\oplus\mathfrak{q}_2^+)U_r(\mathfrak{q}_0^+)\cdot v_k$. Therefore the image of (4.3.5) is stable under the action of $(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\times(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)$, and (4.3.5) is a bijection, which implies that

$$\prod_{l=1}^n \det{}_l\left(\widehat{\mu}_0^+\right)^{t_l-t_{l+1}}\cdot v_k\notin M_{k,|\underline{t}|-nk-1}. \tag{4.3.6}$$

At the same time the bijection (4.3.5) gives

$$\begin{aligned} M_{k,r}/M_{k,r-1} &\cong U(\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}})\otimes_{U\left((\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\times(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\right)}\left(\mathrm{Sym}^r(\mathfrak{q}_H^+/\mathfrak{q}_G^+\times\mathfrak{q}_G^+)\otimes\det{}^k\right) \\ &\cong \bigoplus_{\substack{a_1\geq\cdots\geq a_n\geq 0\\ |\underline{a}|=r}}\left(U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)}W_{\underline{a}+k}\right)\boxtimes\left(U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)}W_{\underline{a}+k}\right) \\ &\cong \bigoplus_{\substack{a_1\geq\cdots\geq a_n\geq 0\\ |\underline{a}|=r}}\mathcal{D}_{\underline{a}+k}\boxtimes\mathcal{D}_{\underline{a}+k}. \end{aligned} \tag{4.3.7}$$

The vector $\prod_{l=1}^n\det_l\left(\widehat{\mu}_0^+\right)^{t_l-t_{l+1}}\cdot v_k$ belongs to the $\underline{t}\times\underline{t}$-isotypic part of $M_k|_{\mathfrak{k}_{G,\mathbb{C}}\times\mathfrak{k}_{G,\mathbb{C}}}$, so its image in $M_{k,|\underline{t}|-nk}/M_{k,|\underline{t}|-nk-1}$, which is nonzero by (4.3.6), lands inside $\mathcal{D}_{\underline{t}}(\underline{t})\boxtimes\mathcal{D}_{\underline{t}}(\underline{t})$ under the isomorphism (4.3.7). Now we can conclude that $\prod_{l=1}^n\det_l\left(\widehat{\mu}_0^+\right)^{t_l-t_{l+1}}\cdot v_k$ projects nontrivially to the lowest $\mathfrak{k}_{G,\mathbb{C}}\times\mathfrak{k}_{G,\mathbb{C}}$-type of the $\mathcal{D}_{\underline{t}}\boxtimes\mathcal{D}_{\underline{t}}$-isotypic component of $M_k|_{\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}}$. □

□

4.4. **The q-expansions.** For a Schwartz function α_p on $\mathrm{Sym}(2n,\mathbb{Q}_p)$ whose Fourier transform is supported on the compact set (3.2.3) and takes values inside a number field on $\mathrm{Sym}(2n,\mathbb{Q})$, set

$$f_{\kappa,\underline{\tau}}^{\alpha_p}=\bigotimes_{v\notin S}f_v^{\mathrm{ur}}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\otimes f_{\kappa,\underline{\tau},N}^{\mathrm{vol}}\otimes f_p^{\alpha_p}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\otimes f_{\kappa,\underline{\tau},\infty}.$$

From the discussion in §2 we see that the Siegel Eisenstein series $E^*(\cdot,f_{\kappa,\underline{\tau}}^{\alpha_p})$ on H and its restriction to $G\times G$ are both nearly holomorphic of degree less or equal to $|\underline{t}|-nk$. Since the archimedean section $f_{\kappa,\underline{\tau},\infty}$ belongs to the $\underline{t}\boxtimes\underline{t}$-isotypic component of $I_{Q_H,\infty}(k-\frac{2n+1}{2},\mathrm{Sign}^k)|_{K_{G,\infty}\times K_{G,\infty}}$ and is of weight $(\underline{t},\underline{t})$, we know that the form $A_{n,\phi,k,\chi}^{-1}\cdot E^*(\cdot,f_{\kappa,\underline{\tau}}^{\alpha_p})|_{G\times G}$ lies inside the image of the embedding

$$H^0(X_{G,\Gamma}\times X_{G,\Gamma},\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk}\boxtimes\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk})\otimes_{\mathbb{Q}(\zeta_N)}F\xrightarrow{\varphi_{G\times G}(\cdot,\mathfrak{e}_{\mathrm{can}})}\mathcal{A}(G(\mathbb{Q})\times G(\mathbb{Q})\backslash G(\mathbb{A})\times G(\mathbb{A})/\widehat{\Gamma}\times\widehat{\Gamma})_{\underline{t}\boxtimes\underline{t}},$$

where $\Gamma=\Gamma_1(N,p^m)$ with m sufficiently large, and F is a sufficiently large number field . We denote by $\mathcal{E}_{\kappa,\underline{\tau}}^{\alpha_p}$ the global section of $\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk}\boxtimes\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk}$ over $X_{G,\Gamma}\times X_{G,\Gamma}$ which is mapped to $A_{n,\phi,k,\chi}^{-1}\cdot E^*(\cdot,f_{\kappa,\underline{\tau}}^{\alpha_p})|_{G\times G}$, and consider the ($p$-adic) q-expansion, defined as (2.6.1), of the nearly holomorphic form $\mathcal{E}_{\kappa,\underline{\tau}}^{\alpha_p}$. Let $\Sigma_{N,p,+}$ be the intersection of the set $\Sigma_{p,+}$ and $\bigcap_{v|N}N^{-1}\,\mathrm{Sym}(2n,\mathbb{Z}_v)^*$.

Proposition 4.4.1. *Suppose* $(\kappa, \underline{\tau}) \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ *is an admissible point satisfying the parity condition* $\phi\chi(-1) = (-1)^k$*. Then*

$$\varepsilon_{q,p\text{-adic}}(\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}) = \sum_{\beta_1,\beta_2 \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*} \sum_{\boldsymbol{\beta}=\begin{pmatrix}\beta_1 & \beta_0\\ {}^{\mathrm{t}}\beta_0 & \beta_2\end{pmatrix}\in\Sigma_{N,p,+}} \mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}}(\boldsymbol{\beta}) q^{\beta_1} q^{\beta_2}, \tag{4.4.1}$$

with

$$\begin{aligned}\mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}}(\boldsymbol{\beta}) =& \frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})} \widehat{\alpha}^{\mathrm{vol}}_N(\boldsymbol{\beta}) \cdot \chi^{-1}(C_{\phi_{\boldsymbol{\beta}}}) C_{\phi_{\boldsymbol{\beta}}}^{-k+n+1} \cdot L_N(k-n, \phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1} \cdot L^p(1-k+n, \phi_{\boldsymbol{\beta}}\chi)\\ &\times g^S_{\boldsymbol{\beta}}(k,\phi\chi) \cdot \widehat{\alpha}_p(\boldsymbol{\beta}) \prod_{l=1}^n \det\nolimits_l(2\beta_0)^{t_l - t_{l+1}} \det(2\boldsymbol{\beta})^{k-n-1}.\end{aligned}$$

Proof. The proof is straightforward. All we need to be careful about is to be precise with all representations and maps involved here, instead of looking at isomorphim classes or working up to scalars. We use the symbol $\underline{\tau}_k$ to mean an arithmetic character of $T_n(\mathbb{Z}_p)$ with algebraic part equal to the scalar weight $k = \kappa_{\mathrm{alg}}$. Let $E^{\alpha_p}_{\kappa,\underline{\tau}_k}$ be the inverse image of $A^{-1}_{n,\phi,k,\chi} \cdot E^*(\cdot, f^{\alpha_p}_{\kappa,\underline{\tau}_k})$ under the map $\varphi_H(\cdot, \mathfrak{e}_{\mathrm{can}})$, which is a global section of the sheaf $\omega_k = \mathcal{V}^0_k$ over $X_{H,\Gamma}$. It follows from the definition of polynomial q-expansions, the canonical test object carried by $\mathbb{H}_{2n}$ and Proposition 3.5.1 that

$$E^{\alpha_p}_{\kappa,\underline{\tau}_k}(q, \underline{Y}) = \sum_{\boldsymbol{\beta}\in\Sigma_{N,p,+}} \mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}_k}(\boldsymbol{\beta}) \cdot v_k q^{\boldsymbol{\beta}},$$

where v_k is a basis of the representation $\det^k$. Let $\underline{\boldsymbol{X}} = (\boldsymbol{X}_{ij})_{\leq i,j\leq 2n}$ be the basis of the representation τ_{2n} defined as in the paragraphs above Proposition 2.4.1, and we write it in $n \times n$ blocks as $\begin{pmatrix}\underline{X}_1 & \underline{X}_0\\ {}^{\mathrm{t}}\underline{X}_0 & \underline{X}_2\end{pmatrix}$. Applying Proposition 2.6.1 we get

$$(D^e_k E^{\alpha_p}_{\kappa,\underline{\tau}_k})(q, 0) = \sum_{\boldsymbol{\beta}\in\Sigma_{N,p,+}} \mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}_k}(\boldsymbol{\beta}) \cdot v_k \otimes \left(\sum_{1\leq i\leq j\leq 2n} (2-\delta_{ij})\boldsymbol{\beta}_{ij}\boldsymbol{X}_{ij}\right)^e q^{\boldsymbol{\beta}}. \tag{4.4.2}$$

Let $\tau_{2n,0}$ be the direct summand of $\tau_{2n}|_{\mathrm{GL}(n)\times\mathrm{GL}(n)}$ generated by entries of $\underline{X}_0$. For $\underline{a} \in X(T_n)_+$ with $|\underline{a}| = e$ and $a_n \geq 0$, put $a_{n+1} = 0$ and fix the morphism of $\mathrm{GL}(n) \times \mathrm{GL}(n)$-representations

$$\det\nolimits^k \otimes \operatorname{Sym}^e \tau_{2n,0} \longrightarrow W_{\underline{a}+k} \boxtimes W_{\underline{a}+k} \tag{4.4.3}$$

sending $v_k \otimes \prod_{l=1}^n \det_l(\underline{X}_0)^{a_l - a_{l+1}}$ to the vector $w_{\underline{a}+k} \boxtimes w_{\underline{a}+k}$. Here for each $\underline{b} \in X(T_n)_+$ the function $w_{\underline{b}} : \mathrm{GL}(n)/N_n \to \mathbb{A}^1$ is defined as $w_{\underline{b}}(g) = \det(g)^{-b_1} \prod_{l=1}^{n-1} \det_{n-l}(g)^{b_l - b_{l+1}}$. Recall that $V^r_{k\otimes\operatorname{Sym}^e\tau_{2n}} = \det^k \otimes \operatorname{Sym}^e \tau_{2n}[\underline{\boldsymbol{Y}}]_{\leq r}$. Similarly to $\underline{\boldsymbol{X}}$ we write $\underline{\boldsymbol{Y}} = \begin{pmatrix}\underline{Y}_1 & \underline{Y}_0\\ {}^{\mathrm{t}}\underline{Y}_0 & \underline{Y}_2\end{pmatrix}$. It is easy to check that modulo $\underline{X}_1$, $\underline{X}_2$, $\underline{Y}_0$ gives rise to a $Q_G \times Q_G$-representation morphism from $V^r_{k\otimes\operatorname{Sym}^e\tau_{2n}}|_{Q_G\times Q_G}$ to $\operatorname{Sym}^e \tau_{2n,0} \otimes (V^r_k \boxtimes V^r_k)$ which, when composed with (4.4.3), gives

$$\pi_{k,\underline{a}} : V^r_{k\otimes\operatorname{Sym}^e\tau_{2n}}\big|_{Q_G\times Q_G} \longrightarrow V^r_{\underline{a}+k} \boxtimes V^r_{\underline{a}+k}.$$

Given Proposition 2.3.1, 2.4.1 it is tautological to check that we have the following commutative diagram

$$
\begin{array}{ccc}
H^0(X_{H,\Gamma}, \omega_k) & \xrightarrow{\quad \varphi_H(\cdot, \mathfrak{e}_{\mathrm{can}}) \quad} & \mathcal{A}(H(\mathbb{Q})\backslash H(\mathbb{A})) \\
\Big\downarrow {\scriptstyle D_k^e} & & \Big\downarrow {\scriptstyle \prod_{l=1}^n \det_l\left(-\frac{1}{4\pi i}\widehat{\mu}_0^+\right)^{a_l - a_{l+1}}} \\
H^0(X_{H,\Gamma}, \mathcal{V}^e_{k\otimes \mathrm{Sym}^e \tau_{2n}}) & \xrightarrow{\quad \varphi_H(\cdot, \mathfrak{e}_{\mathrm{can}} \otimes w_{\underline{a}'}(\underline{X}_0^*)) \quad} & \mathcal{A}(H(\mathbb{Q})\backslash H(\mathbb{A})) \\
\Big\downarrow & & \Big\downarrow \\
H^0(X_{G,\Gamma} \times X_{G,\Gamma}, \iota^* \mathcal{V}^e_{k\otimes \mathrm{Sym}^e \tau_{2n}}) & & \\
\Big\downarrow {\scriptstyle \pi_{k,\underline{a}}} & & \\
H^0(X_{G,\Gamma} \times X_{G,\Gamma}, \mathcal{V}^e_{\underline{a}+k} \boxtimes \mathcal{V}^e_{\underline{a}+k}) & \xrightarrow{\varphi_{G\times G}(\cdot, \mathfrak{e}_{\mathrm{can}})} & \mathcal{A}(G(\mathbb{Q}) \times G(\mathbb{Q})\backslash G(\mathbb{A}) \times G(\mathbb{A})),
\end{array}
$$

where $\underline{a}' = (-a_n, \cdots, -a_1)$. Thus the ($p$-adic) q-expansion of $\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}$ is obtained from applying $w_{\underline{a}'}(\underline{X}_0^*)$, with $\underline{a} = \underline{t} - k$, to (4.4.2) and setting $\underline{\boldsymbol{Y}}$ to zero, $q^{\boldsymbol{\beta}}$ to $q^{\beta_1} q^{\beta_2}$, i.e.

$$
\varepsilon_{q,p\text{-adic}}(\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}) = \sum_{\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_0 \\ {}^{\mathrm{t}}\beta_0 & \beta_2 \end{pmatrix} \in \Sigma_{N,p,+}} \prod_{l=1}^{n} \det_l(2\beta_0)^{t_l - t_{l+1}} \mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}_k}(\boldsymbol{\beta}) q^{\beta_1} q^{\beta_2},
$$

which is exactly (4.4.1). ☐

5. The measure $\mu_{\mathcal{E},q\text{-exp}}$ and local zeta integrals at p

We review briefly the theory of p-adic measures, and then pick suitable $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ such that the $\varepsilon_{q,p\text{-adic}}(\mathcal{E}^{\alpha_{\kappa,\underline{\tau},p}}_{\kappa,\underline{\tau}})$'s amalgamate into an element of $\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F[[N^{-1}\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right)$, where F is a finite extension of $\mathbb{Q}_p$ containing all N-th roots of unity. Then we retrieve $f_{p,\kappa,\underline{\tau}}$ from $\widehat{\alpha}_{p,\kappa,\underline{\tau}}$ and carry out local computations at p.

5.1. **p-adic measures.** Suppose that Y is a compact and totally disconnected topological space. Let R be a p-adic ring, i.e. $R = \varprojlim R/p^n R$, and M be a p-adically complete R-module. Denote by $\mathscr{C}(Y,R)$ the R-algebra of continuous R-valued functions on Y. An M-valued p-adic measure on Y is a continuous R-linear map

$$
\begin{aligned}
\mu : \mathscr{C}(Y,R) &\longrightarrow M \\
f &\longmapsto \mu(f) = \int_Y f\, d\mu,
\end{aligned}
$$

where the topology on $\mathscr{C}(Y,R)$ is the topology of uniform convergence. The set of M-valued p-adic measures on Y is a p-adically complete R-module and is denoted as $\mathcal{M}eas(Y,M)$. For an R-algebra R', which is also p-adically complete, since $\mathscr{C}(Y,R') = \mathscr{C}(Y,R)\widehat{\otimes}R'$, there is a natural map $\mathcal{M}eas(Y,M) \to \mathcal{M}eas(Y, M\widehat{\otimes}R')$ and we view $\mathcal{M}eas(Y,M)$ as a subset of $\mathcal{M}eas(Y, M\widehat{\otimes}R')$ if $R \to R'$ is injective. From definition it is easily seen that we have the following maps

$$
\begin{aligned}
Y &\longrightarrow \mathcal{M}eas(Y,R) \\
y &\longmapsto \delta_y(f) := f(y),
\end{aligned} \tag{5.1.1}
$$

and

$$\begin{aligned} \mathscr{M}eas(Y,M)\times\mathscr{C}(Y,R) &\longrightarrow \mathscr{M}eas(Y,M) \\ (\mu,h) &\longmapsto \mu_h(f):=\int_Y fh\,d\mu. \end{aligned} \tag{5.1.2}$$

Moreover if we assume that Y is equipped with the structure of an abelian group (written multiplicatively), then we can define the convolution on $\mathscr{M}eas(Y,R)$ as

$$\begin{aligned} \mathscr{M}eas(Y,R)\times\mathscr{M}eas(Y,R) &\longrightarrow \mathscr{M}eas(Y,R) \\ (\mu_1,\mu_2) &\longmapsto \mu_1*\mu_2(f):=\int_Y\int_Y f(yz)\,d\mu_1(y)d\mu_2(z). \end{aligned} \tag{5.1.3}$$

If $f\in\mathrm{Hom}_{\mathrm{cont}}(Y,R^\times)$ is a multiplicative character, we have

$$\int_Y f\,d(\mu_1*\mu_2)=\left(\int_Y f\,d\mu_1\right)\left(\int_Y f\,d\mu_2\right). \tag{5.1.4}$$

5.2. **The p-adic measure $\mu_{\mathcal{E},q\text{-}\mathrm{exp}}$ and the section $f_{\kappa,\underline{\tau},p}$.** Now take $Y=\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p)$ and $R=\mathcal{O}_F$. The goal is to select the Schwartz function $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ and construct an element $\mu_{\mathcal{E},q\text{-}\mathrm{exp}}$ inside the space $\mathscr{M}eas\left(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right)$ such that

$$\int_{\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p)}(\kappa,\underline{\tau})\,d\mu_{\mathcal{E},q\text{-}\mathrm{exp}}=\varepsilon_{q,p\text{-}\mathrm{adic}}(\mathcal{E}^{\alpha_{\kappa,\underline{\tau},p}}_{\kappa,\underline{\tau}}).$$

By definition it is enough to construct, for each $\boldsymbol{\beta}\in\Sigma_{N,p,+}$, a measure $\mu_{\mathcal{E},\boldsymbol{\beta}}\in\mathscr{M}eas(Y,\mathcal{O}_F)$ with the property that

$$\begin{aligned} \int_{\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p)}(\kappa,\underline{\tau})\,d\mu_{\mathcal{E},\boldsymbol{\beta}}=\mathfrak{c}^{\alpha_{\kappa,\underline{\tau},p}}_{\kappa,\underline{\tau}}(\boldsymbol{\beta})&=\frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})}\widehat{\alpha}^{\mathrm{vol}}_N(\boldsymbol{\beta})\cdot\chi^{-1}(C_{\phi_{\boldsymbol{\beta}}})C_{\phi_{\boldsymbol{\beta}}}^{-k+n+1}\cdot L_N(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1} \\ &\times L^p(1-k+n,\phi_{\boldsymbol{\beta}}\chi)\cdot g^S_{\boldsymbol{\beta}}(k,\phi\chi) \\ &\times\widehat{\alpha}_{\kappa,\underline{\tau},p}(\boldsymbol{\beta})\prod_{l=1}^n\det{}_l(2\beta_0)^{t_l-t_{l+1}}\det(2\boldsymbol{\beta})^{k-n-1}. \end{aligned} \tag{5.2.1}$$

Because of (5.1.4) we can deal with the RHS of (5.2.1) term by term.

The first term $\frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})}\widehat{\alpha}^{\mathrm{vol}}_N(\boldsymbol{\beta})$ is a constant inside $\mathcal{O}_F$. The second term is interpolated by the measure $C^{n+1}_{\phi_{\boldsymbol{\beta}}}\cdot\delta_{(C^{-1}_{\phi_{\boldsymbol{\beta}}},\mathrm{id})}$, where id is the unity of $T_n(\mathbb{Z}_p)$. Both of the term $L_N(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1}$ and the term $g^S_{\boldsymbol{\beta}}(k,\phi\chi)$ can be written as $\mathcal{O}_F$-linear combinations of $\chi^{-1}(m)m^{-k}$ with some positive integers m prime to p. For each m the measure $\delta_{(m^{-1},\mathrm{id})}$ interpolates $\chi^{-1}(m)m^{-k}$.

Regarding the term $L^p(1-k+n,\phi_{\boldsymbol{\beta}}\chi)=(1-\phi_{\boldsymbol{\beta}}\chi^\circ(p)p^{k-n-1})L(1-k+n,\phi_{\boldsymbol{\beta}}\chi^\circ)$, there is the following theorem on the existence of p-adic Dirichlet L-functions.

Theorem 5.2.1 (Kubota–Leopoldt, [Hid93, Theorem 4.4.1])**.** *Given a nontrivial primitive Dirichlet character ξ with conductor prime to p, there is a unique measure $\mu_\xi\in\mathscr{M}eas\left(\mathbb{Z}_p^\times,\mathbb{Z}_p[\xi]\right)$ such that for all integers $j\geq 1$ and finite order characters $\chi\in\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times,\mathbb{C}^\times)$,*

$$\int_{\mathbb{Z}_p^\times}\chi(y)y^j\,d\mu_\xi(y)=(1-\xi\chi^\circ(p)p^{j-1})L(1-j,\xi\chi^\circ).$$

As for the trivial character, for each fixed prime ℓ prime to p, there is a unique measure $\mu_\ell \in \mathscr{M}eas\left(\mathbb{Z}_p^\times, \mathbb{Z}_p\right)$ such that for all j and χ as before,

$$\int_{\mathbb{Z}_p^\times} \chi(y) y^j \, d\mu_\ell(y) = (1-\chi(\ell)^{-1}\ell^{-j})(1-\chi^\circ(p)p^{j-1})L(1-j,\chi^\circ).$$

For simplicity we assume that $\phi^2 \neq 1$ from now on, so that $\phi_{\boldsymbol{\beta}}$ will always be nontrivial. Without this assumption, for a fixed prime ℓ prime to p, we can interpolate $(1-\chi(\ell)^{-1}\ell^{-k+n}) \cdot \mathcal{E}_{\kappa,\underline{\tau}}$ instead of $\mathcal{E}_{\kappa,\underline{\tau}}$. Then everything in the following goes the same, and we get the measure $\mu_{\mathcal{C},\ell,\phi,\beta_1,\beta_2}$ as described in Remark 1.0.2.

Let $h_n(y) = y^{-n}$. Using (5.1.2) we get the measure $\mu_{\phi_{\boldsymbol{\beta}},h_n}$ on $\mathbb{Z}_p^\times$ with $\mu_{\phi_{\boldsymbol{\beta}},h_n}(\kappa) = L^p(1-k+n,\phi_{\boldsymbol{\beta}}\chi)$, whose direct product with the measure δ_{id} on $T_n(\mathbb{Z}_p)$ gives the desired p-adic interpolation of $L^p(1-k+n,\phi_{\boldsymbol{\beta}}\chi)$.

It remains to treat the term $\widehat{\alpha}_{\kappa,\underline{\tau},p}(\boldsymbol{\beta})\prod_{l=1}^n \det_l(2\beta_0)^{t_l-t_{l+1}}\det(2\boldsymbol{\beta})^{k-n-1}$ by selecting suitable $\widehat{\alpha}_{\kappa,\underline{\tau},p}$. Due to the density of polynomial functions inside $\mathscr{C}(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), F)$, the measure interpolating this expression must be $\det(2\boldsymbol{\beta})^{-n-1}\cdot\delta_{(b_0,b_1,\cdots,b_n)}$, where $b_0 = \det(2\boldsymbol{\beta})\det(2\beta_0)^{-1}$, $b_1 = \det_1(2\beta_0)$, $b_l = \det_{l-1}(2\beta_0)^{-1}\det_l(2\beta_0)$ for $2 \le l \le n$, and we must require all the $\det_l(2\beta_0)$ to lie inside $\mathbb{Z}_p^\times$. Accordingly we see that a natural choice of the Schwartz function $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ on $\mathrm{Sym}(2n,\mathbb{Q}_p)$ is

$$\begin{aligned}&\widehat{\alpha}_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}\beta_1 & \beta_0\\ {}^{\mathrm{t}}\beta_0 & \beta_2\end{pmatrix}\right)\\ &= \mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\prod_{l=1}^n \mathbb{1}_{\mathrm{GL}_l(\mathbb{Z}_p)}((2\beta_0)_l)\cdot\chi(\det(2\boldsymbol{\beta}))\prod_{l=1}^n \psi_l\psi_{l+1}^{-1}(\det_l(2\beta_0)),\end{aligned}$$

where (similar to how we have put $t_{n+1}=k$) we set $\psi_{n+1}=\chi$, and $(2\beta_0)_l$ stands for the upper left $l\times l$ minor of $2\beta_0$. In fact the only freedom in the choice is to vary the support.

The inverse Fourier transform of the above defined $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ gives $\alpha_{\kappa,\underline{\tau},p}$, and our choice of $f_{\kappa,\underline{\tau},p}$ is the "big cell" section $f_{\kappa,\underline{\tau}}^{\alpha_{\kappa,\underline{\tau},p}}(s,\xi) \in I_{Q_H,p}(s,\xi)$ associated to $\alpha_{\kappa,\underline{\tau},p}$, evaluated at $s = k-\frac{2n+1}{2}$ and $\xi = \phi^{-1}\chi^{\circ-1}$. Now it is clear that the desired measure $\mu_{\mathcal{E},\boldsymbol{\beta}}$ in (5.2.1) exists. One also notices that its evaluation at $(\kappa,\underline{\tau})$ with $\phi\chi(-1)\neq(-1)^k$ is 0.

So far for all admissible $(\kappa,\underline{\tau})$ satisfying $\phi\chi(-1)=(-1)^k$, we have made our choices of $f_{\kappa,\underline{\tau},v} \in I_{Q_H,v}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})$ for all places v. From now on we write $f_{\kappa,\underline{\tau}}$ to mean the product of all the local sections we have selected for admissible $(\kappa,\underline{\tau})$ if $\phi\chi(-1)=(-1)^k$, and simply 0 if the parity condition does not hold. We denote by $\mathcal{E}_{\kappa,\underline{\tau}}$ the global section of $\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk}\boxtimes\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk}$ over $X_{G,\Gamma}\times X_{G,\Gamma}$ that is mapped to $A_{n,\phi,k,\chi}^{-1}\cdot E^*(\cdot,f_{\kappa,\underline{\tau}})|_{G\times G}$ by the map $\varphi_{G\times G}(\cdot,\mathfrak{e}_{\mathrm{can}})$.

Theorem 5.2.2. *There is a measure $\mu_{\mathcal{E},q\text{-}\mathrm{exp}} \in \mathscr{M}eas\left(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p), \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right)$ satisfying*

$$\int_{\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p)}(\kappa,\underline{\tau})\,d\mu_{\mathcal{E},q\text{-}\mathrm{exp}} = \varepsilon_{q,p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}})$$

for all admissible $(\kappa,\underline{\tau}) \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$.

Explicit computation results on the local zeta integrals for $v \nmid p\infty$ have been obtained in Theorem 4.1.2 and Proposition 4.2.1. For the archimedean place we the nonvanishing result is shown in Proposition 4.3.1. It remains to carry out the local computations at p, which occupy the rest of this section. All the results are summarized in the following Proposition 5.2.3, which gives the interpolation properties of the $(n+1)$-variable p-adic L-function we will finally construct.

By the reasoning near the end of §2.5 we can define $(e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}$, the ordinary projection of $\mathcal{E}_{\kappa,\underline{\tau}}$ on the first factor. For an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$ with $\pi_\infty \cong$

$\mathcal{D}_{\underline{t}}$, denote by $\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ the subspace of π consisting of automorphic forms whose archimedean components, under an isomorphism $\pi \cong \bigotimes_v' \pi_v$ are the highest weight vector inside the lowest $K_{G,\infty}$-type $\underline{t}$, invariant under the right translation of $\widehat{\Gamma}_1(N,p^m)$, and acted on by the character $\underline{\psi}$ by the group $T_G(\mathbb{Z}_p)$.

Proposition 5.2.3. *Let $\varphi \in \pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ be a weight $\underline{t}$ ordinary cuspidal Siegel modular form. Regarding the Petersson inner product of $\overline{\varphi}$ with the automorphic form $\varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}},\mathfrak{e}_{\mathrm{can}})$ on its first factor, we have*

$$\begin{aligned}&\left\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}},\mathfrak{e}_{\mathrm{can}})(\cdot,g),\,\overline{\varphi}\right\rangle\\ &=\phi(-1)^n\mathrm{vol}\left(\widehat{\Gamma}(N)\right)\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}\cdot\frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}}\cdot\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty},v_{\underline{t}}^\vee,v_{\underline{t}})}{\langle v_{\underline{t}}^\vee,v_{\underline{t}}\rangle}\\ &\quad\times E_p(k-n,\pi\times\phi^{-1}\chi^{-1})\cdot L^{Np\infty}(k-n,\pi\times\phi^{-1}\chi^{-1})\cdot eW(\varphi)(g),\end{aligned}$$

where the modified Euler factor $E_p(k-n,\pi\times\phi^{-1}\chi^{-1})$ is defined by (1.0.1), and the operator $W:\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\to\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is defined as

$$W(\varphi)(g):=\left(\int_{N_G(\mathbb{Z}_p)}R_p(u)\overline{\varphi}^{\vartheta}du\right)(g)=\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}(\vartheta gu\vartheta)\,du. \tag{5.2.2}$$

Thanks to the multiplicity one theorem for symplectic groups, the operator W preserves π and $\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$. However this W is not $\mathbb{C}$-linear.

In the unitary case such local zeta integrals are calculated in [Wan15, EHLS16]. The restrictive conditions in [Wan15] amount to $c_{\chi\psi_1}>c_{\chi\psi_2}>\cdots>c_{\chi\psi_n}$ here, in particular missing the most interesting cases where $c_{\chi\psi_1}=c_{\chi\psi_2}=\cdots=c_{\chi\psi_n}=0$. Computations in [EHLS16] are done in a different way from ours below, applying the Godement–Jacquet local functional equation, but without considering the ordinary projection.

5.3. **An observation of Böcherer–Schmidt.** The first step of the calculation is to compute the inverse Fourier transform of $\widehat{\alpha}_{\kappa,\underline{\tau},p}$. However this computation is in fact not very convenient because of the term $\chi(\det(2\boldsymbol{\beta}))$. The observation of Böcherer–Schmidt is that, for computing local zeta integrals, instead of using $\widehat{\alpha}_{\kappa,\underline{\tau},p}$, we may use the Schwartz function modified from it by changing $\chi(\det(2\boldsymbol{\beta}))$ to $\chi(-1)^n\chi^2(\det(2\beta_0))$, i.e.

(5.3.1)

$$\begin{aligned}&\widehat{\alpha}'_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}\beta_1&\beta_0\\ {}^{\mathrm{t}}\beta_0&\beta_2\end{pmatrix}\right)\\ &=\mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\prod_{l=1}^n\mathbb{1}_{\mathrm{GL}_l(\mathbb{Z}_p)}((2\beta_0)_l)\cdot\chi(-1)^n\chi^2(\det(2\beta_0))\prod_{l=1}^n\psi_l\psi_{l+1}^{-1}(\det_l(2\beta_0))\\ &=\mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\prod_{l=1}^n\mathbb{1}_{\mathrm{GL}_l(\mathbb{Z}_p)}((2\beta_0)_l)\cdot\chi(-1)^n\prod_{l=1}^n\psi'_l\psi'^{-1}_{l+1}(\det_l(2\beta_0)),\end{aligned}$$

where $\psi'_l=\chi\psi_l$ if $1\leq l\leq n$ and ψ'_{n+1} is the trivial character. Let $f'_{\kappa,\underline{\tau},p}\in I_{Q_H,p}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})$ be the section associated to $\alpha'_{\kappa,\underline{\tau},p}$, the inverse Fourier transform of $\widehat{\alpha}'_{\kappa,\underline{\tau},p}$.

Recall that we have defined the adelic $\mathbb{U}_p$-operators in (2.5.2) and (2.5.3). For $\underline{a}\in C_n^+$, with the embedding $\iota: G\times G\hookrightarrow H$, we can make $U_{p,\underline{a}}$ act on smooth functions on $H(\mathbb{A})$ simply by the formula (2.5.3) on the first factor. We use $U_{p,\underline{a}}\times 1$ to denote this action, and it is easily seen to be compatible with restriction by ι and the operator $U_{p,\underline{a}}\times 1$ on $G\times G$. The operator $U_{p,n}$ is the one with $\underline{a}=(1,\cdots,1)$,

Proposition 5.3.1. *If m is a positive integer such that the conductor of χ divides p^{2m}, then*

$$(U_{p,n}^m\times 1)E^*(\cdot,f_{\kappa,\underline{\tau}})=(U_{p,n}^m\times 1)E^*(\cdot,f'_{\kappa,\underline{\tau}}).$$

Proof. Let $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f_{\kappa,\underline{\tau}})$ be $E^{*}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f_{\kappa,\underline{\tau}})$ with the factor $W_{\boldsymbol{\beta},p}(1_p,f_{\kappa,\underline{\tau},p})$ removed. The $\boldsymbol{\beta}$-th Fourier coefficient of $(U_{p,n}^m\times 1)E^*(\cdot,f_{\kappa,\underline{\tau}})$ at $h_{\boldsymbol{z}}$ is equal to $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f_{\kappa,\underline{\tau}})W_{\boldsymbol{\beta},p}(1_p,(U_{p,n}^m\times 1)f_{\kappa,\underline{\tau},p})$. We define similarly $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f'_{\kappa,\underline{\tau}})$, and it is obvious that $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f_{\kappa,\underline{\tau}})=E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f'_{\kappa,\underline{\tau}})$. Therefore all we need to show is that

$$W_{\boldsymbol{\beta},p}(1_p,(U_{p,n}^m\times 1)f_{\kappa,\underline{\tau},p})=W_{\boldsymbol{\beta},p}(1_p,(U_{p,n}^m\times 1)f'_{\kappa,\underline{\tau},p}) \tag{5.3.2}$$

for all $\boldsymbol{\beta}\in\mathrm{Sym}(2n,\mathbb{Q})$. Let $S_n=\mathrm{Sym}(n,\mathbb{Q}_p)$, $M_n=M_n(\mathbb{Q}_p)$ and for element $\varsigma\in S_{2n}$ we write it in $n\times n$ blocks as $\begin{pmatrix}\varsigma_1&\varsigma_0\\ {}^{\mathrm t}\varsigma_0&\varsigma_1\end{pmatrix}$. One easily computes

$$\begin{aligned}
&p^{-(|\underline{t}|-n(n+1))m}W_{\boldsymbol{\beta},p}(1_p,(U_{p,n}^m\times 1)f_{\kappa,\underline{\tau},p})\\
&=\sum_{u\in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})}\int_{S_{2n}}f_{\kappa,\underline{\tau},p}\left(wu(\varsigma)\iota\left(\begin{pmatrix}p^m&up^{-m}\\0&p^{-m}\end{pmatrix},1\right)\right)\mathbf{e}_p(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\,d\varsigma\\
&=\sum_{u\in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})}\int_{M_n}\int_{S_n}\int_{S_n}f_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}0&0&-p^{-m}&0\\0&0&0&-1\\p^m&0&(\varsigma_1+u)p^{-m}&\varsigma_0\\0&1&{}^{\mathrm t}\varsigma_0p^{-m}&\varsigma_2\end{pmatrix}\right)\mathbf{e}_p(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\,d\varsigma_1\,d\varsigma_2\,d\varsigma_0\\
&=\left(\phi(p)p^k\right)^{nm}\sum_{u\in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})}\int_{M_n}\int_{S_n}\int_{S_n}\alpha_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}(\varsigma_1+u)p^{-2m}&\varsigma_0p^{-m}\\{}^{\mathrm t}\varsigma_0p^{-m}&\varsigma_2\end{pmatrix}\right)\mathbf{e}_p(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\,d\varsigma_1\,d\varsigma_2\,d\varsigma_0\\
&=\left(\phi(p)p^{k-2n-1}\right)^{nm}\sum_{u\in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})}\mathbf{e}_p(\operatorname{Tr}\beta_1u)\int_{S_{2n}}\alpha_{\kappa,\underline{\tau},p}(\varsigma)\mathbf{e}_p\left(-\operatorname{Tr}\begin{pmatrix}\beta_1p^{2m}&\beta_0p^m\\{}^{\mathrm t}\beta_0p^m&\beta_2\end{pmatrix}\varsigma\right)d\varsigma\\
&=\left(\phi(p)p^{k-n}\right)^{nm}\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(\beta_1)\widehat{\alpha}_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}\beta_1p^{2m}&\beta_0p^m\\{}^{\mathrm t}\beta_0p^m&\beta_2\end{pmatrix}\right),
\end{aligned}$$

and similarly

$$p^{-(|\underline{t}|-n(n+1))m}W_{\boldsymbol{\beta},p}(1_p,(U_{p,n}^m\times 1)f'_{\kappa,\underline{\tau},p})=\left(\phi(p)p^{k-n}\right)^{nm}\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(\beta_1)\widehat{\alpha}'_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}\beta_1p^{2m}&\beta_0p^m\\{}^{\mathrm t}\beta_0p^m&\beta_2\end{pmatrix}\right).$$

It is easily seen that if β_1, β_0p^m and β_2 are all integral, then

$$\det\left(\begin{pmatrix}\beta_1p^{2m}&\beta_0p^m\\{}^{\mathrm t}\beta_0p^m&\beta_2\end{pmatrix}\right)\equiv(-1)^n\det\left(\beta_0p^m\right)^2\mod p^{2m},$$

so when the conductor of χ divides p^{2m}, the functions $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ and $\widehat{\alpha}'_{\kappa,\underline{\tau},p}$ take the same value at such $\begin{pmatrix}\beta_1p^{2m}&\beta_0p^m\\{}^{\mathrm t}\beta_0p^m&\beta_2\end{pmatrix}$, and (5.3.2) is true for all $\boldsymbol{\beta}\in\mathrm{Sym}(2n,\mathbb{Q})$. ☐

Due to the above proposition we know that for $\underline{a}\in C_n^+$ with $\Delta\underline{a}\gg 0$ and $\varphi\in\pi$ we have

$$\left\langle(U_{p,\underline{a}}\times 1)E^*\left(\iota(\cdot,g),f_{\kappa,\underline{\tau}}\right),\overline{\varphi}\right\rangle=\left\langle(U_{p,\underline{a}}\times 1)E^*\left(\iota(\cdot,g),f'_{\kappa,\underline{\tau}}\right),\overline{\varphi}\right\rangle,$$

where the Petersson inner product is taken on the first factor of the restricted Siegel Eisenstein series. We will compute the local zeta integral for $f'_{\kappa,\underline{\tau},p}$.

5.4. **The inverse Fourier transform of $\widehat{\alpha}'_{\kappa,\underline{\tau},p}$.** In this subsection we regard characters of $\mathbb{Z}_p^\times$ (including the trivial character) as functions on $\mathbb{Q}_p$ by making them take the value 0 outside $\mathbb{Z}_p^\times$. Given characters of $\mathbb{Z}_p^\times$ of finite order $\underline{\xi}=(\xi_1,\cdots,\xi_n)$ whose conductors are $p^{c_{\xi_1}},\cdots,p^{c_{\xi_n}}$, for each $1\leq l\leq n$, define the Schwartz function $\Psi_{\underline{\xi},l}$ on $M_l(\mathbb{Q}_p)$ as

$$\Psi_{\underline{\xi},l}(x)=\mathbb{1}_{M_l(\mathbb{Z}_p)}(x)\cdot\prod_{j=1}^{l-1}\xi_j\xi_{j+1}^{-1}(\det_j(-x))\cdot\xi_l(\det_l(-x)).$$

Denote by $\mathcal{F}^{-1}\Psi_{\underline{\xi},l}$ the inverse Fourier transform of $\Psi_{\underline{\xi},l}$. First we give an inductive formula for $\mathcal{F}^{-1}\Psi_{\underline{\xi},l}$. Set

$$\Phi_{\underline{\xi},l}(\varsigma)=\begin{cases}\xi_l^{-1}\left(p^{c_{\xi_l}}\varsigma_l\right)\mathcal{F}^{-1}\Psi_{\underline{\xi},l-1}(\varsigma') & \text{if } \varsigma\in\begin{pmatrix}I_{l-1}&\mathbb{Z}_p^{l-1}\\0&1\end{pmatrix}\begin{pmatrix}\varsigma'&\\&\varsigma_l\end{pmatrix}\begin{pmatrix}I_{l-1}&0\\{}^{\mathrm{t}}\mathbb{Z}_p^{l-1}&1\end{pmatrix}\\ & \text{with } \varsigma'\in M_{l-1}(\mathbb{Q}_p),\ \varsigma_l\in\mathbb{Q}_p,\\ 0 & \text{otherwise.}\end{cases}$$

and

$$\Phi'_{\underline{\xi},l}(\varsigma)=\begin{cases}\mathcal{F}^{-1}\Psi_{\underline{\xi},l-1}(\varsigma') & \text{if } \varsigma\in\begin{pmatrix}\varsigma'&\mathbb{Z}_p^{l-1}\\{}^{\mathrm{t}}\mathbb{Z}_p^{l-1}&\mathbb{Z}_p\end{pmatrix} \text{ with } \varsigma'\in M_{l-1}(\mathbb{Q}_p),\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 5.4.1. *We have*

(1) if ξ_l is nontrivial, then

$$\mathcal{F}^{-1}\Psi_{\underline{\xi},l}(\varsigma)=p^{-lc_{\xi_l}}G(\xi_l)\Phi_{\underline{\xi},l}(\varsigma),\tag{5.4.1}$$

(2) if ξ_l is the trivial character, then

$$\mathcal{F}^{-1}\Psi_{\underline{\xi},l}(\varsigma)=-p^{-l}\Phi_{\underline{\xi},l}(\varsigma)+(1-p^{-1})\Phi'_{\underline{\xi},l}(\varsigma).\tag{5.4.2}$$

Proof. Write $\varsigma=\begin{pmatrix}\varsigma'&\eta\\{}^{\mathrm{t}}\mu&\lambda\end{pmatrix}$ and $x=\begin{pmatrix}x'&y\\{}^{\mathrm{t}}z&w\end{pmatrix}$ with $\varsigma',x'\in M_{l-1}(\mathbb{Q}_p)$ and $\lambda,w\in\mathbb{Q}_p$. Then

$$\begin{aligned}&\mathcal{F}^{-1}\Psi_{\underline{\xi},l}(\varsigma)\\
=&\int_{M_l(\mathbb{Q}_p)}\Psi_{\underline{\xi},l}(x)\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x\varsigma\right)dx\\
=&\int_{M_{l-1}(\mathbb{Z}_p)\times\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p}\Psi_{\underline{\xi},l}\left(\begin{pmatrix}x'&y\\{}^{\mathrm{t}}z&w\end{pmatrix}\right)\mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}x'\varsigma'+{}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+w\lambda)\right)dx'\,dy\,dz\,dw\\
=&\int_{M_{l-1}(\mathbb{Z}_p)}\Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right)\int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p}\xi_l(-w+{}^{\mathrm{t}}zx'^{-1}y)\mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+w\lambda)\right)dy\,dz\,dw\,dx'\\
=&\int_{M_{l-1}(\mathbb{Z}_p)}\Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right)\int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}}\mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+{}^{\mathrm{t}}zx'^{-1}y{}^{\mathrm{t}}\lambda)\right)\int_{\mathbb{Z}_p}\xi_l(-w)\mathbf{e}_p\left(\mathrm{Tr}(w\lambda)\right)dw\,dy\,dz\,dx'.\end{aligned}$$

First assume that ξ_l is nontrivial. Then

$$\int_{\mathbb{Z}_p}\xi_l(-w)\mathbf{e}_p\left(\mathrm{Tr}(w\lambda)\right)dw=p^{-c_{\xi_l}}G(\xi_l)\xi_l^{-1}(\det(p^{c_{\xi_l}}\lambda)).\tag{5.4.3}$$

Hence $\mathcal{F}^{-1}\Psi_{\underline{\xi}}(\varsigma)$ is 0 unless λ belongs to $p^{-c_{\xi_l}}\mathbb{Z}_p^\times$. Suppose $\lambda\in p^{-c_{\xi_l}}\mathbb{Z}_p^\times$, and we write

$$\begin{pmatrix}\varsigma'&\eta\\{}^{\mathrm{t}}\mu&\lambda\end{pmatrix}=\begin{pmatrix}1&\eta\lambda^{-1}\\0&1\end{pmatrix}\begin{pmatrix}\tilde{\varsigma}'&0\\0&\lambda\end{pmatrix}\begin{pmatrix}1&0\\\lambda^{-1}{}^{\mathrm{t}}\mu&1\end{pmatrix}$$

with $\tilde{\varsigma}' = \varsigma' - \eta\lambda^{-1}{}^{\mathrm{t}}\mu$. Then

(5.4.4)

$$\begin{aligned}
&\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right)\int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu + {}^{\mathrm{t}}y\eta + {}^{\mathrm{t}}zx'^{-1}y\lambda)\right)\,dy\,dz\,dx' \\
&= \int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\tilde{\varsigma}'\right)\int_{\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}(\lambda^{-1}{}^{\mathrm{t}}\eta x'(\mu + x'^{-1}y\lambda))\right)\int_{\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z(\mu + x'^{-1}y\lambda))\right)\,dz\,dy\,dx' \\
&= p^{-(l-1)c_{\xi_l}}\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\lambda^{-1}{}^{\mathrm{t}}\mu)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\eta\lambda^{-1})\cdot\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\tilde{\varsigma}'\right)\,dx'.
\end{aligned}$$

Combining (5.4.3) and (5.4.4) we get (5.4.1). Now if ξ_l is the trivial character, then

$$\int_{\mathbb{Z}_p} \xi_l(-w)\mathbf{e}_p\left(\mathrm{Tr}(w\lambda)\right)\,dw = -p^{-1}\mathbb{1}_{p^{-1}\mathbb{Z}_p^\times}(\lambda) + (1-p^{-1})\mathbb{1}_{\mathbb{Z}_p}(\lambda). \tag{5.4.5}$$

When $\lambda \in p^{-1}\mathbb{Z}_p^\times$, (5.4.4) holds with c_{ξ_l} replaced by 1. When $\lambda \in \mathbb{Z}_p$,

$$\begin{aligned}
&\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right)\int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu + {}^{\mathrm{t}}y\eta + {}^{\mathrm{t}}zx'^{-1}y\lambda)\right)\,dy\,dz\,dx' \\
&= \int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right)\int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu + {}^{\mathrm{t}}y\eta)\right)\,dy\,dz\,dx' \\
&= \mathcal{F}^{-1}\Psi_{\underline{\xi},l-1}(\varsigma')\cdot\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\eta)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\mu).
\end{aligned} \tag{5.4.6}$$

We see that (5.4.2) follows from (5.4.5),(5.4.4) (with c_{ξ_l} replaced by 1) and (5.4.6). □

Recall that for an n-tuple of integers $\underline{c} = (c_1, \cdots, c_n)$ we have defined $p^{\underline{c}}$ to be the element $\mathrm{diag}(p^{c_1}, \cdots, p^{c_n}, p^{-c_1}, \cdots, p^{-c_n})$ inside $G(\mathbb{Q}_p)$, so $p^{\underline{c_{\chi\psi}}}$ gives a diagonal matrix in $G(\mathbb{Q}_p)$. When $\xi_1, \cdots, \xi_n$ are all nontrivial, the induction formula in Proposition 5.4.1 easily gives formulas for $\mathcal{F}^{-1}\Psi_{\underline{\xi},l}$, and hence formulas for the section $f'^{d}_{\kappa,\underline{\tau},p}$.

Corollary 5.4.2. *As a function on $G(\mathbb{Q}_p)$ the smooth function $f'^{d}_{\kappa,\underline{\tau},p}(\iota(\cdot,1))$ is supported on the compact open subset*

$$N_G^-(\mathbb{Z}_p)p^{\underline{c_{\chi\psi}}}\begin{pmatrix} p^{-1}I_n & 0 \\ 0 & pI_n\end{pmatrix}B_G(\mathbb{Z}_p)\begin{pmatrix} pI_n & 0 \\ 0 & p^{-1}I_n\end{pmatrix},$$

and takes the value

$$\prod_{l=1}^n \psi_l(x_l)\cdot p^{-n(n+1)-\sum_{l=1}^n lc_{\chi\psi_l}}\left(p^k\phi(p)\right)^{\sum_{l=1}^n c_{\chi\psi_l}}\prod_{l=1}^n G(\chi\psi_l)$$

at the element $u^- p^{\underline{c_{\chi\psi}}}\mathrm{diag}(x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1})u$, with $x_l \in \mathbb{Z}_p^\times$, $u^- \in N_G^-(\mathbb{Z}_p)$ and $u \in \begin{pmatrix} p^{-1}I_n & 0 \\ 0 & pI_n\end{pmatrix}N_G(\mathbb{Z}_p)\begin{pmatrix} pI_n & 0 \\ 0 & p^{-1}I_n\end{pmatrix}$.

Proof. Write $g \in G(\mathbb{Q})$ as $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, using (4.2.1) we get

$$\begin{aligned}
f'^{d}_{\kappa,\underline{\tau},p}(\iota(g,1)) &= f'_{\kappa,\underline{\tau},p}(\mathcal{S}\iota(g,1)) \\
&= \phi_p\chi_p\left(\det\begin{pmatrix} c & -I_n \\ -a & 0\end{pmatrix}\right)\left|\det\begin{pmatrix} c & -I_n \\ -a & 0\end{pmatrix}\right|_p^{-k}\alpha'_{\kappa,\underline{\tau},p}\left(\begin{pmatrix} a^{-1}b & -a^{-1} \\ -d+ca^{-1}b & -ca^{-1}\end{pmatrix}\right) \\
&= \chi(-1)^n\chi^{-1}\left(\det(a)|\det(a)|_p\right)\phi^{-1}\left(|\det(a)|_p\right)|\det(a)|_p^{-k}p^{-n(n+1)} \\
&\quad\cdot\mathbb{1}_{p^{-2}\,\mathrm{Sym}(n,\mathbb{Z}_p)}(a^{-1}b)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(ca^{-1})\cdot\chi(-1)^n\mathcal{F}^{-1}\Psi_{\chi\underline{\psi}}(a^{-1}),
\end{aligned}$$

and the statement follows by applying Proposition 5.4.1. □

We will say that an admissible point $(\kappa, \underline{\tau})$ belongs to the ramified cases if none of the characters $\chi\psi_1, \cdots, \chi\psi_n$ is trivial.

5.5. **The $\mathbb{U}_p$-operators and the theory of Jacquet modules.** Before starting the computation of the zeta integrals at p ,we state some facts that follow easily from the theory of Jacquet modules and are useful in the study of p-adic automorphic forms of finite slopes. One can also consult the treatment in [Hid04, §5.1].

Let $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible cuspidal automorphic representation with a fixed isomorphism $\pi \cong \bigotimes'_v \pi_v$. Assume that $\pi_\infty \cong \mathcal{D}_{\underline{t}}$. For each $\varphi \in \pi$ its ordinary projection $e\varphi$ is defined by the discussion in §2.5. Put $\pi_{\mathrm{ord}} = e\pi$. By Proposition 2.5.5 we know that π_{ord} is contained inside the subspace of holomorphic forms inside π.

The facts we show below and will be of use later are: if π_{ord} is nonzero, then π_p is isomorphic to a composition factor of certain principal series, and the projection of π_{ord} to π_p is one dimensional, and the action of the $\mathbb{U}_p$-operators on $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a}}(\pi_p)$, the intersection of the images of all the $\mathbb{U}_p$-operators acting on π_p, is semisimple.

Given an admissible representation Π of $G(\mathbb{Q}_p)$, define $U_{p,\underline{a},\mathrm{loc}} = \int_{N_G(\mathbb{Z}_p)} \Pi(up^{\underline{a}})\, du$ (in a purely local situation we do not care about the normalization). Let $\Pi(N_G(\mathbb{Q}_p))$ be the subspace of Π spanned by $\Pi(u)v - v$ for all $u \in N_G, v \in \Pi$. The Jacquet module $\Pi_{N_G(\mathbb{Q}_p)}$ is defined to be the quotient of Π by $\Pi(N_G(\mathbb{Q}_p))$.

It follows from Jacquet's Lemma [Cas95, Theorem 4.1.2, Proposition 4.1.4] that the restriction of the projection $\Pi \to \Pi_{N_G(\mathbb{Q}_p)}$ to $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a},\mathrm{loc}}(\Pi)$ is an isomorphism of $T_G(\mathbb{Z}_p)$-representations. It is also easy to check that the action of $U_{p,\underline{a}}, \underline{a} \in C_n^+$ on $U_{p,\underline{a},\mathrm{loc}}(\Pi)$ translates to the action of $p^{\underline{a}} \in T_G(\mathbb{Q}_p)$ on the Jacquet module $\Pi_{N_G(\mathbb{Q}_p)}$. Let δ_{B_G} be the modulus character associated to B_G. It takes the value $\prod_{j=1}^n |x_j|_p^{2(n+1-j)}$ on $\mathrm{diag}(x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}) \in B_G(\mathbb{Q}_p)$. There is the Frobenius reciprocity indicating $\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\Pi, \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right) \cong \mathrm{Hom}_{T_G(\mathbb{Q}_p)}\left(\Pi_{N_G(\mathbb{Q}_p)}, \underline{\theta}\delta_{B_G}^{1/2}\right)$ where $\underline{\theta} = (\theta_1, \cdots, \theta_n)$ is a character of $T_G(\mathbb{Q}_p)$ and $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$ is the normalized induction. Therefore one concludes that as long as the operator $U_p = U_{p,\rho_G}$ acting on π has a nonzero eigenvalue, the $G(\mathbb{Q}_p)$-representation π_p can be embedded into a principal series representation $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$ for some $\underline{\theta}$. More precisely we have the following proposition.

Proposition 5.5.1. *Suppose that there are $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathcal{O}_{\overline{\mathbb{Q}}_p}\backslash\{0\}$ and an automorphic form $\varphi \in \pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ on which the operator $U_{p,\underline{a}}$ acts by $\prod_{j=1}^n \mathfrak{a}_j^{a_j}$ for all $\underline{a} \in C_n^+$. Let $\underline{\theta}$ be the character of $T_G(\mathbb{Q}_p)$ whose restriction to $T_G(\mathbb{Z}_p)$ is $\underline{\psi}$ and $\theta_j(p) = \alpha_j = p^{-(t_j-j)}\mathfrak{a}_j$. Then π_p can be embedded into the principal series representation $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$.*

Notice that when π_{ord} is nonzero, the p-adic evaluations of the above defined $\alpha_1, \cdots, \alpha_n, \alpha_1^{-1}, \cdots, \alpha_n^{-1}$ are pairwise distinct, and are among $\pm(t_1 - 1), \cdots, \pm(t_n - n)$.

The information regarding the $\mathbb{U}_p$-operators acting on $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a}}(\pi_p)$ can be deduced from the knowledge of the action of $T_G(\mathbb{Q}_p)$ on $\left(\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right)_{N_G(\mathbb{Q}_p)}$, the Jacquet module of the principal series that contains π_p. According to [Cas95, Proposition 6.3.1, Proposition 6.3.3], the composition series of $\left(\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right)_{N_G(\mathbb{Q}_p)}$ consists of $|W_G|$ characters of $T_G(\mathbb{Q}_p)$, which are $(\underline{\theta} \circ w) \cdot \delta_{B_G}^{1/2}$, $w \in W_G$, where W_G is the Weyl group of G with respect to its maximal torus T_G. The non-triviality of π_{ord} implies that these $|W_G|$ characters are pairwise distinct, so the $T_G(\mathbb{Q}_p)$-action on

$\left(\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}\right)_{N_G(\mathbb{Q}_p)}$, as well as $\pi_{p,N_G(\mathbb{Q}_p)}$, is semisimple. By a simple examination of the corresponding p-adic valuations one also sees that there is only one w in W_G having the property that for all $\underline{a}\in C_n^+$ the number $\underline{\theta}(w(p^{\underline{a}}))\delta_{B_G}^{1/2}(p^{\underline{a}})$ has p-adic valuation less or equal to $-\langle \underline{t}+2\rho_{G,c},\underline{a}\rangle$.

Proposition 5.5.2. *If π_{ord} is nonzero, then the action of $\mathbb{U}_p$-operators on $\bigcap_{\underline{a}\in C_n^+}U_{p,\underline{a}}(\pi_p)$ is semisimple. Let $\pi_{p,\mathrm{ord}}$ be the image of the projection of π_{ord} to π_p. Then $\pi_{p,\mathrm{ord}}$ is one dimensional.*

From now on when π_{ord} is nonzero, we put $\mathfrak{a}_1,\cdots,\mathfrak{a}_n\in\mathcal{O}_{\mathbb{Q}_p}^\times$ to be the p-adic integers such that the $\mathbb{U}_p$-operator $U_{p,\underline{a}}$ acts on π_{ord} by $\prod_{j=1}^n\mathfrak{a}_j^{a_j}$ for $\underline{a}=(a_1,\cdots,a_n)\in C_n^+$. We will also assume that the group $T_G(\mathbb{Z}_p)$ acts on π_{ord} by the character $\underline{\psi}$. For $1\le j\le n$, the number α_j and the character θ_j of $\mathbb{Q}_p^\times$ are defined from $\mathfrak{a}_j$ and ψ_j as above, i.e. $\alpha_j=p^{-(t_j-j)}\mathfrak{a}_j$ and $\theta_j|_{\mathbb{Z}_p^\times}=\psi_j$ with $\theta_j(p)=\alpha_j$.

5.6. The proof of Prop 5.2.3 for the ramified cases.

Proof (the ramified cases). Assume that $\chi\psi_1,\cdots,\chi\psi_n$ are all nontrivial, and $\varphi\in\pi_{\underline{t}}^{\Gamma_1(N,p^m)}$ is ordinary. The computation is straightforward. For $\Delta\underline{a}\gg 0$, by definition of the operator $U_{p,\underline{a}}$ (2.5.2) and Proposition 5.3.1,

$$\begin{aligned}\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^\vartheta)&=\left(T_{(U_{p,\underline{a}}\times 1)f'_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^\vartheta)\\&=p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q}_p)}\int_{N_G(\mathbb{Z}_p)}f'^{d}_{\kappa,\underline{\tau},p}(\iota(g'up^{\underline{a}},1))\overline{\varphi}(g^\vartheta g')du dg'\\&=p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q}_p)}f'^{d}_{\kappa,\underline{\tau},p}(\iota(g',1))\overline{\varphi}(g^\vartheta g'p^{-\underline{a}})dg.\end{aligned}$$

Abbreviate the scalar $p^{-\sum_{l=1}^n l\cdot c_{\chi\psi_l}}\left(p^k\phi(p)\right)^{\sum_{l=1}^n c_{\chi\psi_l}}\prod_{l=1}^n G(\chi\psi_l)$ as $b_{k,\phi,\chi\underline{\psi}}$. Then applying Corollary 5.4.2 we get

$$\begin{aligned}&b_{k,\phi,\chi\underline{\psi}}^{-1}\mathrm{vol}\left(N_G^-(\mathbb{Z}_p)B_G(\mathbb{Z}_p)\right)^{-1}\cdot\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^\vartheta)\\=&p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle-\left\langle 2\rho_G,\,\underline{c_{\chi\psi}}\right\rangle}\int_{N_G^-(\mathbb{Z}_p)}\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}\left(g^\vartheta u^-p^{\underline{c_{\chi\psi}}}\begin{pmatrix}p^{-1}I_n&0\\0&pI_n\end{pmatrix}u\begin{pmatrix}pI_n&0\\0&p^{-1}I_n\end{pmatrix}p^{-\underline{a}}\right)du\,du^-\\=&p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle-\left\langle 2\rho_G,\,\underline{c_{\chi\psi}}\right\rangle}\int_{N_G^-(\mathbb{Z}_p)}\overline{\varphi}\left(g^\vartheta u^-p^{\underline{c_{\chi\psi}}-\underline{a}}\right)du^-\\=&p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle-\left\langle 2\rho_G,\,\underline{c_{\chi\psi}}\right\rangle}\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}\left(\vartheta gup^{\underline{a}-\underline{c_{\chi\psi}}}\vartheta\right)du\\=&p^{\left\langle \underline{t}-2\rho_{G,nc},\,\underline{c_{\chi\psi}}\right\rangle}\left(U_{p,\underline{a}-\underline{c_{\chi\psi}}}\overline{\varphi}^\vartheta\right)(g)\end{aligned}$$

Using $\mathrm{vol}\left(N_G^-(\mathbb{Z}_p)B_G(\mathbb{Z}_p)\right)=\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}$, we get

$$\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^\vartheta)=b_{k,\phi,\chi\underline{\psi}}\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}p^{\left\langle \underline{t}-2\rho_{G,nc},\,\underline{c_{\chi\psi}}\right\rangle}\cdot\left(U_{p,\underline{a}-\underline{c_{\chi\psi}}}\overline{\varphi}^\vartheta\right)(g).\tag{5.6.1}$$

The automorphic form $\overline{\varphi}^\vartheta\in\pi$ in general is not fixed by $N_G(\mathbb{Z}_p)$, and $W(\varphi)$ by definition equals its average over $N_G(\mathbb{Z}_p)$. We have

$$(U_{p,\underline{a}}\overline{\varphi}^\vartheta)(g)=\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}^\vartheta\left(gup^{\underline{a}}\right)du=\int_{N_G(\mathbb{Z}_p)}\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}^\vartheta\left(gup^{\underline{a}}u'\right)du\,du'=\left(U_{p,\underline{a}}W(\varphi)\right)(g),$$

so (5.6.1) becomes

$$\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^{\vartheta}) = b_{k,\phi,\chi\underline{\psi}}\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^{n}(p^{2l}-1)}\,p^{\left\langle \underline{t}-2\rho_{G,nc},\,\underline{c_{\chi\psi}}\right\rangle}\cdot\left(U_{p,\underline{a}-\underline{c_{\chi\psi}}}W(\varphi)\right)(g),$$

which, together with Theorem 4.1.2, Proposition 4.2.1, the fact that an ordinary nearly holomorphic form must be holomorphic and Proposition 5.5.2, implies Proposition 5.2.3 in the ramified case. ☐

5.7. **The proof of Prop 5.2.3 for general cases.** We first state a proposition whose proof is postponed to the end of §6.2.

Proposition 5.7.1. *For each admissible point $(\kappa,\underline{\tau})$ the nearly holomorphic form $(e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}$ is ordinary on both factors.*

The idea of the proof is simple. The statement is true in the ramified cases by results in §5.6. The admissible points belonging to the ramified cases are Zariski sense inside the weight space $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ and the statement for the general cases follows from a p-adic family argument.

Another proposition that will be useful for us verifies the nonvanishing of the ordinary projection of $W(\varphi)$ for a nonzero ordinary Siegel modular form φ.

Proposition 5.7.2. *If $\varphi\in\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ is nonzero ordinary, then $eW(\varphi)$ is nonzero.*

Proof. Take $\varphi'\in\pi$ invariant under the right translation of $N_G^-(\mathbb{Z}_p)$. We consider the Petersson inner product of $eW(\varphi)$ with φ'^{ϑ}.

$$\begin{aligned}\left\langle U_{p,\underline{a}}W(\varphi),\varphi'^{\vartheta}\right\rangle =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)\times N_G(\mathbb{Z}_p)}\overline{\varphi}(\vartheta g u p^{\underline{a}}u'\vartheta)\varphi'(\vartheta g\vartheta)\,dg\,du\,du'\\ =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(\vartheta g u p^{\underline{a}}\vartheta)\varphi'(\vartheta g\vartheta)\,dgdu.\end{aligned}$$

Making the change of variable $g\mapsto\vartheta g\vartheta$, we get

$$\begin{aligned}\left\langle U_{p,\underline{a}}W(\varphi),\varphi'^{\vartheta}\right\rangle =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(g\vartheta u p^{\underline{a}}\vartheta)\varphi'(g)\,dg\,du\\ =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(g)\varphi(g\vartheta p^{-\underline{a}}u\vartheta)\,dg\,du\\ =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\overline{\varphi}(g)\varphi(gp^{\underline{a}})\,dg\\ =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(gu)\varphi(gp^{\underline{a}})\,dg\,du\\ =& p^{\langle \underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(g)\varphi(gup^{\underline{a}})\,dg\,du\\ =& \left\langle\overline{\varphi},U_{p,\underline{a}}\varphi'\right\rangle,\end{aligned}$$

from which it follows that

$$\left\langle eW(\varphi),\varphi'^{\vartheta}\right\rangle=\lim_{r\to\infty}\left\langle U_p^{r!}W(\varphi),\varphi'\right\rangle=\lim_{r\to\infty}\left\langle\overline{\varphi},U_p^{r!}\varphi'\right\rangle=\left\langle\overline{\varphi},e\varphi'\right\rangle.$$

For fixed φ and φ', there are finite dimensional subspaces (viewed as both over $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$) of π and $\overline{\pi}$ which contain all the automorphic forms appearing in the above identity, and we regard the Petersson inner product as a bi-$\overline{\mathbb{Q}}_p$-linear pairing between them. Hence the limits with respect to the p-adic topology are well defined and commute with the Petersson inner product.

Now take $\varphi' = R_p(p^{\underline{c}})\varphi$ with $\underline{c} \in C_n^+$ and $\Delta\underline{c}$ sufficiently large such that $R_p(p^{\underline{c}})\varphi$ is fixed by $N_G^-(\mathbb{Z}_p)$. Combining the above computation and the fact $U_{p,\underline{a}}R_p(p^{\underline{c}}) = U_{p,\underline{a}+\underline{c}}$, we see that

$$\left\langle eW(\varphi), (R_p(p^{\underline{c}})\varphi)^{\vartheta} \right\rangle = \langle \overline{\varphi}, U_{p,\underline{c}}\varphi \rangle \neq 0,$$

and the nonvanishing of $eW(\varphi)$ follows. □

Now we begin the proof of Proposition 5.2.3 for general cases.

Proof (general cases). Assume that $\varphi \in \pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ is nonzero ordinary. Let $W(\varphi)^p$ be the image of $W(\varphi)$ under the map $\pi \xrightarrow{\sim} \bigotimes' \pi_v \to \bigotimes_{v\neq p} \pi_v$. By the doubling method formula Theorem 4.1.2 and Proposition 4.2.1, 2.5.5, 5.7.1, we deduce that the image of the automorphic form $\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}})(\cdot, g), \overline{\varphi}\rangle$ in $\bigotimes' \pi_v$ lies inside $W(\varphi)^p \otimes \pi_{p,\mathrm{ord}}$. By by Proposition 5.5.2, we know that $W(\varphi)^p \otimes \pi_{p,\mathrm{ord}}$ is a one dimensional $\mathbb{C}$-vector space, so the nonvanishing of $eW(\varphi)$ implies that there exists a complex number $C_{\phi,\kappa,\underline{\tau},\pi} \in \mathbb{C} \cong \overline{\mathbb{Q}}_p$ such that

$$\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}})(\cdot, g), \overline{\varphi}\rangle = C_{\phi,\kappa,\underline{\tau},\pi} \cdot eW(\varphi)(g).$$

Let

$$B_{\phi,\kappa,\underline{\tau},\pi} = A_{n,\phi,k,\chi}^{-1} \cdot \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \cdot \frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})}{\langle v_{\underline{t}}^\vee, v_{\underline{t}}\rangle} \cdot L^{Np\infty}(k-n, \pi\times\phi^{-1}\chi^{-1}),$$

where $A_{n,\phi,k,\chi}$ is defined as (3.5.4). This $B_{\phi,\kappa,\underline{\tau},\pi}$ is a finite complex number because of the absolute convergence of the archimedean zeta integral and the fact that the partial standard L-function $L^{Np\infty}(s,\pi\times\phi^{-1}\chi^{-1})$ does not have a pole at $k-n$. Let $\alpha_1,\cdots,\alpha_n$ and $\underline{\theta} = (\theta_1,\cdots,\theta_n)$ be the invariants associated to π_p at the end of §5.5. Define

$$R_p(s,\theta_j,\phi^{-1}) := \frac{1-(\chi\psi_j)^\circ(p)\cdot\phi(p)\alpha_j^{-1}p^{s-1}}{1-(\chi\psi_j)^\circ(p)\cdot\phi(p)^{-1}\alpha_j p^{-s}} \cdot \left(\phi(p)\alpha_j^{-1}p^{s-1}\right)^{c_{\chi\psi_j}} G(\chi\psi_j),$$

where by convention $(\chi\psi_j)^\circ(p) = \begin{cases} 1 & \text{if } \chi\psi_j \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$. The ordinarity condition on π implies that $R_p(s,\theta_j,\phi^{-1})$, $1\le j\le n$, dose not have a pole at $s = k-n$. Our goal is to show that

$$C_{\phi,\kappa,\underline{\tau},\pi} = B_{\phi,\kappa,\underline{\tau},\pi} \cdot \mathrm{vol}\left(B_G(\mathbb{Z}_p)N_G^-(\mathbb{Z}_p)\right) \prod_{j=1}^n R_p(k-n,\theta_j,\phi^{-1}). \tag{5.7.1}$$

Let $f'^d_{\kappa,\underline{\tau},p}(s) = f_p^{d,\alpha'_{\kappa,\underline{\tau},p}}(s-\frac{1}{2},\phi^{-1}\chi^{-1})$, the "big cell" section inside $I_{P_H,p}(s-\frac{1}{2},\phi^{-1}\chi^{\circ-1})$ (defined as (3.2.1)) , associated to the Schwartz function $\alpha'_{\kappa,\underline{\tau},p}$ whose Fourier transform is (5.3.1). We have $f'^d_{\kappa,\underline{\tau},p} = f'^d_{\kappa,\underline{\tau},p}(k-n)$. We add the parameter s here due to convergence consideration, because in general $f_{\kappa,\underline{\tau},p}(\iota(\cdot,1))$ is not compactly supported. In the following we assume $\mathrm{Re}(s) \gg 0$ whenever necessary, and the computation results will be easily seen to admit meromorphic continuations with respect to s.

For $\underline{c} \in C_n^+$, we write $\langle \varphi_{G\times G}\left((U_{p,\underline{c}}\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}}\right)(\cdot,g), \overline{\varphi}\rangle$ as $\varphi_{\underline{c}}(g)$. Given $\varphi' \in \pi_{\underline{t}}^{N_G^-(\mathbb{Z}_p)}$, we have

$$\begin{aligned}
\left\langle \varphi_{\underline{c}}, \varphi'^{\vartheta} \right\rangle &= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_p)} \int_{N_G(\mathbb{Z}_p)} f'^d_{\kappa,\underline{\tau},p}(s)(\iota(g'up^{\underline{c}},1))\overline{\varphi}(g^\vartheta g')\varphi'(g^\vartheta)\,du\,dg'\,dg \\
&= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_p)} f'^d_{\kappa,\underline{\tau},p}(\iota(g',1))\overline{\varphi}(g^\vartheta g' p^{-\underline{c}})\varphi'(g^\vartheta)\,dg'\,dg \\
&= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_p)} f'^d_{\kappa,\underline{\tau},p}(\iota(g',1))\overline{\varphi}(g)\varphi'(gp^{\underline{c}}g'^{-1})\,dg'\,dg
\end{aligned}$$

$$= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\, \underline{c}\rangle} \mathrm{vol}\left(Q_G(\mathbb{Z}_p)U_G^-(\mathbb{Z}_p)\right) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\varphi}(g) \int_{\mathrm{GL}(n,\mathbb{Q}_p)} p^{-n(n+1)} \cdot \eta_{\kappa,\underline{\tau},n}(s,a)$$
$$\cdot \int_{U_G(\mathbb{Z}_p)\times U_G^-(\mathbb{Z}_p)} \varphi'\left(gp^{\underline{c}}\begin{pmatrix} p^{-1}I_n & 0 \\ 0 & pI_n\end{pmatrix} u \begin{pmatrix} pI_n & 0 \\ 0 & p^{-1}I_n\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix} u^-\right) du\, du^-\, da\, dg,$$
$$= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} \frac{\mathrm{vol}\left(B_G(\mathbb{Z}_p)N_G^-(\mathbb{Z}_p)\right)}{\mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)} \cdot p^{\langle \underline{t}+2\rho_{G,c}, \underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\varphi}(g) \int_{N_G(\mathbb{Z}_p)\times \mathrm{GL}(n,\mathbb{Q}_p)} \eta_{\kappa,\underline{\tau},n}(s,a)$$
$$\cdot\, \varphi'\left(gup^{\underline{c}}\begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\right) du\, da\, dg,$$

where for $1 \le l \le n$ we define the Schwartz function $\eta_{\kappa,\underline{\tau},l}(s,\cdot)$ on $M_l(\mathbb{Q}_p)$, supported on $\mathrm{GL}(n,\mathbb{Q}_p)$, as

(5.7.2) $$\eta_{\kappa,\underline{\tau},l}(s,a) = \chi(\det(a)|\det(a)|_p)\cdot \phi(|\det(a)|_p)\cdot |\det(a)|_p^{s-1}\cdot \mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}(a)$$

for $a \in M_l(\mathbb{Q}_p)$. Define the operator

$$\mathcal{T}_{\kappa,\underline{\tau},p}(s) = \int_{\mathrm{GL}(n,\mathbb{Q}_p)} \eta_{\kappa,\underline{\tau},n}(s,a)\pi_p\left(\begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\right) da$$

(certainly in general as an operator acting on π or a model of π_p, its absolute convergence requires $\mathrm{Re}\,(s)$ to be sufficiently large). Then

$$\left\langle \varphi_{\underline{c}},\, \varphi'^{\vartheta}\right\rangle = B_{\phi,\kappa,\underline{\tau},\pi}\frac{\mathrm{vol}\left(B_G(\mathbb{Z}_p)N_G^-(\mathbb{Z}_p)\right)}{\mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)} \lim_{s\to k-n}\left\langle \overline{\varphi},\, U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle.$$

At the same time it follows from the computation in the last proposition that

$$\left\langle U_{p,\underline{c}}W(\varphi),\, \varphi'^{\vartheta}\right\rangle = \left\langle \overline{\varphi},\, U_{p,\underline{c}}\varphi'\right\rangle.$$

There exists a polynomial $R(X) \in \mathbb{C}[X]$ such that $(e\times 1)\mathcal{E}_{\kappa,\underline{\tau}} = (R(U_p)\times 1)\mathcal{E}_{\kappa,\underline{\tau}}$ and $eW(\varphi) = R(U_p)\varphi$. Thus we have

$$\left\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}}),\, \overline{\varphi}\otimes\varphi'^{\vartheta}\right\rangle = B_{\phi,\kappa,\underline{\tau},\pi}\frac{\mathrm{vol}\left(B_G(\mathbb{Z}_p)N_G^-(\mathbb{Z}_p)\right)}{\mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)} \lim_{s\to k-n}\left\langle \overline{\varphi},\, R(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle,$$
$$\left\langle eW(\varphi),\, \varphi'^{\vartheta}\right\rangle = \left\langle \overline{\varphi},\, R(U_p)\varphi'\right\rangle.$$

Now one sees that in order to verify (5.7.1), it suffices to show that there exists some $\varphi' \in \pi^{N_G^-(\mathbb{Z}_p)}$ with $e\varphi' \neq 0$, such that, as a function in s, $\left\langle \overline{\varphi},\, Q(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle$ admits a meromorphic continuation and

(5.7.3) $$\lim_{s\to k-n}\frac{\left\langle \overline{\varphi},\, R(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle}{\left\langle \overline{\varphi},\, R(U_p)\varphi'\right\rangle} = \mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)\prod_{j=1}^{n} R_p(k-n,\theta_j,\phi^{-1}).$$

If we fix φ' it is not difficult to check that there exists an open compact subgroup $K_p \subset G(\mathbb{Z}_p)$ such that π^{K_p} contains $U_{p,\underline{c}}\varphi'$, $U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'$ for all $\underline{c}\in C_n^+$ and $s\in\mathbb{C}$ with $\mathrm{Re}\,(s)$ large enough. Therefore we can assume that the polynomial $R(X)$ satisfies $e\varphi' = R(U_p)\varphi'$ and $e\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi' = R(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'$. In this case the value of the left hand side of (5.7.3) dose not change if we replace φ by any $\varphi''\in\pi$ with $\left\langle \overline{\varphi''},\, e\varphi'\right\rangle \neq 0$. Thus we have a big freedom in choosing φ' and φ'' to compute the left hand side of (5.7.3). The requirements on φ' and φ'' are $\varphi' \in \pi_{\underline{t}}^{N_G^-(\mathbb{Z}_p)}$ and $\left\langle \overline{\varphi''},\, e\varphi'\right\rangle \neq 0$. It is also clear that the computation can be reduced to a local situation using any model of π_p.

Let $\mathfrak{f}_{N_G^-} \in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ be the section supported $B_G(\mathbb{Q}_p)N_G^-(\mathbb{Z}_p)$ and taking the value 1 on $N_G^-(\mathbb{Z}_p)$. Fix an open compact subgroup K_p of $G(\mathbb{Z}_p)$ sufficiently small such that the vectors

$U_{p,\underline{c}}\mathfrak{f}_{N_G^-}$, $U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}$, with $\underline{c}\in C_n^+$, $\mathrm{Re}\,(s)\gg 0$, are all fixed by the right translation of K_p and the restriction of $\underline{\theta}$ to $B_G(\mathbb{Q}_p)\cap K_p$ is the trivial character. Let $\tilde{\mathfrak{f}}_{K_p}\in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}^{-1}$ be the section supported on $B_G(\mathbb{Q}_p)K_p$ and taking the value 1 on K_p . (5.7.3) will follow from the equality

$$\frac{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}\right\rangle}{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathfrak{f}_{N_G^-}\right\rangle} = \mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)\prod_{j=1}^{n} R_p(s,\theta_j,\phi^{-1}) \tag{5.7.4}$$

for all $\underline{c}\in C_n^+$. Note that although π_p is a sub-representation of $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ and in general they are not equal, by the discussion in §5.5, under the normalization for the $\mathbb{U}_p$-operators associated to $\underline{t}$, the ordinary subspace in $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ is one dimensional and certainly coincides with that of π_p.

The pairing between a section $\mathfrak{f}\in\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ and a section $\tilde{\mathfrak{f}}\in\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}^{-1}$ is given as $\langle\mathfrak{f},\tilde{\mathfrak{f}}\rangle=\int_{G(\mathbb{Z}_p)}\mathfrak{f}(g)\tilde{\mathfrak{f}}(g)\,dg$. We have

$$\begin{aligned}\frac{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}\right\rangle}{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathfrak{f}_{N_G^-}\right\rangle} &= \frac{\left(U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}\right)(1)}{\left(U_{p,\underline{c}}\mathfrak{f}_{N_G^-}\right)(1)}\\ &= \frac{\int_{\mathrm{GL}(n,\mathbb{Q}_p)}\int_{N_G(\mathbb{Z}_p)}\mathfrak{f}_{N_G^-}\left(up^{\underline{c}}\left(\begin{smallmatrix}a&0\\0&{}^{\mathrm{t}}a^{-1}\end{smallmatrix}\right)\right)\eta_{\kappa,\underline{\tau},n}(s,a)\,du\,da}{\int_{N_G(\mathbb{Z}_p)}\mathfrak{f}_{N_G^-}\left(up^{\underline{c}}\right)}\\ &= \int_{\mathrm{GL}(n,\mathbb{Q}_p)}\mathfrak{f}_{N_G^-}\left(\begin{pmatrix}a&0\\0&{}^{\mathrm{t}}a^{-1}\end{pmatrix}\right)\eta_{\kappa,\underline{\tau},n}(s,a)\,da\\ &= \int_{\mathrm{GL}(n,\mathbb{Q}_p)}\mathfrak{w}_n(a)\,\eta_{\kappa,\underline{\tau},n}(s,a)\,da,\end{aligned}$$

where for $1\le l\le n$ we define $\mathfrak{w}_l:\mathrm{GL}(l,\mathbb{Q}_p)\to\mathbb{C}$ to be the smooth function supported on $B_n(\mathbb{Q}_p)N_n^-(\mathbb{Z}_p)$ such that $\mathfrak{w}_l(bu^-)=\prod_{j=1}^{l}\theta_j(b_j)|b_j|_p^{l+1-j}$ for $b\in\mathrm{diag}(b_1,\cdots,b_l)N_l(\mathbb{Q}_p)$ and $u^-\in N_l^-(\mathbb{Z}_p)$. The desired equality (5.7.4) can be deduced from the induction relation

$$\begin{aligned}&\int_{GL(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\ &=\mathrm{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)R_p(s,\theta_l,\phi^{-1})\int_{GL(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a)\eta_{\kappa,\underline{\tau},l-1}(s,a)\,da,\end{aligned} \tag{5.7.5}$$

where $B_{l-1,1}(\mathbb{Z}_p)=\begin{pmatrix}\mathrm{GL}(l-1,\mathbb{Z}_p)&\mathbb{Z}_p^{l-1}\\0&\mathbb{Z}_p^\times\end{pmatrix}$ and $N_{l-1,1}(\mathbb{Z}_p)=\begin{pmatrix}I_{l-1}&0\\{}^{\mathrm{t}}\mathbb{Z}_p^{l-1}&1\end{pmatrix}$.

From the definition of $\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}$, we see that it is invariant under the right (resp. left) translation of $N_l^-(\mathbb{Z}_p)$ (resp. $N_l(\mathbb{Z}_p)$). By the definition of $\mathfrak{w}_l$ and that of $\eta_{\kappa,\underline{\tau},l}$ (5.7.2), we have

$$\begin{aligned}
&\mathrm{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)^{-1}\int_{GL(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&=\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)\times\mathbb{Q}_p^\times\times N_{l-1,1}(\mathbb{Q}_p)\times N_{l-1,1}^-(\mathbb{Z}_p)}\mathfrak{w}_l\left(u\begin{pmatrix}a'&0\\0&a_l\end{pmatrix}u^-\right)\left|\det(a')\right|_p^{-1}|a_l|_p^{l-1}\\
&\qquad\cdot\eta_{\kappa,\underline{\tau},l}\left(s,u\begin{pmatrix}a'&0\\0&a_l\end{pmatrix}u^-\right)da'\,da_l\,du\,du^-\\
&=\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)\times\mathbb{Q}_p^\times\times\mathbb{Q}_p^{l-1}}\left(\phi(p)^{-1}\alpha_lp^{-s}\right)^{\mathrm{val}_p(a_l)}\chi\psi_l(a_l|a_l|_p)\cdot\mathfrak{w}_{l-1}(a')\\
&\qquad\cdot\chi\left(\det(a')|\det(a')|_p\right)\cdot\phi(\det(a')|_p)\cdot|\det(a')|_p^{s-1}\cdot\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}\left(\begin{pmatrix}a'&y\\0&a_l\end{pmatrix}\right)da'\,da_l\,dy.
\end{aligned}\tag{5.7.6}$$

Next we split the proof of (5.7.5) into two cases depending on whether the character $\chi\psi_l$ is trivial or not. First we look at the case when $\chi\psi_l$ is trivial. Using Proposition 5.4.1 we get

$$\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}\left(\begin{pmatrix}a'&y\\0&a_l\end{pmatrix}\right)=\left(-p^{-l}\mathbb{1}_{p^{-1}\mathbb{Z}_p^\times}(a_l)\mathbb{1}_{a_l\mathbb{Z}_p^{l-1}}(y)+(1-p^{-1})\mathbb{1}_{\mathbb{Z}_p}(a_l)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(y)\right)\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l-1}(a'),$$

and (5.7.6) becomes

$$\begin{aligned}
&\mathrm{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)^{-1}\int_{\mathrm{GL}(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&=\int_{\mathbb{Q}_p^\times\times\mathbb{Q}_p^{l-1}}\left(\phi(p)^{-1}\alpha_lp^{-s}\right)^{\mathrm{val}_p(a_l)}\left(-p^{-l}\mathbb{1}_{p^{-1}\mathbb{Z}_p^\times}(a_l)\mathbb{1}_{p^{-1}\mathbb{Z}_p^{l-1}}(y)+(1-p^{-1})\mathbb{1}_{\mathbb{Z}_p}(a_l)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(y)\right)da_l\,dy\\
&\quad\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da'\\
&=\left(-p^{-1}\cdot\phi(p)\alpha_l^{-1}p^s+(1-p^{-1})\sum_{j=0}^{\infty}\left(\phi(p)^{-1}\alpha_lp^{-s}\right)^j\right)\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da'\\
&=\frac{1-\phi(p)\alpha_l^{-1}p^{s-1}}{1-\phi(p)^{-1}\alpha_lp^{-s}}\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da',
\end{aligned}$$

which is exactly (5.7.5) in the case when $\chi\psi_l$ is trivial. Now assume that $\chi\psi_l$ is nontrivial. Again using Proposition 5.4.1 we get

$$\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}\left(\begin{pmatrix}a'&y\\0&a_l\end{pmatrix}\right)=p^{-lc_{\chi\psi_l}}G(\chi\psi_l)\cdot\mathbb{1}_{p^{-c_{\chi\psi_l}}\mathbb{Z}_p^\times}(a_l)\mathbb{1}_{a_l\mathbb{Z}_p^{l-1}}(y)\cdot\left(\chi\psi_l(p^{c_{\chi\psi_l}}a_l)\right)^{-1}\cdot\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l-1}(a'),$$

which together with (5.7.6) gives

$$\begin{aligned}
&\mathrm{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)^{-1}\int_{\mathrm{GL}(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&=p^{-lc_{\chi\psi_l}}G(\chi\psi_l)\cdot\mathrm{vol}\left(p^{-c_{\chi\psi_l}}\mathbb{Z}_p^{l-1}\right)\cdot\left(\phi(p)^{-1}\alpha_lp^{-s}\right)^{-c_{\chi\psi_l}}\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da'\\
&=\left(\phi(p)\alpha_l^{-1}p^{s-1}\right)^{c_{\chi\psi_l}}G(\chi\psi_l)\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da',
\end{aligned}$$

and proves (5.7.5) in the case when $\chi\psi_l$ is nontrivial. ∎

6. The measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ valued in Hida families

From the previously constructed measure $\mu_{\mathcal{E},q\text{-exp}}$ on $\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)$, we apply Hida theory to produce, for each character $\underline{\imath}$ of $T_n(\mathbb{Z}/p\mathbb{Z})$, a measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ on $\mathbb{Z}_p^\times$, valued in n-variable Hida families of $G \times G$.

6.1. Brief review of Hida theory for G.

Usually Hida theory is formulated with $\mathbf{G}$ instead of G, but it should be clear that by restricting to a connected component we get a good theory for G.

6.1.1. *The Igusa tower.* Let $Y_{\mathbf{G},N}$ be the Siegel moduli scheme defined over $\mathbb{Z}_p$, parametrizing principally polarized abelian schemes (A, λ) of dimension n with a principal level N structure ψ_N over $\mathrm{Spec}(\mathbb{Z}_p)$, and $X_{\mathbf{G},N}$ be a smooth toroidal compactification of $Y_{\mathbf{G},N}$ with boundary C, over which there is the semi-abelian scheme $\mathcal{G} \to X_{\mathbf{G},N}$ extending the universal abelian scheme $\mathcal{A} \to Y_{\mathbf{G},N}$. Let $\mathrm{Ha} = \mathrm{Ha}(\mathcal{G}[p^\infty])$ be the Hasse invariant, which is a global section of the invertible sheaf $(\det \omega(\mathcal{G}/X_{\mathbf{G},N}))^{\otimes p-1}$ over the reduction $X_{\mathbf{G},N/\mathbb{F}_p}$. The push-forward of $\det \omega(\mathcal{G}/X_{\mathbf{G},N})$ to the minimal compactification $X^*_{\mathbf{G},N}$ is ample. For a sufficiently large integer c we can lift Ha^c to a section over $X_{\mathbf{G},N}$, and we denote by E such a lift.

Now let F be a finite extension of $\mathbb{Q}_p$ containing all the N-th roots of unity, and $X_{G,N}$ be a connected component of the base change of $X_{\mathbf{G},N}$ to $\mathcal{O}_F$. Define $S = X_{G,N}[1/E]$ and $S_l = S \otimes_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}/p^l\mathbb{Z})$. Let $T_{l,m} = \underline{\mathrm{Isom}}_{S_l}\left((\mathcal{G}[p^m])^{D,\text{ét}}, (\mathbb{Z}/p^m\mathbb{Z})^n\right)$ where the superscript D means the Cartier dual. The scheme $T_{l,m}$ is étale over S_l with Galois group $\mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$. The inverse system $\cdots \to T_{l,m} \to T_{l,m-1} \to \cdots \to T_{l,1} \to S_l$ is called the Igusa tower. By abuse of notation the pullback of the divisor C to $T_{m,l}$ will also be written as C.

6.1.2. *p-adic (cuspidal) Siegel modular forms.* Define

$$V_{l,m} := H^0\left(T_{l,m}, \mathcal{O}_{T_{l,m}}(-C)\right)^{N_n(\mathbb{Z}/p^m\mathbb{Z})},$$

and set $V_{l,\infty} = \varinjlim_m V_{l,m}$. By taking the inverse and direct limits of $V_{l,\infty}$ one defines

$$V = \varprojlim_l V_{l,\infty}, \qquad \mathcal{V} = \varinjlim_l V_{l,\infty}.$$

Elements in V are called (cuspidal) p-adic Siegel modular forms (of tame principal level N) . The space $\mathcal{V}$ will be used to construct Hida families. We also define the space V' in the same way as V but without requiring the cuspidality condition. The evaluation at the Mumford object (whose construction is explained in §2.6) defines the q-expansion map

$$\varepsilon_{q,l} : V'_{l,\infty} \longrightarrow \mathcal{O}_F/p^l\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]],$$

and the p-adic q-expansion map for p-adic Siegel modular forms

$$\varepsilon_{q,p\text{-adic}} : V' \longrightarrow \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]]. \tag{6.1.1}$$

The injectivity of $\varepsilon_{q,l}$ and $\varepsilon_{q,p\text{-adic}}$ follows from the irreducibility of the Igusa tower $\varprojlim_m T_{1,m}$ [FC90, V.7], and is called the q-expansion principle for p-adic Siegel modular forms.

For each continuous character $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ (also called a p-adic weight), let $V[\underline{\tau}]$ (resp. $\mathcal{V}[\underline{\tau}]$) be the $\underline{\tau}$-isotypic part of $V \otimes_{\mathcal{O}_F} \mathcal{O}_{F(\underline{\tau})}$ (resp. $\mathcal{V} \otimes_{\mathcal{O}_F} \mathcal{O}_{F(\underline{\tau})}$) under the action of $T_n(\mathbb{Z}_p)$, where $F(\underline{\tau})$ is the field obtained by adjoining to F the values of the character $\underline{\tau}$. Elements inside the space $V[\underline{\tau}]$ are called (cuspidal) p-adic Siegel modular forms of (p-adic) weight $\underline{\tau}$. Thanks to the Hodge–Tate map

$$(\mathcal{G}[p^\infty])^{D,\text{ét}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_l} \xrightarrow{\sim} \omega(\mathcal{G}/S_l), \tag{6.1.2}$$

for an algebraic weight $\underline{t}$, there is the canonical embedding

$$H^0(X_{G,N}, \omega_{\underline{t}}(-C)) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^l\mathbb{Z} \hookrightarrow H^0(S_l, \omega_{\underline{t}}(-C)) \hookrightarrow V_{l,\infty}[\underline{t}].$$

The cuspidality condition guarantees that the following standard condition for Hida theory is satisfied,

$$\text{(Hyp)} \qquad H^0(S, \omega_{\underline{t}}(-C)) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^l\mathbb{Z} \xrightarrow{\sim} H^0(S_l, \omega_{\underline{t}}(-C))$$

for all dominant algebraic weight $\underline{t}$, from which the density theorem follows, saying that the space of classical forms $\bigoplus_{\underline{t}\gg 0} H^0(X_{G,N}, \omega_{\underline{t}}(-C))[1/p] \cap V$ is dense inside V [Hid02, §3.5].

The action of $\mathbb{U}_p$-operators can be defined for $V[\underline{\tau}]$, $\mathcal{V}[\underline{\tau}]$ via algebraic correspondence [Hid04, §8.3] [Liu15, §2.9.5], and is compatible with all the $\mathbb{U}_p$-operators we have defined before (in fact it is the $\mathbb{U}_p$-action on V that has a canonical normalization, and the normalizations of the $\mathbb{U}_p$-action in other circumstances are chosen to agree with it). Recall that we have set $U_p = U_{p,\rho_G}$ to be the operator associated to $\rho_G = (n, n-1, \cdots, 1) \in C_n^+$. By the discussion on §2.5, the ordinary projector

$$e = \lim_{r\to\infty} U_p^{r!}$$

is well defined on $\bigoplus_{\underline{t}\geq 0} H^0(X_{G,N}, \omega_{\underline{t}}(-C))$. Then the density theorem indicates that the operator e extends to V and $\mathcal{V}$. It projects the spaces V and $\mathcal{V}$ to their subspaces where all the eigenvalues of $\mathbb{U}_p$-operators are p-adic units. Put

$$V_{\mathrm{ord}} = eV, \qquad \mathcal{V}^*_{\mathrm{ord}} = \mathrm{Hom}_{\mathcal{O}_F}(e\mathcal{V}, F/\mathcal{O}_F).$$

The group $T_n(\mathbb{Z}_p)$ naturally acts on both V_{ord} and $\mathcal{V}^*_{\mathrm{ord}}$ and equip them with an $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure. Besides (Hyp) the other two conditions for the axiomatic vertical control theorem are

(C) $e(Ef) = Ee(f)$ for all $f \in H^0(S_1, \omega_{\underline{t}})$.

(F) $\dim_F eH^0\left(X_{G,N}, \omega_{\underline{t}} \otimes \det^k \omega(\mathcal{G}/X_{G,N})\right)$ is bounded independent of k.

The condition (C) can be easily checked using the q-expansion principle and the condition (F) follows from results in [TU99].

6.1.3. *Hida families and the vertical control theorem.* The group $T_n(\mathbb{Z}_p)$ decomposes as $\Gamma_{T_n} \times T_n(\mathbb{Z}/p\mathbb{Z})$ with Γ_{T_n} being the p-profinite part. Set $\Lambda_n = \mathcal{O}_F[[\Gamma_{T_n}]]$. The $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module of Hida families of cuspidal p-adic Siegel modular forms of tame principal level N is defined as

$$\text{(6.1.3)} \qquad \mathcal{M}_{\mathrm{ord}} = \mathrm{Hom}_{\Lambda_n}\left(\mathcal{V}^*_{\mathrm{ord}}, \Lambda_n\right).$$

Given $\underline{\tau} \in \mathrm{Hom}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ put $\mathbf{p}_{\underline{\tau}} : \mathcal{O}_F[[T_n(\mathbb{Z}_p)]] \to \mathcal{O}_{F(\underline{\tau})}$ to be the map sending $\gamma \in T_n(\mathbb{Z}_p)$ to $\underline{\tau}(\gamma)$.

Theorem 6.1.1 (Vertical Control Theorem [Hid04, Theorem 8.13])**.** *As a Λ_n-module, the space $\mathcal{M}_{\mathrm{ord}}$ of Hida families is free of finite rank. For each p-adic weight $\underline{\tau}$ we have the Hecke-equivariant isomorphism $\mathcal{M}_{\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]], \mathbf{p}_{\underline{\tau}}} \mathcal{O}_{F(\underline{\tau})} \cong V_{\mathrm{ord}}[\underline{\tau}]$. When $\underline{t}$ is a sufficiently regular algebraic weight, $V_{\mathrm{ord}}[\underline{t}] = eH^0(X_{G,N}, \omega_{\underline{t}}(-C))$.*

The unramified Hecke operators and $\mathbb{U}_p$-operators act on $\mathcal{M}_{\mathrm{ord}}$ and we denote by $\mathbb{T}^N_{\mathrm{ord}}$ the subalgebra of $\mathrm{End}_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]}(\mathcal{M}_{\mathrm{ord}})$ generated by them. The natural map $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}}) \to \mathrm{Spec}(\mathcal{O}_F[[T_n(\mathbb{Z}_p)]])$ is called the weight projection map.

The finite group $T_n(\mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{M}_{\mathrm{ord}}$ and we have the decomposition of free Λ_n-modules

$$\mathcal{M}_{\mathrm{ord}} = \bigoplus_{\underline{\imath}\in \mathrm{Hom}(T_n(\mathbb{Z}/p\mathbb{Z}), \mu_{p-1})} \mathcal{M}_{\mathrm{ord},\underline{\imath}},$$

such that $T_n(\mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{M}_{\mathrm{ord},\underline{\imath}}$ by the character $\underline{\imath}$.

6.1.4. *The spaces* $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural$ *and* $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord}^\Delta)^\natural$. The group $T_n(\mathbb{Z}_p)$ acts on itself by multiplication and induces a natural $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure on the space $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F)$. We define $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')^\natural$ to be the subspace of $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')$ consisting of continuous maps $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F) \to V'$ that are not only $\mathcal{O}_F$-linear but further $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear. An equivalent description for the elements of the subspace $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')^\natural$ is that the evaluations at all $\underline{\tau} \in \mathrm{Hom}_{\rm cont}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ belong to $V'[\underline{\tau}]$. Let $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural$ be the ordinary cuspidal part of $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')^\natural$. For each character $\underline{\imath}$ of $T_n(\mathbb{Z}/p\mathbb{Z})$, we construct a morphism $\Phi_{\underline{\imath}}$ mapping $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural$ into the space of Hida families.

Unfolding the definitions one easily sees that there is a natural pairing $V_{\rm ord} \times \mathcal{V}_{\rm ord}^* \xrightarrow{\langle,\rangle} \mathcal{O}_F$ such that the following diagram commutes if $\underline{\imath} = \underline{\tau}\,|_{T_n(\mathbb{Z}/p\mathbb{Z})}$

$$\begin{array}{ccc}
\mathcal{M}_{{\rm ord},\underline{\imath}} \times \mathcal{V}_{\rm ord}^* & \longrightarrow & \Lambda_n \\
\downarrow{\scriptstyle \mathbf{s}_{\underline{\tau}}\times 1} & & \downarrow{\scriptstyle \mathbf{p}_{\underline{\tau}}} \\
V_{\rm ord}[\underline{\tau}] \times \mathcal{V}_{\rm ord}^* \hookrightarrow \left(V_{\rm ord} \otimes_{\mathcal{O}_F} \mathcal{O}_{F(\underline{\tau})}\right) \times \mathcal{V}_{\rm ord}^* & \xrightarrow{\langle,\rangle} & \mathcal{O}_{F(\underline{\tau})},
\end{array}$$

where $\mathbf{s}_{\underline{\tau}}$ is the specialization map

$$\mathbf{s}_{\underline{\tau}} : \mathcal{M}_{\rm ord} \longrightarrow \mathcal{M}_{\rm ord} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]], \mathbf{p}_{\underline{\tau}}} \mathcal{O}_{F(\underline{\tau})} \xrightarrow{\sim} V_{\rm ord}[\underline{\tau}]. \tag{6.1.4}$$

This pairing induces an $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear pairing

$$\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural \times \mathcal{V}_{\rm ord}^* \longrightarrow \mathscr{M}eas(T_n(\mathbb{Z}_p), \mathcal{O}_F), \tag{6.1.5}$$

where the $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure on $\mathscr{M}eas(T_n(\mathbb{Z}_p), \mathcal{O}_F)$ comes from that of $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F)$.

Now fix a character $\underline{\imath}$ of the finite group $T_n(\mathbb{Z}/p\mathbb{Z})$. Let $\mathfrak{u}$ be a generator of $1 + p\mathbb{Z}_p$ and we associate to it the p-adic logarithm function $\log_{\mathfrak{u}} : 1 + p\mathbb{Z}_p \to \mathbb{Z}_p$ such that the value at $\mathfrak{u}$ is 1 and we extend $\log_{\mathfrak{u}}$ to $\mathbb{Z}_p^\times$ by requiring it to take value 0 on μ_{p-1} ($\mathbb{Z}_p^\times$ canonically decomposes as $\mu_{p-1} \times (1 + p\mathbb{Z}_p)$). Denote by γ_i the element of $T_n(\mathbb{Z}_p)$ whose i-th component is $\mathfrak{u}$ and other components are 1. Then $\gamma_1, \cdots, \gamma_n$ topologically generate Γ_{T_n}. The p-adic Mellin transform with respect to $\underline{\imath}$ is the map

$$\begin{aligned}
&\mathscr{M}eas(T_n(\mathbb{Z}_p), \mathcal{O}_F) \longrightarrow \Lambda_n \\
&\qquad \mu \longmapsto \int_{T_n(\mathbb{Z}_p)} \underline{\imath}(x_1, \cdots, x_n) \prod_{i=1}^n \gamma_i^{\log_{\mathfrak{u}}(x_i)}\, d\mu(x_1, \cdots, x_n),
\end{aligned} \tag{6.1.6}$$

where $\gamma_i^{\log_{\mathfrak{u}}(x_i)}$ is the element $\sum_{m=0}^\infty \binom{\log_{\mathfrak{u}} x_i}{m} (\gamma_i - 1)^m \in \Lambda_n$ with the binomial coefficient $\binom{\log_{\mathfrak{u}} x_i}{m}$ defined as $\frac{\log_{\mathfrak{u}} x_i (\log_{\mathfrak{u}} x_i - 1) \cdots (\log_{\mathfrak{u}} x_i - m + 1)}{m!}$. One can check that this p-adic Mellin transform with respect to $\underline{\imath}$ is Λ_n-linear. Combining it with (6.1.5) we get a Λ_n-linear pairing

$$\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural \times \mathcal{V}_{\rm ord}^* \longrightarrow \Lambda_n,$$

and therefore the desired morphism of Λ_n-modules

$$\Phi_{\underline{\imath}} : \mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural \to \mathcal{M}_{{\rm ord},\underline{\imath}}. \tag{6.1.7}$$

Moreover for each point $\underline{\tau} \in \mathrm{Hom}_{\rm cont}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ whose restriction to $T_n(\mathbb{Z}/p\mathbb{Z})$ is $\underline{\imath}$ and $\mu \in \mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\rm ord})^\natural$, we have

$$\int_{T_n(\mathbb{Z}_p)} \underline{\tau}\, d\mu = \mathbf{s}_{\underline{\tau}} \circ \Phi_{\underline{\imath}}(\mu).$$

For our applications we define the $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module V^Δ, which as an $\mathcal{O}_F$-module is the subspace of $V \otimes_{\mathcal{O}_F} V$ generated by the elements killed by $\gamma \otimes 1 - 1 \otimes \gamma$ for all $\gamma \in T_n(\mathbb{Z}_p)$. The action of $T_n(\mathbb{Z}_p)$ on V^Δ via either factor agrees with the other, so V^Δ has a well-defined $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure. Denote by V^Δ_{ord} the sub-$\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module of V^Δ obtained by taking the ordinary projection on both factors, and we define the $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module $\mathscr{M}eas(T_n(\mathbb{Z}_p), V^\Delta_{\mathrm{ord}})^\natural$ to be the space of continuous $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear maps from $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F)$ to V^Δ_{ord}. Through the same argument as above we see that there exists a canonical $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear pairing

$$V^\Delta_{\mathrm{ord}} \times \left(\mathcal{V}^*_{\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{V}^*_{\mathrm{ord}}\right) \longrightarrow \mathcal{O}_F$$

whose restriction to either factor agrees with the previous pairing $V_{\mathrm{ord}} \times \mathcal{V}^*_{\mathrm{ord}} \xrightarrow{\langle,\rangle} \mathcal{O}_F$. It induces a morphism of Λ_n-modules

$$\Phi^\Delta_{\underline{\imath}} : \mathscr{M}eas(T_n(\mathbb{Z}_p), V^\Delta_{\mathrm{ord}})^\natural \to \mathcal{M}_{\mathrm{ord},\underline{\imath}} \otimes_{\Lambda_n} \mathcal{M}_{\mathrm{ord},\underline{\imath}}, \tag{6.1.8}$$

with the property

$$\int_{T_n(\mathbb{Z}_p)} \underline{\tau}\, d\mu = (\mathbf{s}_{\underline{\tau}} \times \mathbf{s}_{\underline{\tau}}) \circ \Phi^\Delta_{\underline{\imath}}(\mu)$$

for all $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ whose restriction to $T_n(\mathbb{Z}/p\mathbb{Z})$ is $\underline{\imath}$ and $\mu \in \mathscr{M}eas(T_n(\mathbb{Z}_p), V^\Delta_{\mathrm{ord}})^\natural$.

6.1.5. *The q-expansions of Hida families.* For each $\beta \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}$, the maps $\varepsilon_{q,\beta} : V_{l,\infty} \to \mathcal{O}_F/p^l\mathcal{O}_F$, $l \geq 1$, of taking the β-th coefficient of the q-expansion patch to an $\mathcal{O}_F$-linear map $\varepsilon_{q,\beta} : \mathcal{V} \to F/\mathcal{O}_F$, which gives an element of $\mathcal{V}^*_{\mathrm{ord}}$. Thus by definition there is a Λ_n-linear map

$$\varepsilon_{q,\beta} : \mathcal{M}_{\mathrm{ord}} \longrightarrow \Lambda_n$$

which makes the following diagram

$$\begin{array}{ccccc}
\mathcal{M}_{\mathrm{ord},\underline{\imath}} & & \xrightarrow{\quad\varepsilon_{q,\beta}\quad} & & \Lambda_n \\
\Big\downarrow{\scriptstyle \mathbf{s}_{\underline{\tau}}} & & & & \Big\downarrow{\scriptstyle \mathbf{p}_{\underline{\tau}}} \\
V_{\mathrm{ord}}[\underline{\tau}] & \xrightarrow{\varepsilon_{q,p\text{-adic}}} & \mathcal{O}_{F(\underline{\tau})}[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}]] & \xrightarrow{\beta\text{-th coefficient}} & \mathcal{O}_{F(\underline{\tau})}
\end{array}$$

commute for $\underline{\tau}$ that restricts to $\underline{\imath}$ on $T_n(\mathbb{Z}/p\mathbb{Z})$. From $\varepsilon_{q,\beta}$, for $(\beta_1,\beta_2) \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$ we define the Λ_n-linear map

$$\varepsilon_{q,\beta_1,\beta_2} : \mathcal{M}_{\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{M}_{\mathrm{ord}} \longrightarrow \Lambda_n.$$

6.2. Construct $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ from $\mu_{\mathcal{E},q\text{-exp}}$.

6.2.1. *Embedding nearly holomorphic forms into p-adic forms.* Let $T_{\infty,m}$ be the formal scheme $\varinjlim_l T_{l,m}$ defined over $\mathcal{O}_F$. When $m=0$ the formal scheme $T_{\infty,0}$ is the completion of $S = X_{G,N}[1/E]$ along its special fibre. Over $T_{\infty,0}$ the Hodge filtration admits a splitting

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{G,N})^{\mathrm{can}}\big|_{T_{\infty,0}} = \omega(\mathcal{G}/T_{\infty,0}) \oplus \mathcal{U}_{\mathcal{H}},$$

called the unit root splitting, which is constructed by considering the F-crystal structure of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{G,N})$ [Kat73, Theorem 4.1]. Take the generic fibre $T_{\mathrm{rig},m}$ of the formal scheme $T_{\infty,m}$. It is a rigid analytic subspace of the rigid analytic space $X^{\mathrm{an}}_{G,\Gamma(Np^m)}$ associated to the scheme $X_{G,\Gamma(Np^m)}$ over F. Pulling back the unit root splitting from level $\Gamma(N)$ to $\Gamma(Np^m)$ yields a projection $\mathcal{V}^r_{\underline{t}} \to \omega_{\underline{t}}$ of coherent sheaves over the rigid analytic space $T_{\mathrm{rig},m}$, from which one gets, combining with the Hodge–Tate map (6.1.2), the following Hecke equivariant map

$$\iota_{p\text{-adic}} : H^0(X_{G,\Gamma_1(N,p^m)}, \mathcal{V}^r_{\underline{t}})[\underline{\psi}] \longrightarrow H^0(T_{\infty,m}, \omega_{\underline{t}})^{N_n(\mathbb{Z}/p^m\mathbb{Z})}[\underline{\psi}][1/p] \longrightarrow V'[\underline{\tau}][1/p],$$

where $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ is an arithmetic weight with algebraic part $\underline{t}$ dominant and finite part $\underline{\psi}$ valued in $\mu_{(p-1)p^{m-1}}$. The symbol $[\underline{\psi}]$ means the $\underline{\psi}$ equivariant part under the natural action of $T_n(\mathbb{Z}_p)$. The injectivity of $\iota_{p\text{-adic}}$ results from the fact that the unit root splitting agrees with the C^∞ splitting at ordinary CM points and the analytic density of ordinary CM points (see also [Liu15, Proposition 3.12.1]).

The map $\iota_{p\text{-adic}}$ embeds nearly holomorphic forms into the space of p-adic forms Hecke equivariantly and gives an integral structure to the space $H^0(X_{G,\Gamma_1(N,p^m)}, \mathcal{V}_{\underline{t}}^r)$ which is preserved by the $\mathbb{U}_p$-operators. Moreover we have

Proposition 6.2.1. $eH^0(X_{G,\Gamma_1(N,p^m)}, \mathcal{V}_{\underline{t}}^r) = eH^0(X_{G,\Gamma_1(N,p^m)}, \omega_{\underline{t}})$.

Proof. Proposition 2.5.5 says that the composition $E_{\underline{t}}e$ is 0, or equivalently the image of e is killed by the operator $E_{\underline{t}}$, so holomorphic. □

6.2.2. *The measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$.* The composition of $\iota_{p\text{-adic}}$ with (6.1.1) is exactly the (p-adic) q-expansion map for nearly holomorphic forms defined in (2.6.1), because the basis $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$ is compatible with the unit root splitting as $\delta_{j,\mathrm{can}}$, $1 \leq j \leq n$ are horizontal for the Gauss–Manin connection. Now Proposition 4.4.1 together with the q-expansion principle implies that $\iota_{p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}})$ lies inside $V'[\underline{\tau}]$ for all admissible $(\kappa, \underline{\tau})$. One direct corollary of the q-expansion principle is that the space V' of p-adic forms (of tame principal level N) is a closed subspace, under the induced topology, of the space $\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}]]$. Then the density of all the admissible points inside $\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ with respect to the p-adic topology indicates that the measure $\mu_{\mathcal{E},q\text{-exp}}$ in Theorem 5.2.2 belongs to the image of the embedding of $\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), V'^{\Delta}\right)$ into $\mathcal{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{\geq 0}]]\right)$, induced by the q-expansion map.

This is not sufficient for us. Before we continue we must make sure that $\mu_{\mathcal{E},q\text{-exp}}$ actually is contained in the image of the cuspidal part. Thanks to the cuspidality result Theorem 4.2.2 we know that $\iota_{p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}})$ is cuspidal if $t_1 = t_2 = \cdots = t_n = k > 2n+1$. The p-adic density of such points guarantees that $\mu_{\mathcal{E},q\text{-exp}}$ lies inside the image of the injective map

$$\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), V^{\Delta}\right) \longrightarrow \mathcal{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right),$$

and we denote by $\mu_{\mathcal{E}}$ the preimage of $\mu_{\mathcal{E},q\text{-exp}}$.

Now by applying the ordinary projection $e \times e : V^{\Delta} \to V^{\Delta}_{\mathrm{ord}}$ to $\mu_{\mathcal{E}}$, we obtain the measure $\mu_{\mathcal{E},\mathrm{ord}}$ inside $\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), V^{\Delta}_{\mathrm{ord}}\right) = \mathcal{M}eas\left(\mathbb{Z}_p^\times, \mathcal{M}eas(T_n(\mathbb{Z}_p), V^{\Delta}_{\mathrm{ord}})\right)$. Using (6.1.8), we define

$$\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}} = \Phi^{\Delta}_{\underline{\imath}}\left(\mu_{\mathcal{E},\mathrm{ord}}\right).$$

This $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ lies inside $\mathcal{M}eas(\mathbb{Z}_p^\times, \mathcal{M}_{\mathrm{ord},\underline{\imath}} \otimes_{\Lambda_n} \mathcal{M}_{\mathrm{ord},\underline{\imath}})$ and satisfies

$$(\mathbf{s}_{\underline{\tau}} \times \mathbf{s}_{\underline{\tau}})\left(\int_{\mathbb{Z}_p^\times} \kappa\, d\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}\right) = (e \times e)\mathcal{E}_{\kappa,\underline{\tau}}$$

for all admissible $(\kappa, \underline{\tau})$ such that the restriction of $\underline{\tau}$ to $T_n(\mathbb{Z}/p\mathbb{Z})$ is $\underline{\imath}$.

Before we ending this section we give the proof of Proposition 5.7.1.

Proof of Proposition 5.7.1. We show $(e \times e)(\mu_{\mathcal{E}}) - (e \times 1)(\mu_{\mathcal{E}}) = 0$, which can be implied by the vanishing of its image $\nu_{\beta_1,\beta_2} \in \mathcal{M}eas(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F)$ under the map

$$\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), V^{\Delta}\right) \longrightarrow \mathcal{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right)$$
$$\xrightarrow{(\beta_1,\beta_2)\text{-th coefficient}} \mathcal{M}eas(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F),$$

for all $(\beta_1,\beta_2)\in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$. The p-adic Mellin transform (defined similarly as (6.1.6)) gives an isomorphism between $\mathscr{M}eas(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\mathcal{O}_F)$ and $\mathcal{O}_F[[\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p)]]$. It is not difficult to see that the vanishing of ν_{β_1,β_1} follows from the Zariski density of the subset inside $\operatorname{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\right)$ consisting of those points (by (5.6.1) including all admissible points with $\chi\psi_1,\cdots,\chi\psi_n$ nontrivial) at which the evaluations of ν_{β_1,β_2} are zero. □

7. The p-adic L-function for ordinary families and its interpolation properties

The p-adic L-function for a given ordinary family of Hecke eigen-systems is constructed by projecting the Hida-family-valued measure $\mu_{\mathcal{E},\mathrm{ord},\underline{t}}$ to the corresponding eigenspace for that ordinary family and then taking a nonvanishing Fourier coefficient.

The universal ordinary Hecke algebra $\mathbb{T}^N_{\mathrm{ord}}$ of tame principal level N is finite torsion free over Λ_n, and reduced because of Proposition 5.5.2.

Given a point $x\in\operatorname{Spec}(\mathbb{T}^N_{\mathrm{ord}})(\overline{\mathbb{Q}}_p)$ whose projection to the weight space $\underline{\tau}\in\operatorname{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ is arithmetic with dominant algebraic part $\underline{t}\in X(T_n)_+$, define $\mathfrak{S}_x$ to be the finite dimensional $F(\underline{\tau})$-vector space consisting of cuspidal holomorphic Siegel modular forms which are contained in $H^0(X_{G,\Gamma_1(N,p^m)},\omega_{\underline{t}}(-C))[\underline{\psi}]$ for some m, and belong to the eigenspace parametrized by x for the unramified Hecke operators and $\mathbb{U}_p$-operators. The space $\mathfrak{S}_x$ is stable under the operator eW, the composition of the ordinary projector and the operator W defined as (5.2.2). Let $\mathfrak{a}_{x,j}\in\mathcal{O}^\times_{\overline{\mathbb{Q}}_p}$, $1\leq j\leq n$, be the p-adic integers such that for each $\underline{a}=(a_1,\cdots,a_n)\in C_n^+$, the eigenvalue of the operator $U_{p,\underline{a}}$ parametrized by x is given by $\prod_{j=1}^n\mathfrak{a}_{x,j}^{a_j}$. If $\pi\subset\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is an irreducible cuspidal automorphic representation generated by an element inside $\mathfrak{S}_x$, then for $v\nmid Np$, it is clear that the isomorphism class of π_v is completely determined by x. At the same time the isomorphism class of the component π_p is also determined by $\underline{\psi}=\underline{\tau}_{\mathrm{f}}$ and $\mathfrak{a}_{x,1},\cdots,\mathfrak{a}_{x,n}$ (see §5.5). Thus the isomorphism class of the $G(\mathbb{A}^N)$-representation $\bigotimes'_{v\nmid N}\pi_v$ is determined by x and we denote it by π_x^N. Set $\alpha_{x,j}=p^{-(t_j-j)}\mathfrak{a}_{x,j}$.

To π_x^N and Dirichlet characters ϕ, χ, we associate the partial standard L-function $L^{Np\infty}(s,\pi_x^N\otimes\phi^{-1}\chi^{-1})$, and the modified Euler factor at p

$$E_p(s,\pi_x^N\times\phi^{-1}\chi^{-1})=\frac{\left(1-\chi^\circ(p)\cdot\phi(p)p^{s-1}\right)\prod_{j=1}^n\left(1-(\chi\psi_j)^\circ(p)\cdot\phi(p)\alpha_{x,j}^{-1}p^{s-1}\right)}{\left(1-\chi^\circ(p)\cdot\phi(p)^{-1}p^{-s}\right)\prod_{j=1}^n\left(1-(\chi\psi_j)^\circ(p)\cdot\phi(p)^{-1}\alpha_{x,j}p^{-s}\right)}$$
$$\times\left(\phi(p)p^{s-1}\right)^{c_\chi}G(\chi)\prod_{j=1}^n\left(\phi(p)\alpha_{x,j}^{-1}p^{s-1}\right)^{c_{\chi\psi_j}}G(\chi\psi_j).$$

Let $\mathcal{C}$ be a geometrically irreducible component of $\operatorname{Spec}(\mathbb{T}^N_{\mathrm{ord}}\otimes_{\mathcal{O}_F}F)$. Set $F_{\mathcal{C}}$ to be the function field of $\mathcal{C}$ and $\mathbb{I}_{\mathcal{C}}$ to be the integral closure of Λ_n inside $F_{\mathcal{C}}$. Denote by $\lambda_{\mathcal{C}}:\mathbb{T}^N_{\mathrm{ord}}\to\mathbb{I}_{\mathcal{C}}$ the homomorphism of Λ_n-algebras corresponding to $\mathcal{C}$. The group $T_n(\mathbb{Z}/p\mathbb{Z})$ acts on $\mathbb{T}^N_{\mathrm{ord}}$ and its action on $\mathbb{I}_{\mathcal{C}}$ is by a character $\underline{\iota}_{\mathcal{C}}$.

There is an isomorphism of $F_{\mathcal{C}}$-algebras

$$\mathbb{T}^N_{\mathrm{ord}}\otimes_{\Lambda_n}F_{\mathcal{C}}=F_{\mathcal{C}}\oplus R_{\mathcal{C}}$$

such that the projection of $\mathbb{T}^N_{\mathrm{ord}}\otimes_{\Lambda_n}\mathbb{I}_{\mathcal{C}}$ onto the first factor coincides with $\lambda_{\mathcal{C}}$. Define $\mathbb{1}_{\mathcal{C}}\in\mathbb{T}^N_{\mathrm{ord}}\otimes_{\Lambda_n}F_{\mathcal{C}}$ to be the idempotent corresponding to the first factor. For a finite extension F' of F, write $\Lambda_{n,F'}$ (resp. $\mathbb{T}^N_{\mathrm{ord},F'}$, $\mathbb{I}_{\mathcal{C},F'}$) to be the base change of Λ_n (resp. $\mathbb{T}^N_{\mathrm{ord}}$, $\mathbb{I}_{\mathcal{C}}$) from $\mathcal{O}_F$ to F'. If the weight projection map $\Lambda_{n,F'}\to\mathbb{T}^N_{\mathrm{ord},F'}$ is étale at the point $x\in\mathcal{C}(F')$, put $x'\in\operatorname{Spec}(\mathbb{T}^N_{\mathrm{ord}}\otimes_{\Lambda_n}\mathbb{I}_{\mathcal{C},F',x})$ to be the maximal ideal generated by $T\otimes 1-1\otimes\lambda_{\mathcal{C}}(T)$ for all $T\in\mathbb{T}^N_{\mathrm{ord}}$ and $1\otimes a$ for all a

inside the maximal ideal corresponding to x. It follows from [Sta15, Tag 00UE, Tag 00U8] that $\left(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x}\right)_{x'} = \mathbb{I}_{\mathcal{C},F',x}$, so the localization map $\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x} \to \left(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x}\right)_{x'}$ is surjective, and there exists the decomposition of $\mathbb{I}_{\mathcal{C},F',x}$-algebras

$$\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x} = \mathbb{I}_{\mathcal{C},F',x} \oplus R'_{\mathcal{C},x}$$

with the first projection being $\lambda_{\mathcal{C}}$. Thus the projector $\mathbb{1}_{\mathcal{C}}$ lies inside $\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x}$ as long as the weight projection map is étale at $x \in \mathcal{C}(F')$.

Remark 7.0.1. It is still an open problem to decide the exact conditions on a classical point $x \in \mathcal{C}(\overline{\mathbb{Q}}_p)$ in order for the connected component $\mathcal{C}$ to be étale at x. In [AIP15, §8.3], an example is given in the case $n = 2$ and the tame level is 1, where if the automorphic representation π_x associated to x is tempered and ordinary, $\pi_{x,p}$ is unramified and the weight satisfies $t_1 \geq t_2 > 3$, it is shown that the weight projection map is étale at x. The argument is for eigenvariety of finite slope forms and follows that in [Che11] where the étaleness results are proved for non-critical regular classical tempered crystalline points with distinct Frobenius eigenvalues on the eigenvarieties for unitary groups under certain conditions. The argument relies on the classcity results for p-adic forms, the analysis of the Galois representations associated to classical points in a neighborhood of x at the primes dividing the tame level together with the compatibility with the local Langlands correspondence, and the multiplicity one results. In our case of $\mathrm{Sp}(2n)$, the multiplicity one result is known [Art13] and some classcity theorems are obtained in [BPS16, Pil11]. It is possible that, using some arguments in [Til06, Pil12a] on analyzing the Galois representations at primes dividing the tame level, one can prove some étaleness results for tempered classical points with weights $t_1 \geq \cdots \geq t_n > \frac{n(n+1)}{2}$ under certain suitable assumptions.

Now applying the Hecke projector $\mathbb{1}_{\mathcal{C}}$ to the measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{r}}_{\mathcal{C}}}$ constructed in §6.2.2 gives an element inside $\mathscr{M}eas(\mathbb{Z}_p^\times, \mathcal{M}_{\mathrm{ord},\underline{\mathfrak{r}}} \otimes_{\Lambda_n} \mathcal{M}_{\mathrm{ord},\underline{\mathfrak{r}}}) \otimes_{\Lambda_n} F_{\mathcal{C}}$ on which the Hecke operators act by $\lambda_{\mathcal{C}}$. Suppose that the point $x \in \mathcal{C}(F')$ projects to an arithmetic point $\underline{\mathfrak{r}}$ in the weight space whose algebraic part is dominant and the weight projection map is étale at x. Let $\mathbf{s}_x : \mathcal{M}_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x} \to V_{\mathrm{ord}}[\underline{\mathfrak{r}}] \otimes_{\mathcal{O}_{F(\underline{\mathfrak{r}})}} \mathcal{O}_{F'}$ be the specialization map defined from (6.1.4) by extension of scalars. Fix an orthogonal basis $\mathfrak{s}_x = \{\varphi_1, \cdots, \varphi_d\}$ of the vector space $\mathfrak{S}_x$, i.e. $\varphi_1, \cdots, \varphi_d$ span $\mathfrak{S}_x$ and satisfy $\langle \varphi_i, \overline{\varphi_j} \rangle = 0$ if $i \neq j$. Then for each arithmetic $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ with $t_1 \geq \cdots \geq t_n \geq k \geq n+1$ and $\kappa(-1) = \phi(-1)$, we know by construction that the specialization at x of the Hida family $\mathbb{1}_{\mathcal{C}} \int_{\mathbb{Z}_p^\times} \kappa \, d\mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{r}}_{\mathcal{C}}}$ is a classical cuspidal holomorphic Siegel modular form on $G \times G$. By Proposition 5.2.3 we have

$$\begin{aligned} &\mathbf{s}_x \left(\mathbb{1}_{\mathcal{C}} \int_{\mathbb{Z}_p^\times} \kappa \, d\mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{r}}_{\mathcal{C}}} \right) \\ &= \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n (p^{2l}-1)} \cdot \frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}} \cdot \frac{Z_\infty(f_{\kappa,\underline{\mathfrak{r}},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})}{\langle v_{\underline{t}}^\vee, v_{\underline{t}} \rangle} \\ &\quad \times E_p(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}) \cdot L^{Np\infty}(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}) \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\varphi \otimes eW(\varphi)}{\langle \varphi, \overline{\varphi} \rangle}, \end{aligned} \tag{7.0.1}$$

where $v_{\underline{t}}$ is the highest weight vector inside the lowest $K_{G,\infty}$-type of the holomorphic discrete series $\mathcal{D}_{\underline{t}}$ and $v_{\underline{t}}^\vee$ is taken to be its dual vector.

For each $(\beta_1, \beta_2) \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$, define (recall that for simplicity we have assumed $\phi^2 \neq 0$)

$$\mu_{\mathcal{C},\phi,\beta_1,\beta_2} = \varepsilon_{q,\beta_1,\beta_2}\left(\mathbb{1}_{\mathcal{C}} \cdot \mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{r}}_{\mathcal{C}}}\right) \in \mathscr{M}eas(\mathbb{Z}_p^\times, \Lambda_n) \otimes_{\Lambda_n} F_{\mathcal{C}}.$$

Contrary to the case of $\mathrm{GL}(2)_{/\mathbb{Q}}$, where for an algebraic eigenform the first Fourier coefficient always has the smallest p-adic evaluation, in our situation there is no such canonical choice for

β_1, β_2. By construction we know that the measure $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}$ vanishes at all $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ with $\kappa(-1) \neq \phi(-1)$.

Theorem 7.0.2. *Assume that the weight projection map* $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}}) \to \mathrm{Spec}(\mathcal{O}_F[[T_n(\mathbb{Z}_p)]])$ *is étale at the point* $x \in \mathcal{C}(\overline{\mathbb{Q}}_p)$. *Then the measure* $\mu_{\mathcal{C},\phi,\beta_1,\beta_2} \in \mathcal{M}eas(\mathbb{Z}_p^\times, \Lambda_n) \otimes_{\Lambda_n} F_{\mathcal{C}}$ *has no poles at* x. *Let* $\underline{\tau}$ *be the projection of* x *to the weight space* $\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$. *For* $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ *with* $\kappa(-1) = \phi(-1)$. *If* $(\kappa, \underline{\tau})$ *is admissible, i.e. arithmetic with* $t_1 \geq \cdots t_n \geq k \geq n+1$, *then we have*

$$\left(\int_{\mathbb{Z}_p^\times} \kappa\, d\mu_{\mathcal{C},\phi,\beta_1,\beta_2}\right)(x) = \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n (p^{2l}-1)} \cdot \frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}}$$
$$\times \frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})}{\langle v_{\underline{t}}^\vee, v_{\underline{t}}\rangle} \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\mathfrak{c}(\varphi, \beta_1)\mathfrak{c}(eW(\varphi), \beta_2)}{\langle \varphi, \overline{\varphi}\rangle}$$
$$\times E_p(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}) \cdot L^{Np\infty}(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}).$$

Here $\mathfrak{c}(\cdot, \beta_i)$ *stands for the* β_i*-th Fourier coefficient,* $i = 1, 2$.

The nonvanishing of the archimedean zeta integral term $Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})$ is guaranteed by Proposition 4.3.1, and the nonvanishing of $eW(\varphi)$, the ordinary projection of $\varphi \in \mathfrak{s}_x$, follows from Proposition 5.7.2.

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