NEARLY OVERCONVERGENT SIEGEL MODULAR FORMS

ZHENG LIU

Abstract. We introduce a sheaf-theoretic formulation of Shimura's theory of nearly holomorphic Siegel modular forms and differential operators. We use it to define and study nearly overconvergent Siegel modular forms and their p-adic families.

Contents

1. Introduction

Shimura developed his theory of nearly holomorphic forms in his study on the algebraicity of special L-values and Klingen Eisenstein series [Shi76, Shi00]. With the goal of combining this useful tool with Hida and Coleman–Mazur theories for p-adic families of modular forms to study special L-values and Selmer groups by using p-adic congruences and deformations, Urban [Urb14] introduced a sheaf-theoretic formulation of Shimura's theory in the $\mathrm{GL}(2)_{/\mathbb{Q}}$ case. Such a formulation enables him to define and study some basic properties of nearly overconvergent modular forms.

In this article we generalize Urban's work to Siegel modular forms. In the construction of automorphic sheaves over Siegel varieties equipped with integrable connections, we take a different approach from [Urb14] by using a canonical $\mathbf{Q}$-torsor over the Siegel variety and $(\mathfrak{g}, \mathbf{Q})$-modules. Here $\mathfrak{g}$ is the Lie algebra of the algebraic group $\mathbf{G} = \mathrm{GSp}(2n)_{/\mathbb{Z}}$ and $\mathbf{Q}$ is the standard Siegel parabolic subgroup of $\mathbf{G}$. Compared to $\mathbf{G}$-representations, $(\mathfrak{g}, \mathbf{Q})$-modules are more adaptive for p-adic deformations. Combining the ideas and techniques in [AIP15] with our sheaf-theoretic formulation of nearly holomorphic Siegel modular forms and differential operators, we introduce the space of nearly overconvergent Siegel modular forms and their p-adic families.

One of the main motivations for considering differential operators and nearly holomorphic forms and their p-adic theory is for arithmetic applications of various integral representations of L-functions or L-values, the algebraicity results on special L-values and Klingen Eisenstein series by the doubling method [ShE97, Shi00, Har97, Har07], the construction of p-adic L-functions by evaluating Eisenstein series at CM points [Kat78] and by Rankin-Selberg method [Hid88], and the study of p-adic regulators of Heegner cycles by the Waldspurger formula [BDP13], just to name a few. The results in this article are applied in [Liu15] to construct p-adic L-functions for ordinary families on symplectic groups using the doubling method, generalizing [BS00]. The construction of p-adic L-functions for unitary groups by the doubling method has also been carried out in [EW16, EHLS16].

As we know, the algebraicity of an automorphic representation is mainly related with its archimedean component. When utilizing integral representations to study special L-values, differential operators and nearly holomophic forms naturally show up in the analysis of archimedean zeta integrals. Over the field of complex numbers, roughly speaking, cuspidal nearly holomorphic forms are automorphic forms inside cuspidal automorphic representations whose archimedean components are isomorphic to holomorphic discrete series. The holomorphic forms are those whose archimedean components belong to the lowest K_∞-types of the holomorphic discrete series. The Maass–Shimura differential operators correspond to the Lie algebra action on the archimedean components. The theory of nearly holomorphic forms and differential operators aims to introduce nice algebraic or even integral structure to the complex vector space of nearly holomorphic forms and to the action of the Lie algebra, as well as provide explicit formulas for evaluating archimedean zeta integrals. Besides Shimura, the differential operators and nearly holomorphic forms have also been studied in [Har86], [Nap92], [Böc85], [Ibu99] and [PSS15] through different approaches.

In Shiumra's theory of nearly holomorphic Siegel modular forms, there are three main ingredients. Let $\mathfrak{h}_n$ be the genus n Siegel upper half space, $\Gamma \subset \mathrm{Sp}(2n, \mathbb{Z})$ be a congruence subgroup, and (ρ, W_ρ) be an algebraic $\mathrm{GL}(n)$-representation of finite rank. Shimura defined

(1) the space $N^r_\rho(\mathfrak{h}_n, \Gamma)$ of $W_\rho(\mathbb{C})$-valued nearly holomorphic forms on $\mathfrak{h}_n$ of level Γ and (non-holomorphy) degree r, together with its algebraic structure defined by using CM points,
(2) the Maass–Shimura differential operator $D_{\mathfrak{h}_n,\rho} : N^r_\rho(\mathfrak{h}_n, \Gamma) \to N^{r+1}_{\rho\otimes\tau}(\mathfrak{h}_n, \Gamma)$, where τ is the symmetric square of the standard representation of $\mathrm{GL}(n)$,
(3) a holomorphic projection $N^r_\kappa(\mathfrak{h}_n, \Gamma) \to N^0_\kappa(\mathfrak{h}_n, \Gamma)$ for a generic weight κ.

Both the differential operators and the holomorphic projection preserve the algebraic structure in (1), and they play important roles in choosing desirable archimedean sections in arithmetic applications of various integral representations of L-functions and L-values.

This paper consists of two parts. In the first part, we construct the automorphic quasi-coherent sheaf $\mathcal{V}_\rho$ over a smooth toroidal compactification of the Siegel modular variety Y of level Γ defined over $\mathbb{Z}[1/N]$ for some positive integer N. This automorphic sheaf $\mathcal{V}_\rho$ has an increasing filtration $\mathcal{V}^r_\rho$

and we construct a connection

$$\mathcal{V}_\rho^r \longrightarrow \mathcal{V}_\rho^{r+1} \otimes_{\mathcal{O}_X} \Omega_X^1(\log(X-Y)). \tag{1.0.1}$$

Composing this connection with the Kodaira–Spencer isomorphism, we get the differential operator $D_\rho : \mathcal{V}_\rho^r \to \mathcal{V}_{\rho\otimes\tau}^{r+1}$. We show in §2.5 that $\mathcal{V}_\rho^r$ together with D_ρ recovers the first two ingredients in Shimura's theory, and there is the commutative diagram

$$\begin{array}{ccccc} H^0(X_{\mathbb{C}}^\circ, \mathcal{V}_\rho^r) & \xrightarrow{\sim} & N_\rho^r(\mathfrak{h}_n, \Gamma) & \hookrightarrow & C^\infty(\Gamma\backslash \mathbf{G}^\circ(\mathbb{R})) \\ \downarrow D_\rho & & \downarrow D_{\mathfrak{h}_n,\rho} & & \downarrow \mathfrak{q}^+\text{-action} \\ H^0(X_{\mathbb{C}}^\circ, \mathcal{V}_{\rho\otimes\tau}^{r+1}) & \xrightarrow{\sim} & N_{\rho\otimes\tau}^{r+1}(\mathfrak{h}_n, \Gamma) & \hookrightarrow & C^\infty(\Gamma\backslash \mathbf{G}^\circ(\mathbb{R})), \end{array} \tag{1.0.2}$$

where $X_{\mathbb{C}}^\circ$ is a connected component of the base change of X to $\mathbb{C}$, $\mathbf{G}^\circ = \mathrm{Sp}(2n)$, and $\mathfrak{q}^+ = \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix} (\mathrm{Lie}\,\mathbf{Q})_{\mathbb{C}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}^{-1}$.

Automorphic sheaves are defined over X using algebraic $\mathbf{Q}$-representations free of finite rank and the canonical $\mathbf{Q}$-torsor $T_{\mathcal{H}}^\times = \underline{\mathrm{Isom}}_X\left(\mathcal{O}_X^{2n}, \mathcal{H}_{dR}^1(\mathcal{A}/Y)^{\mathrm{can}}\right)$, where $\mathcal{A} \to Y$ is the principally polarized universal abelian scheme, and the isomorphisms are required to respect the Hodge filtration and preserve the symplectic pairing of $\mathcal{H}_{dR}^1(\mathcal{A}/Y)^{\mathrm{can}}$ up to similitude. Given an algebraic $\mathbf{Q}$-representation V, the associated automorphic sheaf is defined as the contracted product $\mathcal{V} = T_{\mathcal{H}}^\times \times^{\mathbf{Q}} V$.

If one wants to consider automorphic sheaves further equipped with integrable connections that induce Hecke equivariant maps on global sections, we show in §2.2 that the right objects to consider are $(\mathfrak{g}, \mathbf{Q})$-modules. It is the $\mathfrak{g}$-module structure combined with the Gauss–Manin connection on $\mathcal{H}_{dR}^1(\mathcal{A}/Y)^{\mathrm{can}}$ that gives rise to the desired connection. Then in order to construct the sheaves of nearly holomorphic Siegel modular forms with differential operators, it remains to select suitable $(\mathfrak{g}, \mathbf{Q})$-modules. In §2.3 we define, for each algebraic $\mathrm{GL}(n)$-representation ρ locally free of finite rank, a $(\mathfrak{g}, \mathbf{Q})$-module V_ρ. As a $\mathbf{Q}$-module V_ρ has an increasing filtration $V_\rho^r, r \geq 0$ such that $\mathfrak{g} \cdot V_\rho^r \subset V_\rho^{r+1}$. We define the sheaf of nearly holomorphic forms of weight ρ and (non-holomorphy) degree r as $\mathcal{V}_\rho^r = T_{\mathcal{H}}^\times \times^{\mathbf{Q}} V_\rho^r$. The general construction in §2.2 equips $\mathcal{V}_\rho$ with the connection (1.0.1). The construction of holomorphic projections is postponed to §3.7 where it is done in the more general setting of nearly overconvergent families.

In the second part, combining the ideas and techniques in [AIP15] with our construction in the first part, we define and study some basic properties of the space of nearly overconvergent forms and p-adic families of nearly overconvergent forms. When replacing dominant algebraic weights by general p-adic anaylitic weights, it is convenient to construct the corresponding representations of the Lie algebra, which can be viewed as a p-adic deformation of the Lie algebra representations attached to dominant algebraic weights. However, these Lie algebra representations do not integrate to representations of the algebraic group, but only integrate to certain p-adic analytic representations of some rigid analytic subgroup of the rigid analytification of the algebraic group. In order to construct sheaves of p-adic automorphic forms with p-adic analytic weights, one natural approach is to modify the torsor of the algebraic group to a p-adic analytic torsor of its rigid analytic subgroup, and form the contracted product of the p-adic analytic torsor with the representation of the rigid analytic subgroup.

In [AIP15], for $v, w \geq 0$ within a certain range, over the strict neighborhood $\mathcal{X}_{\mathrm{Iw}}(v)$ of the ordinary locus of the compactifed Iwahori-level Siegel variety X_{Iw}, an Iwahori-like space $\mathcal{T}_{\mathcal{F},w}^\times(v)$ inside the $\mathrm{GL}(n)_{\mathrm{an}}$-torsor $T_{\omega,\mathrm{an}}^\times = \underline{\mathrm{Isom}}_X\left(\mathcal{O}_X^n, \omega(\mathcal{A}/Y)^{\mathrm{can}}\right)_{\mathrm{an}}$ is constructed by using canonical subgroups. Here the subscript"an" means the rigid analytification. This $\mathcal{T}_{\mathcal{F},w}^\times(v)$ can be viewed as a torsor of a rigid analytic subgroup $\mathcal{I}_w$ inside $\mathrm{GL}(n)_{\mathrm{rig}}$, the rigid analytic fibre of the completion of $\mathrm{GL}(n)$ along its special fibre. For a w-analytic weight $\kappa \in \mathrm{Hom}_{\mathrm{cont}}\left((\mathbb{Z}_p^\times)^n, \mathbb{C}_p^\times\right)$, there corresponds a natural representation $W_{\kappa,w}$ of Lie(GL(n)) which integrates to a representation of $\mathcal{I}_w$. The Banach sheaf $\omega^{\dagger}_{\kappa,w}$ of overconvergent modular forms of the w-analytic weight κ over $\mathcal{X}_{\mathrm{Iw}}(v)$ is obtained as the contracted product of $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ and $W_{\kappa,w}$.

Taking ρ to be the trivial representation and $r=1$, the construction in §2 gives an automorphic coherent sheaf $\mathcal{J}=\mathcal{V}^1_{\mathrm{triv}}$. The quick way to define the Banach sheaf of w-analytic weight κ degree r nearly overconvergent forms is to set $\mathcal{V}^{\dagger,r}_{\kappa,w} := \omega^{\dagger}_{\kappa,w}\otimes\mathrm{Sym}^r\,\mathcal{J}$ (this is similar to the way of defining $\mathcal{H}^r_k$, $\mathcal{H}^r_{\mathfrak{U}}$ in [Urb14]). For the convenience of defining differential operators and holomorphic projections as in §3.6, 3.7, we need a contracted product interpretation for $\mathcal{V}^{\dagger,r}_{\kappa,w}$. Associated to the p-adic analytic weight κ, generalizing the previous V_ρ, there is a natural $\mathfrak{g}$-module $V_{\kappa,w}$ which integrates to a $(\mathfrak{g},\mathcal{Q}_w)$-module, where $\mathcal{Q}_w\subset\mathbf{Q}_{\mathrm{an}}$ is the rigid analytic group defined as the preimage of $\mathcal{I}_w$ of the projection $\mathbf{Q}_{\mathrm{an}}\to\mathrm{GL}(n)_{\mathrm{an}}$. We define the $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ as the subspace of $T^{\times}_{\mathcal{H},\mathrm{an}}$ whose image under the projection $T^{\times}_{\mathcal{H},\mathrm{an}}\to T^{\times}_{\omega,\mathrm{an}}$ lies inside $\mathcal{T}^{\times}_{\omega,w}(v)$. Then $\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ together with $V_{\kappa,w}$ gives the desired contracted product interpretation for the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$.

Now let $\mathcal{U}$ be an affinoid subdomain of the weight space whose $\mathbb{C}_p$-points are all w-analytic. The construction above works for the universal weight as well and produces the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa^{un},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U}$. In §3.5 we show that the $\mathcal{A}(\mathcal{U})$-Banach module $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} := H^0\left(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U},\mathcal{V}^{\dagger,r}_{\kappa^{un},w}(-C)\right)$ is projective. §3.9 is devoted to defining the $\mathbb{U}_p$-operators and showing the compactness of the operator $U_p=\mathrm{res}\circ U_{p,n}\circ\cdots\circ U_{p,1}$ acting on $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$. Then the Coleman–Riesz–Serre spectral theory is applied to give the slope decomposition of $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} := \bigcup_{r\geq 0} N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ in §3.11.

The p-adic theory of differential operators and nearly holomorphic forms has also been considered in [Eis12,EFMV] (unitary case) and [Ich15] (simplectic case). They define nearly holomorphic forms as global sections of $\left(\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}\right)^{\otimes m}$ for some positive integer m, and the differential operators are then the connections induced from the Gauss–Manin connection on $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}$. In order to consider p-adic deformations, their method relies on unit root splitting of $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}$ over the ordinary locus and the q-expansion or Serre–Tate expansion principle, and does not extend to nearly overconvergent forms. We believe that our method here works also for Shimura varieties for unitary groups. In [HX14], a construction of the Gauss–Manin connections for nearly overconvergent forms is given in the $\mathrm{GL}(2)_{/\mathbb{Q}}$ case, where they consider the action of GL(1) (the Levi subgroup of the Siegel parabolic of GL(2)) instead of that of Lie(GL(2)). Note that besides constructing differential operators acting on nearly overconvergent forms of general p-adic analytic weight, there is another problem of taking the differential operator to a p-adic analytic power. This is easy for p-adic forms over the ordinary locus by using the q-expansion principle, but for nearly overconvergent forms there seems no obvious approach. Recently this problem has been addressed for families of nearly overconvergent modular forms in [AI17]. It is expected that some ideas there extend to our case of nearly overconvergent Siegel modular forms.

Acknowledgement. I am very grateful to my advisor Eric Urban for his guidance and insightful suggestions. I would also like to thank Michael Harris, Vincent Pilloni for their helpful advice, and Adrian Iovita, Johan de Jong, Ellen Eischen, Kai-Wen Lan, Liang Xiao for useful conversations.

Notation. Let $\mathbf{G}$ be the rank n symplectic similitude group

$$\mathrm{GSp}(2n)_{/\mathbb{Z}}=\left\{g\in\mathrm{GL}(2n)_{/\mathbb{Z}}: {}^{\mathrm{t}}g\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}g=\nu(g)\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\right\}$$

with the multiplier character $\nu:\mathbf{G}\to\mathbb{G}_m$. Denote by $\mathbf{Q}$ the standard Siegel parabolic subgroup of $\mathbf{G}$ consisting of matrices whose lower left $n\times n$ block is 0 and $\mathbf{T}$ the maximal torus consisting of diagonal matrices. Write $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{U}$ with $\mathbf{M}$ and $\mathbf{U}$ as its Levi subgroup and unipotent radical. Fix the embedding $\mathrm{GL}(n) \hookrightarrow \mathbf{M}$ sending $a \in \mathrm{GL}(n)$ to $\begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix}$. Let $\mathbf{G}^\circ = \mathrm{Sp}(2n)_{/\mathbb{Z}}$ be the kernel of the multiplier character ν with maximal torus $\mathbf{T}^\circ \cong \mathbb{G}_m^n$ and standard Siegel parabolic $\mathbf{Q}^\circ = \mathbf{M}^\circ \ltimes \mathbf{U}$. The embedding $\mathrm{GL}(n) \hookrightarrow \mathbf{M}$ gives an isomorphism of $\mathrm{GL}(n)$ onto $\mathbf{M}^\circ$. The maximal torus $\mathbf{T}^\circ$ of $\mathrm{Sp}(2n)$ can also be regarded as a maximal torus of $\mathbf{M}^\circ \cong \mathrm{GL}(n)$. We use $\mathbf{B}$ to denote the Borel subgroup of $\mathbf{M}^\circ$ corresponding to the subgroup of upper triangular matrices in $\mathrm{GL}(n)$ and $\mathbf{N}$ to denote the unipotent radical of $\mathbf{B}$. For an algebra E, let $\mathrm{Rep}_E \mathbf{Q}$ (resp. $\mathrm{Rep}_{E,f} \mathrm{GL}(n)$) stand for the category of algebraic representations of the group $\mathbf{Q}$ (resp. $\mathrm{GL}(n)$) base changed to E on locally free E-modules (resp. locally free E-modules of finite rank). The projection $\mathbf{Q} \to \mathrm{GL}(n)$ mapping $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbf{Q}$ to $a \in \mathrm{GL}(n)$ defines a functor $\mathrm{Rep}_{E,f} \mathrm{GL}(n) \to \mathrm{Rep}_E \mathbf{Q}$ and we regard every object in $\mathrm{Rep}_{E,f} \mathrm{GL}(n)$ also as a $\mathbf{Q}$-representation. The congruence subgroup $\{\gamma \in \mathbf{G}^\circ(\mathbb{Z}) : \gamma \equiv I_{2n} \mod N\}$ of $\mathbf{G}^\circ(\mathbb{Z})$ is denoted by $\Gamma(N)$.

2. Nearly holomorphic forms

2.1. **Automorphic sheaves over Siegel varieties.** Let $Y = Y_{\mathbf{G},\Gamma(N)}$ be the Siegel variety parametrizing principally polarized abelian schemes of relative dimension n with principal level N structure with $N \geq 3$ defined over $\mathbb{Z}[1/N]$. Over it there is the universal abelian scheme $\mathbf{p} : \mathcal{A} \to Y$. Take a smooth toroidal compactification X of Y with boundary $C = X - Y$. Then $\mathbf{p} : \mathcal{A} \to Y$ extends to a semi-abelian scheme $\mathbf{p} : \mathcal{G} \to X$. Let $\omega(\mathcal{G}/X)$ be the pullback of $\Omega^1_{\mathcal{G}/X}$ along the zero section of $\mathbf{p}$. According to [Lan12, Proposition 6.9], the locally free sheaf $\mathcal{H}^1_{dR}(\mathcal{A}/Y) = R^1\mathbf{p}_*(\Omega^\bullet_{\mathcal{A}/Y})$ has a canonical extension $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}} \cong \mathcal{H}^1_{\log\text{-dR}}(\mathcal{G}/X)$ which is a locally free subsheaf of $(Y \to X)_*\mathcal{H}^1_{dR}(\mathcal{A}/Y)$. This canonical extension $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}$ is endowed with a symplectic pairing under which $\omega(\mathcal{G}/X)$ is maximally isotropic. The Hodge filtration of $\mathcal{H}^1_{dR}(\mathcal{A}/Y)$ also extends to

$$0 \longrightarrow \omega(\mathcal{G}/X) \longrightarrow \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}} \longrightarrow \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X) \longrightarrow 0$$

where ${}^{\mathrm{t}}\mathcal{G}/X$ is the dual semi-abelian scheme of $\mathcal{G}/X$.

There is a standard way to construct, from a representation in $\mathrm{Rep}_{\mathbb{Z}} \mathbf{Q}$, a quasi-coherent sheaf over X whose global sections are equipped with Hecke actions. The free sheaf $\mathcal{O}_X^{2n}$ can be equipped with a two-step filtration with the first n copies as the subsheaf, and a symplectic pairing using the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Define the right $\mathbf{Q}$-torsor over X

$$T^\times_{\mathcal{H}} = \underline{\mathrm{Isom}}_X \left(\mathcal{O}_X^{2n}, \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}\right)$$

to be the isomorphisms respecting the filtrations and the symplectic pairings up to similitude. The right $\mathbf{Q}$-action is given as

$$(b \cdot \phi)(v) = (\phi \circ b)(v) = \phi(bv)$$

for any open subscheme $U = \mathrm{Spec}(R) \subset X$, $\phi \in T^\times_{\mathcal{H}}(U)$, $v \in R^{2n}$ and $b \in \mathbf{Q}(R)$.

With this right $\mathbf{Q}$-torsor, by forming contracted product, one can define the functor

$$\mathcal{E} : \mathrm{Rep}_{\mathbb{Z}} \mathbf{Q} \longrightarrow \mathrm{QCoh}(X)$$
$$V \longmapsto T^\times_{\mathcal{H}} \times^{\mathbf{Q}} V$$

from the category of algebraic representations of $\mathbf{Q}$ on locally free $\mathbb{Z}$-modules to that of quasi-coherent sheaves over X. Let us give a more detailed description of $\mathcal{E}(V)$ in local affine charts. Let $U = \mathrm{Spec}(R)$ be an affine open subscheme of X such that $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}(U)$ is free over R. We identify elements in $T^\times_{\mathcal{H}}(U)$ with ordered basis $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ of $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}(U)$, which gives rise to isomorphisms between R^{2n} and $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}(U)$ preserving both the Hodge filtration and symplectic paring up to similitude. Then $\mathcal{E}(V)(U)$ is the set of maps $v : T^\times_{\mathcal{H}}(U) \to V \otimes R$ such that $v(\alpha g) = g^{-1} \cdot v(\alpha)$ for all $g \in \mathbf{Q}(R)$ and $\alpha \in T^\times_{\mathcal{H}}(U)$.

Moreover for all $V \in \mathrm{Rep}_{\mathbb{Z}}\,\mathbf{Q}$ the global sections of the associated quasi-coherent sheaf $\mathcal{E}(V)$ come with a Hecke action constructed via algebraic correspondence (cf. [FC90, §VII.3]). Such an $\mathcal{E}(V)$ together with the Hecke action on its global sections is often called an automorphic sheaf. Morphisms between algebraic $\mathbf{Q}$-representations induce Hecke equivariant morphisms between global sections of the corresponding quasi-coherent sheaves. The functor $\mathcal{E}$ is exact and faithful [Lan12, Definition 6.13]. Certainly this functor is not fully faithful (see Example 2.4.5). Let V_{st} be the standard representation of $\mathbf{G}$ restricted to $\mathbf{Q}$ and W_{st} be the standard representation of $\mathrm{GL}(n)$ regarded as a $\mathbf{Q}$-representation. Then immediately from the definition we see $\mathcal{E}(V_{\mathrm{st}}) \cong \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}$ and $\mathcal{E}(W_{\mathrm{st}}) \cong \omega(\mathcal{G}/X)$.

The multiplier character $\nu : \mathbf{G} \to \mathbb{G}_m$ can be seen as an algebraic representation of $\mathbf{Q}$ and we denote its corresponding invertible sheaf over X by $\mathcal{E}(\nu)$. As an invertible sheaf $\mathcal{E}(\nu)$ is isomorphic to the trivial structure sheaf $\mathcal{O}_X$. However the Hecke action differs by a Tate twist. For $V \in \mathrm{Rep}_{\mathbb{Z}}\,\mathbf{Q}$ we define $\mathcal{E}(V)(i)$ to be $\mathcal{E}(V \otimes \nu^i) = \mathcal{E}(V) \otimes \mathcal{E}(\nu)^i$.

Remark 2.1.1. The Hecke actions are only defined on global sections not on the quasi-coherent sheaves. However in the following for simplicity we say a quasi-coherent sheaf with Hecke actions to mean that Hecke operators act on its global sections, and a Hecke equivariant morphism between quasi-coherent sheaves to mean that the induced map on global sections is Hecke equivariant. Also by an isomorphism between two automorphic sheaves we mean a Hecke equivariant one unless otherwise stated.

2.2. **$(\mathfrak{g}, \mathbf{Q})$-modules and Gauss–Manin connection.** Let $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}$, $\mathfrak{q} = \mathrm{Lie}\,\mathbf{Q}$ be the Lie algebras of $\mathbf{G}$ and its Siegel parobolic $\mathbf{Q}$.

Definition 2.2.1. Let E be an algebra. A $(\mathfrak{g}, \mathbf{Q})$-module V over E is an algebraic representation of $\mathbf{Q}$ and $\mathfrak{g}$ base changed to E on locally free E-modules, such that the action of $\mathfrak{q} \subset \mathfrak{g}$ on V is the one induced from that of $\mathbf{Q}$ and for any $g \in \mathbf{Q}$, $X \in \mathfrak{g}$ and $v \in V$,

$$g \cdot X \cdot g^{-1} \cdot v = \big(\mathrm{Ad}(g)X\big) \cdot v.$$

We denote the category of $(\mathfrak{g}, \mathbf{Q})$-modules over E by $\mathrm{Rep}_E(\mathfrak{g}, \mathbf{Q})$.

It is mentioned on [FC90, p.223] that $\mathbf{G}(\mathbb{C})$-equivariant quasi-coherent $\mathcal{D}$-modules over the compact dual $\mathbb{D}^\vee = \mathbf{G}(\mathbb{C})/\,\mathbf{Q}(\mathbb{C})$ correspond to $(\mathfrak{g}, \mathbf{Q})$-modules. We show below that for an object $V \in \mathrm{Rep}_{\mathbb{Z}}(\mathfrak{g}, \mathbf{Q})$, we can equip with its associated automorphic sheaf $\mathcal{E}(V)$ an integrable connection using the $\mathfrak{g}$-module structure on V. If a $(\mathfrak{g}, \mathbf{Q})$-module is of finite rank, then it comes from an algebraic representation of $\mathbf{G}$. However, the $(\mathfrak{g}, \mathbf{Q})$-module we will define in the next section is not of finite rank, but contains sub-$(\mathfrak{g}, \mathbf{Q})$-modules of finite rank. Using those $(\mathfrak{g}, \mathbf{Q})$-modules makes the theory parallel Shimura's theory very well and gives nice formulas for the differential operators. More importantly, the $(\mathfrak{g}, \mathbf{Q})$-module structure compared to $\mathbf{G}$-representation is more convenient for doing p-adic theory.

For the locally free sheaf $\mathcal{H}^1_{dR}(\mathcal{A}/Y) = R^1\mathbf{p}_*\big(\Omega^\bullet_{\mathcal{A}/Y}\big)$ over Y, a canonical integrable connection called the Gauss–Manin connection can be constructed [KO68]. We record the following result on the extension of the Gauss–Manin connection.

Theorem 2.2.2. *([Lan12, Proposition 6.9]) The Gauss–Manin connection $\nabla : \mathcal{H}^1_{dR}(\mathcal{A}/Y) \longrightarrow \mathcal{H}^1_{dR}(\mathcal{A}/Y) \otimes \Omega^1_Y$ extends to an integrable connection with log poles along the boundary*

$$\nabla : \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}} \otimes \Omega^1_X(\log C),$$

satisfying Griffith transversality and compatible with the symplectic pairing on $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}$.

Let U, α be as in our description of the contracted product defining $\mathcal{E}(V)$. Given $D \in T_X(U) = \mathrm{Der}_{\mathbb{Z}[1/N]}(R, R)$, a section of the tangent bundle of X over U, by Theorem 2.2.2 there exists $X(D, \alpha) \in \mathfrak{g}(R)$ (in fact $\mathfrak{g}(\mathrm{Frac}(R))$ with logarithm poles along the boundary if U intersects with the boundary) such that

$$\nabla(D)(\alpha) = \alpha \cdot X(D, \alpha). \tag{2.2.1}$$

For $v \in \mathcal{E}(V)(U)$ we define the operator $\nabla_{\mathcal{E}(V)}(D)$ acting on it as

$$\big(\nabla_{\mathcal{E}(V)}(D)(v)\big)(\alpha) := Dv(\alpha) + X(D, \alpha) \cdot v(\alpha). \tag{2.2.2}$$

Here D acts on $v(\alpha) \in V \otimes R$ through the action of $\mathrm{Der}_{\mathbb{Z}[1/N]}(R, R)$ on R, i.e. by coefficients. The action of $X(D, \alpha)$ on $v(\alpha)$ is the action of the Lie algebra $\mathfrak{g}$ on V.

Proposition 2.2.3. *The above defined $\nabla_{\mathcal{E}(V)}(D)(v)$ belongs to $\mathcal{E}(V)(U)$ and the formula (2.2.2) on local sections patches together to an integrable connection with log poles along the boundary*

$$\nabla_{\mathcal{E}(V)} : \mathcal{E}(V) \longrightarrow \mathcal{E}(V) \otimes \Omega^1_X(\log C).$$

Proof. What we need to show is that for any $g \in \mathbf{Q}(R)$

$$\left(\nabla_{\mathcal{E}(V)}(D)(v)\right)(\alpha \cdot g) = g^{-1} \cdot \left(\nabla_{\mathcal{E}(V)}(D)(v)\right)(\alpha). \tag{2.2.3}$$

The Gauss–Manin connection ∇ satisfies that

$$\begin{aligned}
\nabla(D)(\alpha \cdot g) &= \nabla(D)(\alpha) \cdot g + \alpha \cdot Dg \\
&= (\alpha \cdot g) \cdot (g^{-1} X(D, \alpha) g + g^{-1} Dg) \\
&= (\alpha \cdot g) \cdot (\mathrm{Ad}(g^{-1}) X(D, \alpha) + g^{-1} Dg)
\end{aligned}$$

i.e.

$$X(D, \alpha \cdot g) = \mathrm{Ad}(g^{-1}) X(D, \alpha) + g^{-1} Dg.$$

We compute the left hand side of (2.2.3) by definition,

$$\begin{aligned}
\mathrm{LHS} &= D \cdot v(\alpha \cdot g) + X(D, \alpha \cdot g) \cdot v(\alpha \cdot g) \\
&= D\big(g^{-1} \cdot v(\alpha)\big) + \big(Ad(g^{-1}) X(D, \alpha) + g^{-1} Dg\big) \cdot v(\alpha \cdot g) \\
&= \big((Dg^{-1})g\big) \cdot \big(g^{-1} \cdot v(\alpha)\big) + g^{-1} \cdot \big(Dv(\alpha)\big) + \big(\mathrm{Ad}(g^{-1}) X(D, \alpha) + g^{-1} Dg\big) \cdot \big(g^{-1} \cdot v(\alpha)\big) \\
&= -\big(g^{-1} Dg\big) \cdot \big(g^{-1} \cdot v(\alpha)\big) + g^{-1} \cdot \big(Dv(\alpha)\big) + \big(g^{-1} \cdot X(D, \alpha) \cdot g\big) \cdot \big(g^{-1} \cdot v(\alpha)\big) \\
&\quad + \big(g^{-1} Dg\big) \cdot \big(g^{-1} \cdot v(\alpha)\big) \\
&= g^{-1} \cdot \big(Dv(\alpha) + X(D, \alpha) \cdot v(\alpha)\big),
\end{aligned}$$

which equals to the right hand side. The compatibility of the action of $\mathfrak{g}$ and $\mathbf{Q}$ is used for the fourth equality. The integrability of the Gauss–Manin connection implies that for $D_1, D_2 \in T_X(U)$

$$X([D_1, D_2], \alpha) = D_1 X(D_2, \alpha) - D_2 X(D_1, \alpha) + X(D_1, \alpha) X(D_2, \alpha) - X(D_2, \alpha) X(D_1, \alpha).$$

Also,

$$\begin{aligned}
\nabla_{\mathcal{E}(V)}(D_1) \nabla_{\mathcal{E}(V)}(D_2) &= D_1 D_2 v(\alpha) + (D_1 X(D_2, \alpha)) \cdot v(\alpha) + X(D_2, \alpha) \cdot D_1 v(\alpha) \\
&\quad + X(D_1, \alpha) \cdot D_2 v(\alpha) + X(D_1, \alpha) \cdot X(D_2, \alpha) \cdot v(\alpha), \\
\nabla_{\mathcal{E}(V)}(D_2) \nabla_{\mathcal{E}(V)}(D_1) &= D_2 D_1 v(\alpha) + (D_2 X(D_1, \alpha)) \cdot v(\alpha) + X(D_1, \alpha) \cdot D_2 v(\alpha) \\
&\quad + X(D_2, \alpha) \cdot D_1 v(\alpha) + X(D_2, \alpha) \cdot X(D_1, \alpha) \cdot v(\alpha).
\end{aligned}$$

Thus

$$\begin{aligned}&\left(\nabla_{\mathcal{E}(V)}(D_1)\nabla_{\mathcal{E}(V)}(D_2) - \nabla_{\mathcal{E}(V)}(D_2)\nabla_{\mathcal{E}(V)}(D_1)\right)(\alpha)\\ =&[D_1, D_2]v(\alpha) + (D_1X(D_2,\alpha) - D_2X(D_1,\alpha) + [X(D_1,\alpha), X(D_2,\alpha)])\cdot v(\alpha)\\ =&[D_1,D_2]v(\alpha) + X([D_1,D_2],\alpha)\cdot v(\alpha) = \nabla_{\mathcal{E}(V)}([D_1,D_2]),\end{aligned}$$

i.e. the connection $\nabla_{\mathcal{E}(V)}$ is integrable. □

Remark 2.2.4. If the $(\mathfrak{g},\mathbf{Q})$-module V can be constructed from the standard representation V_{st} of $\mathbf{G}$ by taking tensor products, symmetric powers and wedge products, then applying the same operations to $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}} = \mathcal{E}(V_{\mathrm{st}})$ we get the locally free sheaf $\mathcal{E}(V)$ attached to V, so the Gauss–Manin connection on $\mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}$ immediately induces a connection on $\mathcal{E}(V)$. This is the approach adopted in by E. Eischen in [Eis12]. The point of our construction here is that V does not need to be a representation of $\mathbf{G}$. The construction works for all $(\mathfrak{g},\mathbf{Q})$-modules and therefore can be easily adapted to deal with p-adic analytic weights and the universal weight (see §3.2, 3.4, 3.6). There is another construction for the connection $\nabla_{\mathcal{E}(V)}$ in [Til11, §3.2] using Grothendieck's sheaves of differentials when V is a finite dimensional $\mathbf{G}$-representation. That approach may be modified to deal with the non-algebraic weight except that there might be some issue with taking duality when infinite dimensional representations are involved.

2.3. **The $(\mathfrak{g},\mathbf{Q})$-module V_κ.** Now in order to use the constructions in §2.1 and §2.2 to formulate Shimura's theory of nearly holomorphic forms in a sheaf-theoretic context, what we need is to define a suitable $(\mathfrak{g},\mathbf{Q})$-module for a given algebraic representation of $\mathrm{GL}(n)$.

Let $(\rho, W_\rho)\in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ be an algebraic representation of $\mathrm{GL}(n)$ locally free of finite rank. We define the $(\mathfrak{g},\mathbf{Q})$-module V_ρ as follows. For any algebra R, set

$$V_\rho(R) := W_\rho(R)\otimes_R R[\underline{Y}] = W_\rho(R)\otimes_R R[Y_{ij}]_{1\le i\le j\le n}$$

where $\underline{Y} = (Y_{ij})_{1\le i,j\le n}$ is the symmetric $n\times n$ matrix with the indeterminate $Y_{ij}=Y_{ji}$ in the (i,j) entry. Elements in $V_\rho(R)$ can be regarded as polynomials in the $\frac{n(n+1)}{2}$ variables Y_{ij} with coefficients in $W_\rho(R)$. Define the $\mathbf{Q}$-action on V_ρ by

$$(g\cdot P)(\underline{Y}) = a\cdot P(a^{-1}b + a^{-1}\underline{Y}d) \tag{2.3.1}$$

for $g = \begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in \mathbf{Q}(R)$ and $P(\underline{Y})\in V_\rho(R)$. In order to describe the $\mathfrak{g}$-action on V_ρ, we first pick the following basis of $\mathfrak{g}$

$$\begin{aligned}
&\eta_0 = -\sum_{1\le i\le n} E_{i+n,i+n}, && \eta_{ij} = E_{ij} - E_{j+n,i+n}, && 1\le i,j\le n,\\
&\mu^+_{ii} = E_{i,i+n}, && \mu^-_{ii} = E_{i+n,i}, && 1\le i\le n,\\
&\mu^+_{ij} = E_{i,j+n} + E_{j,i+n}, && \mu^-_{ij} = E_{i+n,j} + E_{j+n,i}, && 1\le i<j\le n,
\end{aligned}$$

where E_{ij} is the $2n\times 2n$ matrix with 1 in the (i,j) entry and 0 elsewhere. Let $\mathfrak{q}$, $\mathfrak{m}$, $\mathfrak{u}$ and $\mathfrak{u}^-$ be the Lie algebras of $\mathbf{Q}$, $\mathbf{M}$, $\mathbf{U}$ and the opposite unipotent $\mathbf{U}^-$. For the $\mathfrak{q}$-action on V_ρ we simply take the one induced from the $\mathbf{Q}$-action defined above. We make $\mathfrak{u}^-$ act by the formulas

(2.3.2)

$$\begin{aligned}
(\mu^-_{ij}\cdot P)(\underline{Y}) &= \sum_{1\le k\le n}(Y_{ki}\varepsilon_{kj} + Y_{kj}\varepsilon_{ki})\cdot P(\underline{Y}) - \sum_{1\le k\le l\le n}(Y_{ki}Y_{jl}+Y_{kj}Y_{il})\frac{\partial}{\partial Y_{kl}}P(\underline{Y}), \quad i\ne j,\\
(\mu^-_{ii}\cdot P)(\underline{Y}) &= \sum_{1\le k\le n}\varepsilon_{ki}\cdot P(\underline{Y}) - \sum_{1\le k\le l\le n} Y_{ki}Y_{il}\frac{\partial}{\partial Y_{kl}}P(\underline{Y}).
\end{aligned}$$

where $\varepsilon_{ij} \in \mathfrak{gl}(n)$ is the $n \times n$ matrix with 1 in the (i,j) entry and 0 elsewhere, and it acts via the $\mathfrak{gl}(n)$-action on the coefficient of $P(\underline{Y})$. It remains to show the compatibility of such defined actions of $\mathbf{Q}$ and $\mathfrak{u}^-$. This can be done by direct computation using the formulas. There is also a more conceptual proof. To describe it we construct a representation of the group

$$I_{\mathbf{G}}(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}(\mathbb{Z}_p) \middle| \ c \equiv 0 \mod p \right\}.$$

Let $\mathbf{Q}^-(\mathbb{Z}_p)$ be the subgroup of $I_{\mathbf{G}}(\mathbb{Z}_p)$ whose elements have 0 as the right upper $n \times n$ corner. we make it act on $W_\rho(\mathbb{Q}_p)$ through its Levi part. Equip $W_\rho(\mathbb{Q}_p)$ with a p-adic norm by choosing a basis of $W_\rho(\mathbb{Q}_p)$, and since it is finite dimensional all norms defined in this way are equivalent. We consider the p-adic analytic induction $\mathrm{Ind}_{\mathbf{Q}^-(\mathbb{Z}_p)}^{I_{\mathbf{G}}(\mathbb{Z}_p)} W_\rho(\mathbb{Q}_p)$. Thanks to the Iwahori decomposition we know

$$\mathrm{Ind}_{\mathbf{Q}^-(\mathbb{Z}_p)}^{I_{\mathbf{G}}(\mathbb{Z}_p)} W_\rho(\mathbb{Q}_p) = W_\rho(\mathbb{Q}_p) \left\langle Y_{ij} \right\rangle_{1 \le i < j \le n} = W_\rho(\mathbb{Q}_p) \left\langle \underline{Y} \right\rangle,$$

with $g \in I_{\mathbf{G}}(\mathbb{Z}_p)$ acting on $P(\underline{Y}) \in W_\rho(\mathbb{Q}_p) \left\langle \underline{Y} \right\rangle$ by

$$(g \cdot P)(\underline{Y}) = (a + \underline{Y}c) \cdot P\left((a + \underline{Y}c)^{-1}(b + \underline{Y}d)\right). \tag{2.3.3}$$

Here $W_\rho(\mathbb{Q}_p) \left\langle \underline{Y} \right\rangle$ is the space of strictly convergent power series in $\underline{Y}$ (i.e. convergent on the closed unit ball). Then the formulas (2.3.1) and (2.3.2) can be deduced from (2.3.3), and the compatibility of the actions of $\mathbf{Q}$ and $\mathfrak{u}^-$ on V_ρ follows.

Remark 2.3.1. One can check the formulas (2.3.1) (2.3.2) actually agree with the formulas (2.11)(2.12) given in [JV79], so as $\mathfrak{g}(\mathbb{C})$-modules, the $V_\rho(\mathbb{C})$ defined here should agree with the $\mathcal{O}^f\left(\mathbf{G}^\circ(\mathbb{R}), K_{\mathbf{G}^\circ(\mathbb{R})}, W_\rho(\mathbb{C})\right)$ defined there, where $K_{\mathbf{G}^\circ(\mathbb{R})} \cong U(n, \mathbb{R})$ is the maximal compact subgroup of $\mathbf{G}^\circ(\mathbb{R})$.

As a $\mathbf{Q}$-representation, V_ρ comes with an increasing filtration

$$\mathrm{Fil}^r V_\rho = V_\rho^r = W_\rho[\underline{Y}]_{\le r}, \tag{2.3.4}$$

where the subscript $\le r$ means polynomials in $\underline{Y}$ of total degree less or equal to r. $\mathrm{Fil}^r V_\rho$ can also be characterized as the sum of generalized η_0-eigenspaces with eigenvalues $\ge -r$ [FC90, p.230]. The eigenvalues of η_0 are also called F-weights there. Regarding the $\mathrm{GL}(n)$-representation W_ρ as a $\mathbf{Q}$-representation we have $V_\rho^0 = W_\rho$. It follows from the definition formulas that

$$\mathfrak{g} \cdot V_\rho^r \subset V_\rho^{r+1}. \tag{2.3.5}$$

Let V_{triv} be the $(\mathfrak{g}, \mathbf{Q})$-module constructed as above by taking ρ to be the trivial representation. Denote by J the $\mathbf{Q}$-representation V_{triv}^1. We note here the following useful isomorphism of $\mathbf{Q}$-representations

$$V_\rho^r \cong V_\rho^0 \otimes \mathrm{Sym}^r J = W_\rho \otimes \mathrm{Sym}^r J. \tag{2.3.6}$$

For a dominant weight $\kappa = (k_1, \ldots, k_2) \in X(\mathbf{T}^\circ)^+$ of $\mathrm{GL}(n)$ with respect to $\mathbf{B}$. Set $\kappa' = (-k_n, \ldots, -k_1)$. We define W_κ to be the algebraic $\mathrm{GL}(n)$-representation

$$\left\{ f : \mathrm{GL}(n) \to \mathbb{A}^1 \ \middle| \ \begin{array}{l} \text{morphism of schemes satisfying } f(gb) = \kappa'(b) f(g) \\ \text{for all } g \in \mathrm{GL}(n) \text{ and } b \in \mathbf{B} \end{array} \right\} \tag{2.3.7}$$

with $\mathrm{GL}(n)$ acting by left inverse translation. Putting $\rho = \kappa$ we get the $(\mathfrak{g}, \mathbf{Q})$-module V_κ and $\mathbf{Q}$-representations V_κ^r, $r \ge 0$. Denote by τ the symmetric square of the standard representation of $\mathrm{GL}(n)$. Let $\tau^\vee$ be dual representation of τ. In the following most $\mathrm{GL}(n)$-representations we consider are tensor products of some κ with symmetric powers of τ and $\tau^\vee$.

Remark 2.3.2. We can twist V_ρ by the i-th power of of the multiplier character ν and denote the resulting $(\mathfrak{g}, \mathbf{Q})$-module by $V_\rho(i)$. Such a twist will change the F-weights by $-i$ and corresponds to a Tate twist [FC90, p.222].

2.4. **The sheaf $\mathcal{V}_\kappa^r$ of nearly holomorphic forms.** Let κ be a dominant weight of $\mathrm{GL}(n)$. With preparations in previous sections we give the following definitions.

Definition 2.4.1. The locally free sheaf over X of weight κ, (non-holomorphy) degree r nearly holomorphic forms is defined to be $\mathcal{V}_\kappa^r = \mathcal{E}(V_\kappa^r)$.

When $r = 0$, we also use ω_κ to denote $\mathcal{V}_\kappa^0$ which is the sheaf of weight κ holomorphic forms. More generally for $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ we define the locally free sheaves $\mathcal{V}_\rho = \mathcal{E}(V_\rho)$, $\mathcal{V}_\rho^r = \mathcal{E}(V_\rho^r)$ and denote $\mathcal{V}_\rho^0$ by ω_ρ. The nearly holomorphic forms are defined to be global sections of the sheaf $\mathcal{V}_\kappa^r$.

Definition 2.4.2. Let R be a $\mathbb{Z}[1/N]$-algebra. The space of nearly holomorphic forms (resp. cuspidal nearly holomorphic forms) over R of weight κ, principal level N and (non-holomorphy) degree r is defined to be $N_\kappa^r(\Gamma(N), R) = H^0\left(X_{/R}, \mathcal{V}_\kappa^r\right)$ (resp. $N_{\kappa,\mathrm{cusp}}^r(\Gamma(N), R) = H^0\left(X_{/R}, \mathcal{V}_\kappa^r(-C)\right)$).

There is the moduli interpretation à la Katz for nearly holomorphic forms. Away from the cusps, a nearly holomorphic form f over R of weight κ, principal level N and degree r is a rule assigning to every quadruple $(A_{/S}, \lambda, \psi_N, \alpha)$ an element $f(A_{/S}, \lambda, \psi_N, \alpha)$ inside $V_\kappa^r(S) = W_\kappa(S)[\underline{Y}]_{\leq r}$, where S is an R-algebra, $(A_{/S}, \lambda)$ is a principally polarized dimension n abelian scheme, ψ_N is a principal level N structure and α is a basis of $\mathcal{H}^1_{dR}(A/S)$ respecting the Hodge filtration and symplectic pairing up to similitude. Taking into account the definition of W_κ (2.3.7), by evaluating $f(A_{/S}, \lambda, \psi_N, \alpha) \in W_\kappa(S)[\underline{Y}]_{\leq r}$ at the identity, one may also formulate Katz's interpretation for f as follows. The nearly holomorphic form f is a rule assigning to each quadruple $(A_{/S}, \lambda, \psi_N, \alpha)$ an element $f^{\mathrm{sc}}(A_{/S}, \lambda, \psi_N, \alpha) \in S[\underline{Y}]_{\leq r}$ such that for each $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbf{Q}$ with a belonging to $\mathbf{B}$, we have $f^{\mathrm{sc}}(A_{/S}, \lambda, \psi_N, \alpha \circ g) = \kappa'(a) f^{\mathrm{sc}}(A_{/S}, \lambda, \psi_N, \alpha)$.

It follows directly from Prop. 2.2.3 and (2.3.5) that the sheaves $\mathcal{V}_\rho, \mathcal{V}_\rho^r$ are equipped with the integrable connections

$$\nabla_\rho : \mathcal{V}_\rho \longrightarrow \mathcal{V}_\rho \otimes \Omega_X^1(\log C)$$

and

$$\nabla_\rho : \mathcal{V}_\rho^r \longrightarrow \mathcal{V}_\rho^{r+1} \otimes \Omega_X^1(\log C). \tag{2.4.1}$$

The global sections of the differential sheaf Ω_X^1 has a natural Hecke action and the extended Kodaira–Spencer isomorphism [Lan12, Proposition 6.9] says that there is the Hecke-equivariant isomorphism

$$\Omega_X^1(\log C) \cong \mathrm{Sym}^2(\omega(\mathcal{G}/X))(-1) \cong \omega_\tau(-1).$$

There is a canonical isomorphism of locally free sheaves $t^+ : \mathcal{V}_{\rho\otimes\tau}^{r+1}(-1) \to \mathcal{V}_{\rho\otimes\tau}^{r+1}$ which is not Hecke equivariant but commutes with Hecke actions up to a twist by the multiplier character. Composing ∇_ρ with it we get the differential operator

$$D_\rho : \mathcal{V}_\rho^r \xrightarrow{\nabla_\rho} \mathcal{V}_\rho^{r+1} \otimes \Omega_X^1(\log C) \xrightarrow{\mathrm{KS}} \mathcal{V}_{\rho\otimes\tau}^{r+1}(-1) \xrightarrow{t^+} \mathcal{V}_{\rho\otimes\tau}^{r+1}.$$

It commutes with Hecke actions up to a multiplier twist (cf. §3.10, [Urb14, §2.5.2, Proposition 3.3.7]).

Put $\mathcal{J} = \mathcal{E}(J)$ and $\mathcal{J}^\vee$ to be its dual. By (2.3.6) we have

Proposition 2.4.3. *$\mathcal{V}_\rho^r \cong \omega_\rho \otimes \mathrm{Sym}^r \mathcal{J}$ as locally free sheaves over X with Hecke actions.*

Remark 2.4.4. In [Urb13, §4.1.2, 4.3.1] Urban defined a locally free sheaf $\mathcal{J}'$ to be the one making the diagram below commutative with bottom row exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega(\mathcal{G}/X) \otimes \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X)^{\vee} & \longrightarrow & \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}} \otimes \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X)^{\vee} & \longrightarrow & \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X) \otimes \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X)^{\vee} & \longrightarrow & 0 \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathrm{Sym}^2\big(\omega(\mathcal{G}/X)\big)(-1) & \longrightarrow & \mathcal{J}' & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

After that he defined the sheaf of weight κ degree r nearly holomorphic forms to be $\omega_\kappa \otimes \mathrm{Sym}^r \mathcal{J}'^{\vee}$. One can show that the sheaf $\mathcal{J}^{\vee}$ satisfies Urban's condition for defining $\mathcal{J}'$. Hence $\mathcal{J} \cong \mathcal{J}'^{\vee}$ and our definition of sheaves of nearly holomorphic forms agrees with his.

We end this section with an example showing that the locally free sheaves associated to two non-isomorphic $\mathbf{Q}$-representations can be isomorphic as locally free sheaves without considering the Hecke actions. It also illustrates that the sheaf $\mathcal{J}$ may have splitting that does not come from the $\mathbf{Q}$-representation and such a splitting can give rise to holomorphic but non-Hecke equivariant differential operators.

Example 2.4.5. Take $n = 1$, $\mathbf{G} = \mathrm{GL}(2)$ and $\mathbf{G}^{\circ} = \mathrm{SL}(2)$. We show that the sheaf $\mathcal{J}^{\vee} = \left(\mathcal{V}^1_{\mathrm{triv}}\right)^{\vee}$ and the first jet sheaf $\mathcal{P}^1(\mathcal{O}_X)$ are isomorphic in $\mathrm{QCoh}(X)$ but their corresponding $\mathbf{Q}$-representations are not isomorphic. Let V_1, V_2 be the $\mathbf{Q}$-representations giving rise to $\mathcal{J}^{\vee}$, $\mathcal{P}^1(\mathcal{O}_X)$ respectively. Write $Y = Y_{11}$. Then $V_1^{\vee} = \mathrm{triv} \otimes \mathbb{Z}[Y]_{\leq 1}$ with basis $\{Y, 1\}$, and the action of $\mathbf{Q}^{\circ}$ is given by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot P(Y) = P(a^{-1}b + a^{-2}Y),$$

or in the matrix form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a^{-2} & 0 \\ a^{-1}b & 1 \end{pmatrix}.$$

Clearly V_1 is indecomposable as a $\mathbf{Q}$-representation. On the other hand by [FC90, Proposition VI 5.1], $V_2^{\vee} \cong U_1(\mathfrak{g}^{\circ}) \otimes_{U(\mathfrak{q}^{\circ})} \mathrm{triv}$ as a $\mathbf{Q}^{\circ}$-representation, where $\mathfrak{g}^{\circ} = \mathrm{Lie}\, \mathbf{G}^{\circ} = \mathfrak{sl}(2) = \mathrm{Span}\{h, x, y\}$ and $\mathfrak{q}^{\circ} = \mathrm{Span}\{h, x\}$ with $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. As a basis of $V_2^{\vee}$ we can take $\{y \otimes 1, 1 \otimes 1\}$, and we have $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ act on them by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (y \otimes 1) = a^{-2} y \otimes 1, \qquad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (1 \otimes 1) = 1 \otimes 1,$$

or in the matrix form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

This is saying that the $\mathbf{Q}$-action on V_2 splits. Hence V_1 and V_2 are not isomorphic as $\mathbf{Q}$-representations.

However as coherent sheaves $\mathcal{J}^{\vee}$ and $\mathcal{P}^1(\mathcal{O}_X)$ are indeed isomorphic, because the nearly holomorphic form E_2 splits $\mathcal{J}^{\vee} \cong \omega(\mathcal{G}/X)^{\otimes 2} \otimes \mathcal{J}$ as locally free sheaves [Urb14, Remark 2.3.7]. Actually this non-Hecke equivariant splitting gives rise to Serre's ∂ operator that acts on a modular form f of weight k by

$$\partial f = 12\theta f - kPf,$$

where $\theta = q\frac{d}{dq}$ and P is the holomorphic funciton on the upper half plane defined as $P(q) = 1 - 24\sum_{m \geq 1} \sigma_1(n) q^n$ with $q = e^{2\pi i z}$ (cf. [Kat73b, §A1.4]). Serre's ∂ operator is a holomorphic differential operator but not Hecke equivariant.

2.5. **Equivalence to Shimura's nearly holomorphic forms and differential operators.** First recall Shimura's definition of nearly holomorphic forms and Maass–Shimura differential operators. Let $\mathfrak{h}_n = \{z \in M_n(\mathbb{C}) : {}^{t}z = z, \operatorname{Im} z > 0\}$ be the genus n Siegel upper half space and $\Gamma \subset \mathbf{G}^\circ(\mathbb{Z}) = \mathrm{Sp}(2n, \mathbb{Z})$ be a congruence subgroup. As usual $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathbf{G}(\mathbb{R})$ acts on $\mathfrak{h}_n$ by $\gamma z = (a_\gamma z + b_\gamma) \cdot (c_\gamma z + d_\gamma)^{-1}$. Put $s(z) = (z - \bar{z})^{-1}$ and $\mu(\gamma, z) = c_\gamma z + d_\gamma$.

For an algebraic representation (ρ, W_ρ) of $\mathrm{GL}(n)$ free of finite rank, Shimura defines [Shi00, §13.11] the space of $W_\rho(\mathbb{C})$-valued nearly holomorphic forms of degree r, denoted by $N^r_\rho(\mathfrak{h}_n, \Gamma)$, to be the set consisting of functions $f \in C^\infty(\mathfrak{h}_n, W_\rho(\mathbb{C}))$ satisfying

(i) $f(z)$ can be written as a degree $\leq r$ polynomial in the components of $s(z)$ with coefficients being holomorphic maps from $\mathfrak{h}_n$ to $W_\rho(\mathbb{C})$, and
(ii) f transforms under $\gamma \in \Gamma$ by $f(\gamma z) = \rho(\mu(\gamma, z)) f(z)$.

When $n = 1$ the function f is also required to satisfy the cusp condition, i.e. for every $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ there exists $a_{in} \in \mathbb{C}$ and $M \in \mathbb{N}$ such that

$$\rho(\mu(\gamma, z))^{-1} f(\gamma z) = \sum_{i=0}^{r} (\pi \operatorname{Im} z)^{-i} \sum_{n=0}^{\infty} a_{in} e^{2\pi i z/M}.$$

The Maass–Shimura differential operator $D_{\mathfrak{h}_n,\rho}$ is defined as [Shi00, §12.9]

$$\begin{aligned} D_{\mathfrak{h}_n,\rho} : N^r_\rho(\mathfrak{h}_n, \Gamma) &\longrightarrow N^{r+1}_{\rho\otimes\tau}(\mathfrak{h}_n, \Gamma) \\ f &\longmapsto \rho(s)\big(d_z(\rho(s^{-1})f)\big). \end{aligned} \tag{2.5.1}$$

Now we show that $N^r_\rho(\mathfrak{h}_n, \Gamma)$, together with the Maass–Shimura differential operator $D_{\mathfrak{h}_n,\rho}$, is nothing but the global sections over $\Gamma\backslash\mathfrak{h}_n$ of the sheaf $\mathcal{V}^r_\rho$ equipped with the differential operator D_ρ defined in the previous sections. Let $Y^\circ_{\mathbb{C}}$ be a connected component of Y base changed to $\mathbb{C}$. Then $Y^\circ_{\mathbb{C}} \cong \Gamma(N)\backslash\mathfrak{h}_n$ as complex manifolds and the universal abelian variety $\mathbf{p} : \mathcal{A}_{\mathbb{C}} \to Y^\circ_{\mathbb{C}}$ is isomorphic to $\mathbf{p} : \Gamma(N)\backslash\mathbb{C}^n \times \mathfrak{h}_n/\mathbb{Z}^{2n} \to \Gamma(N)\backslash\mathfrak{h}_n$. Here $(m_1, m_2) \in \mathbb{Z}^{2n}$ and $\gamma \in \Gamma(N)$ act on $(w, z) \in \mathbb{C}^n \times \mathfrak{h}_n$ by

$$\begin{aligned} (w, z) \cdot (m_1, m_2) &= (w + m_1 z + m_2, z), \\ \gamma \cdot (w, z) &= (w\mu(\gamma, z)^{-1}, \gamma z). \end{aligned}$$

Let $q : \mathfrak{h}_n \to \Gamma(N)\backslash\mathfrak{h}_n$ be the quotient map and $A_{\mathfrak{h}_n} = \mathbb{C}^n \times \mathfrak{h}_n/\mathbb{Z}^{2n} \to \mathfrak{h}_n$ be the pullback of $\mathcal{A}_{\mathbb{C}}$ via q. For each $z = (z_{ij}) \in \mathfrak{h}_n$ the fibre $A_{\mathfrak{h}_n,z} \cong \mathbb{C}^n/\Lambda_z$, where Λ_z is the lattice spanned by e_i, the vector with 1 as the i-th entry and 0 elsewhere, $1 \leq i \leq n$, and $z_j = {}^{t}(z_{1j}, z_{2j}, \ldots, z_{nj})$, $1 \leq j \leq n$. Let $\lambda_{\mathfrak{h}_n}$ (resp. $\psi_{\mathfrak{h}_n,N}$) be the polarization (principal level N structure) of $A_{\mathfrak{h}_n}$ such that its fibre at z is given by the real Riemann form $E_z : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$, defined as $E_z(w_1, w_2) = -\operatorname{Im}\big({}^{t}w_1 (\operatorname{Im} z)^{-1} \overline{(i w_2)}\big)$ (resp. $\frac{1}{N}e_1, \ldots, \frac{1}{N}e_n, \frac{1}{N}z_1, \ldots, \frac{1}{N}z_n$). The $\{e_i, z_j\}_{1\leq i,j\leq n}$ form a basis of $H_1(A_{\mathfrak{h}_n,z}, \mathbb{Z})$. Over $\mathfrak{h}_n$ we have a global basis $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ for the sheaf $q^*\mathcal{H}^1_{dR}(\mathcal{A}_{\mathbb{C}}/Y_{\mathbb{C}}) = \mathcal{H}^1_{dR}(A_{\mathfrak{h}_n}/\mathfrak{h}_n)$ defined as

$$\alpha_i\left(\sum_{j=1}^{n} m_{1,j} z_j + m_{2,j} e_j\right) = m_{2,i}, \qquad \beta_i\left(\sum_{j=1}^{n} m_{1,j} z_j + m_{2,j} e_j\right) = m_{1,i}.$$

The basis (α, β) is horizontal with respect to the Gauss–Manin connection, i.e.

$$\nabla(\alpha_i) = \nabla(\beta_i) = 0, \quad 1 \leq i \leq n.$$

After base changing to $C^\infty(\mathfrak{h}_n, \mathbb{C})$, the Hodge decomposition gives another basis of $\mathcal{H}^1_{dR}(A_{\mathfrak{h}_n}/\mathfrak{h}_n) \otimes C^\infty(\mathfrak{h}_n, \mathbb{C})$, denoted as $(dw, d\bar{w}) = (dw_1, \ldots, dw_n, d\bar{w}_1, \ldots, d\bar{w}_n)$.

Neither $(dw, d\bar{w})$ nor (α, β) gives rise to an element of $\left(q^* T_{\mathcal{H}}^{\times}\right)(\mathfrak{h}_n) \otimes C^{\infty}(\mathfrak{h}_n, \mathbb{C})$. The basis $(dw, d\bar{w})$ does not satisfy the pairing condition, while (α, β) is not compatible with the Hodge filtration. Nevertheless (dw, β) (resp. $(dw, -d\bar{w} \cdot s)$) does give an element of $\left(q^* T_{\mathcal{H}}^{\times}\right)(\mathfrak{h}_n)$ (resp. $\left(q^* T_{\mathcal{H}}^{\times}\right)(\mathfrak{h}_n) \otimes C^{\infty}(\mathfrak{h}_n, \mathbb{C})$), and it is easily checked that

$$(dw, -d\bar{w} \cdot s) = (dw, \beta) \cdot \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}. \tag{2.5.2}$$

By evaluating global sections of $\mathcal{V}_\rho^r$ over $Y_{\mathbb{C}}^{\circ}$ at the test object $\left(A_{\mathfrak{h}_n}/\mathfrak{h}_n, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw, -d\bar{w} \cdot s)\right)$, we define a map

$$\begin{aligned} \phi : H^0(Y_{\mathbb{C}}^{\circ}, \mathcal{V}_\rho^r) &\longrightarrow N_\rho^r(\mathfrak{h}_n, \Gamma(N)) \\ f &\longmapsto f\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{N,\mathfrak{h}_n}, (dw, -d\bar{w} \cdot s)\right)|_{\underline{Y}=0}. \end{aligned} \tag{2.5.3}$$

Proposition 2.5.1. *ϕ is well defined and is an isomorphism.*

Proof. We need to check that the above defined $\phi(f)$ does land inside $N_\rho^r(\mathfrak{h}_n, \Gamma(N))$. First look at the evaluation of f at the test object $\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw, \beta)\right)$. Since (dw, β) is holomorphic we have

$$f\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw, \beta)\right) = P_f(\underline{Y}),$$

a polynomial in $\underline{Y}$ of degree $\leq r$ with coefficients being holomorphic maps from $\mathfrak{h}_n$ to $W_\rho(\mathbb{C})$. Combining (2.3.1) and (2.5.2) we get

$$\begin{aligned} \phi(f) &= f\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw, -d\bar{w} \cdot s)\right)|_{\underline{Y}=0} \\ &= f\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw, \beta) \cdot \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{\underline{Y}=0} \\ &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot f\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw, \beta)\right)\Bigg|_{\underline{Y}=0} \\ &= P_f(\underline{Y} + s)|_{\underline{Y}=0} \\ &= P_f(s). \end{aligned}$$

This shows that $\phi(f)$ satisfies condition (i) in the definition of $N_\rho^r(\mathfrak{h}_n, \Gamma(N))$. Under the isomorphism

$$\begin{aligned} \gamma : A_{\mathfrak{h}_n,z} &\longrightarrow A_{\mathfrak{h}_n,\gamma z} \\ w &\mapsto w \cdot \mu(\gamma, z)^{-1} \end{aligned}$$

for $\gamma \in \Gamma(N)$ we have

$$\gamma^*(dw, -d\bar{w} \cdot s) = (dw, -d\bar{w} \cdot s) \begin{pmatrix} \mu(\gamma, z)^{-1} & 0 \\ 0 & \mu(\gamma, z) \end{pmatrix},$$

from which we see that $\phi(f)$ also has the transformation property required in condition (ii). Finally the bijectivity of ϕ can be seen from the fact that essentially it sends $P_f(\underline{Y})$ to $P_f(s)$ and we can recover one of them from the other. $\square$

We continue to prove the compatibility of D_ρ and $D_{\mathfrak{h}_n,\rho}$ under the map ϕ.

Proposition 2.5.2. *$D_{\mathfrak{h}_n,\rho} \circ \phi = \phi \circ D_\rho$*

Proof. Let $\langle\ ,\ \rangle$ be the canonical pairing between the sheaf of differentials $\Omega_{\mathfrak{h}_n}^1$ and the tangent bundle $T_{\mathfrak{h}_n}$. Take $\partial/\partial z_{ij} \in T_{\mathfrak{h}_n}$ and $f \in H^0(Y_{\mathbb{C}}^{\circ}, \mathcal{V}_\rho^r)$. We show that $\langle D_{\mathfrak{h}_n,\rho} \circ \phi(f),\ \partial/\partial z_{ij}\rangle =$

$\langle \phi \circ D_\rho(f), \partial/\partial z_{ij}\rangle$. Assume $i \neq j$ (the computation for the case $i = j$ is the same and we omit it), the Gauss–Manin connection acts on (dw, β) as

$$\nabla(\partial/\partial z_{ij})(dw,\beta) = (dw,\beta)\cdot\begin{pmatrix}0 & 0\\ E_{ij}+E_{ji} & 0\end{pmatrix} = (dw,\beta)\cdot\mu_{ij}^{-}. \tag{2.5.4}$$

Let $P_f(\underline{Y})$ be as in the above proof. According to the definition of D_ρ by (2.2.2),

$$\begin{aligned}\langle \phi \circ D_\rho(f), \partial/\partial z_{ij}\rangle &= \langle (D_\rho f)\big(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (dw,\beta)\big), \partial/\partial z_{ij}\rangle\big|_{\underline{Y}=s}\\ &= \frac{\partial}{\partial z_{ij}}P_f(\underline{Y}) + (\mu_{ij}^{-}\cdot P_f)(\underline{Y})\Big|_{\underline{Y}=s}\\ &= \frac{\partial}{\partial z_{ij}}P_f(\underline{Y}) + \sum_{1\le k\le n}(Y_{ki}\varepsilon_{kj}+Y_{kj}\varepsilon_{ki})\cdot P_f(\underline{Y}) - \sum_{1\le k\le l\le n}(Y_{ki}Y_{jl}+Y_{kj}Y_{il})\frac{\partial}{\partial Y_{kl}}P_f(\underline{Y})\Big|_{\underline{Y}=s}\\ &= \frac{\partial}{\partial z_{ij}}P_f(\underline{Y})\Big|_{\underline{Y}=s} + \sum_{1\le k\le n}(s_{ki}\varepsilon_{kj}+s_{kj}\varepsilon_{ki})\cdot P_f(s) - \sum_{1\le k\le l\le n}(s_{ki}s_{jl}+s_{kj}s_{il})\frac{\partial}{\partial s_{kl}}P_f(s).\end{aligned}$$

Using

$$\frac{\partial s_{kl}}{\partial z_{ij}} = -\left(s\left(\frac{\partial}{\partial z_{ij}}(z-\bar{z})\right)s\right)_{kl} = -(s_{ik}s_{jl}+s_{il}s_{jk}),$$

we get

$$\begin{aligned}\langle \phi \circ D_\rho(f), \partial/\partial z_{ij}\rangle &= \frac{\partial}{\partial z_{ij}}P_f(\underline{Y})\Big|_{\underline{Y}=s} + \sum_{1\le k\le n}(s_{ki}\varepsilon_{kj}+s_{kj}\varepsilon_{ki})\cdot P_f(s) + \frac{\partial s_{kl}}{\partial z_{ij}}\frac{\partial}{\partial s_{kl}}P_f(s)\\ &= \frac{\partial}{\partial z_{ij}}\big(P_f(s)\big) + \sum_{1\le k\le n}(s_{ki}\varepsilon_{kj}+s_{kj}\varepsilon_{ki})\cdot P_f(s)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f) + \sum_{1\le k\le n}(s_{ki}\varepsilon_{kj}+s_{kj}\varepsilon_{ki})\cdot \phi(f). \qquad (2.5.5)\end{aligned}$$

On the other hand according to the definition of $D_{\mathfrak{h}_n,\rho}$ (2.5.1)

$$\begin{aligned}\langle D_{\mathfrak{h}_n,\rho}\circ\phi(f), \partial/\partial z_{ij}\rangle &= \langle \rho(s)\big(d_z(\rho(s^{-1})\phi(f))\big), \partial/\partial z_{ij}\rangle\\ &= \rho(s)\left(\frac{\partial}{\partial z_{ij}}\big(\rho(s^{-1})\phi(f)\big)\right)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f) + \rho\left(s\frac{\partial s^{-1}}{\partial z_{ij}}\right)\phi(f)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f) + \left(\sum_{k=1}^{n} s_{ki}\varepsilon_{kj}+s_{kj}\varepsilon_{ki}\right)\cdot\phi(f).\end{aligned}$$

Comparing with (2.5.5), we conclude. ☐

It is also explained in [Shi90, §7] [Shi00, Appendix A8] that the Maass–Shimura differential operators correspond to the action of the Lie algebra $(\mathrm{Lie}\,\mathbf{G})_{\mathbb{C}}$ on nearly holomorphic forms. Therefore the sheaf-theoretic definition of the differential operators in §2.4 can be viewed as a geometric interpretation of the Lie algebra action at the archimedean place on automorphic forms whose corresponding automorphic representations have holomorphic discrete series as the archimedean component, and we have the commutative diagram (1.0.2).

2.6. **Polynomial q-expansions.** We first define the Mumford objects. Then using the moduli interpretation of $N^r_\kappa(\Gamma(N),R) = H^0\left(X_{/R}, \mathcal{V}^r_\kappa\right)$, we evaluate a nearly holomorphic form at a Mumford object to get its polynomial q-expansion. We also include formulas for the action of differential operators on the polynomial q-expansions.

Following [FC90, V.1], let $L = \mathbb{Z}^n$ with fixed basis $e_1, \dots, e_n$ and L^* be its dual. Put S_L to be the symmetric quotient of $L \times L$ and $S_{L,\geq 0}$ to be the intersection of S_L with the cone dual to the cone inside $S_L^* \otimes_{\mathbb{Z}} \mathbb{R}$ consisting of semi-positive definite forms. Take a basis $s_1, \dots, s_{n(n+1)/2}$ of S_L lying inside $S_{L,\geq 0}$, and set $\mathbb{Z}((S_{L,\geq 0})) = \mathbb{Z}[[S_{L,\geq 0}]][1/s_1 s_2 \cdots s_{n(n+1)/2}]$. For $\beta \in S_{L,\geq 0}$, the corresponding element in $\mathbb{Z}[[S_{L,\geq 0}]]$ is sometimes written as q^β.

The natural map $L \to S_L \otimes L^*$ defines a period group $L \subset L^* \otimes \mathbb{G}_{m/\mathbb{Z}((S_{L,\geq 0}))}$, principally polarized by the duality between L and L^*. Mumford's construction [FC90] gives an abelian variety $A_{/\mathbb{Z}((S_{L,\geq 0}))}$ with a canonical polarization λ_{can} and a canonical basis $\omega_{\mathrm{can}} = (\omega_{1,\mathrm{can}}, \dots, \omega_{n,\mathrm{can}})$ of $\omega(A/\mathbb{Z}((S_{L,\geq 0})))$. The exact sequence

$$0 \to L^* \otimes \prod_l \varprojlim_m \mu_{l^m} \to \prod_l T_l(A) \to L \otimes \widehat{\mathbb{Z}} \to 0,$$

after base changing to $\mathbb{Z}((N^{-1}S_{L,\geq 0}))[\zeta_N, 1/N]$, gives rise to a principal level N structure $\psi_{N,\mathrm{can}}$ for $A_{/\mathbb{Z}((S_{L,\geq 0}))}$. Let $D_{ij} \in \mathrm{Der}(\mathbb{Z}((S_{L,\geq 0})), \mathbb{Z}((S_{L,\geq 0})))$ be the element dual to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $\delta_{i,\mathrm{can}} = \nabla(D_{ii})\omega_{i,\mathrm{can}}$. For $\beta \in S_{L,\geq 0}$ we have $D_{ij}(q^\beta) = (2-\delta_{ij})\beta_{ij}q^\beta$ with $\delta_{ij} = 0$ if $i \neq j$, and 1 if $i = j$. Then $\delta_{\mathrm{can}} = (\delta_{1,\mathrm{can}}, \dots, \delta_{n,\mathrm{can}})$ together with ω_{can} forms a basis of $\mathcal{H}^1_{dR}(A/\mathbb{Z}((S_{L,\geq 0})))$ respecting both the Hodge filtration and the symplectic pairing.

Evaluating a nearly holomorphic form $f \in N^r_\kappa(\Gamma(N),R)$ at the test object

$$\mathrm{Mum}_N(q) = (A_{/\mathbb{Z}((N^{-1}S_{L,\geq 0}))[\zeta_N, 1/Np]}, \lambda_{\mathrm{can}}, \psi_{N,\mathrm{can}}, \omega_{\mathrm{can}}, \delta_{\mathrm{can}})$$

defines its polynomial q-expansion

$$\begin{aligned} N^r_\kappa(\Gamma(N),R) &\longrightarrow \mathbb{Z}[\zeta_N, 1/N][[N^{-1}S_{L,\geq 0}]] \otimes W_\kappa(R)[\underline{Y}]_{\leq r} \\ f &\longmapsto f(q,\underline{Y}) = f(\mathrm{Mum}_N(q)). \end{aligned} \tag{2.6.1}$$

Next we compute formulas of differential operators in terms of polynomial q-expansions. Let $\underline{X} = (X_{ij})_{1\leq i,j\leq n}$ be the symmetric matrix with the indeterminate $X_{ij} = X_{ji}$ as the ij-th and ji-th entries for $1 \leq i \leq j \leq n$. The X_{ij}'s form a basis of the $\mathrm{GL}(n)$-representation τ. An element $a \in \mathrm{GL}(n)$ acts on $\underline{X}$ by $a \cdot \underline{X} = {}^t a\, \underline{X} a$. Let $X_{ij}^\vee$ be the basis of $\tau^\vee$ dual to X_{ij}. Then under the trivialization $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$, X_{ij} corresponds to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $X_{ij}^\vee$ corresponds to D_{ij}. From the construction of $\mathrm{Mum}_N(q)$ one can see that $\nabla(D_{ij})(\omega_{\mathrm{can}}, \delta_{\mathrm{can}}) = (\omega_{\mathrm{can}}, \delta_{\mathrm{can}})\mu^-_{ij}$, i.e. $X\big(D_{ij}, (\omega_{\mathrm{can}}, \delta_{\mathrm{can}})\big) = \mu^-_{ij}$.

Proposition 2.6.1. *Let $f \in N^r_\kappa(\Gamma(N),R)$ be a nearly holomorphic form with polynomial q-expansion $f(q,\underline{Y}) \in \mathbb{Z}[\zeta_N, 1/N][[N^{-1}S_{L,\geq 0}]] \otimes W_\kappa(R)[\underline{Y}]_{\leq r}$. Then*

$$(D_\kappa f)(q,\underline{Y}) = \sum_{1\leq i\leq j\leq n} \Big(D_{ij} f(q,\underline{Y}) + \mu^-_{ij} \cdot f(q,\underline{Y})\Big) \otimes X_{ij}$$

Example 2.6.2. If we apply the above proposition to the $n = 1$ case where $\kappa = k \in \mathbb{N}$, we recover the formula given in [Urb14, Proposition 2.4.1] for D_k (denoted δ_k there). In this case the image of the polynomial q-expansion belongs $R[\zeta_N, 1/N][[q^{1/N}]][\underline{Y}]_{\leq r}$ and $D_{11} = q\frac{d}{dq}$. Write $Y = Y_{11}$. The representations κ and τ are both one-dimensional and we omit writing down their basis.

$$\begin{aligned}(D_\kappa f)(q,Y) &= D_{11}f(q,Y) + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot f(q,Y)\\ &= q\frac{d}{dq}f(q,Y) + Y\varepsilon_{11}\cdot f(q,Y) - Y^2\frac{\partial}{\partial Y}f(q,Y)\\ &= (q\frac{d}{dq} - Y^2\frac{\partial}{\partial q})f(q,Y) + kYf(q,Y).\end{aligned}$$

2.7. **Holomorphic differential operators.** The purpose of this section is to explain Shimura's construction of holomorphic differential operators in the sheaf-theoretic context. Let $G \hookrightarrow H$ be an embedding of reductive groups over $\mathbb{Q}$, and we assume both $G(\mathbb{R})$ and $H(\mathbb{R})$ have holomorphic discrete series. One of the motivations for studying nearly holomorphic forms is that they help construct differential operators sending holomorphic forms on H of weight κ_0 (often taken to be a scalar weight) to holomorphic forms on G of a specified weight κ. Such holomorphic differential operators have been considered and applied in many works on studying special L-values, e.g. [Shi00, Har97, Har08, BS00, CP04, EHLS16, EW16], just to list a few.

Let $G = \mathbf{G}^\circ = \mathrm{Sp}(2n)_{/\mathbb{Q}}$ and $H = \mathrm{Sp}(4n)_{/\mathbb{Q}}$ with Siegel parabolic subgroups $Q_G = \mathbf{Q}^\circ$ and Q_H. The Shimura variety Y_G (resp. Y_H) of principal level N is defined over $\mathbb{Q}(\zeta_N)$ and is a connected component of $Y = Y_{\mathbf{G},\Gamma(N)}$ (resp. the Siegel variety parametrizing principally polarized abelian schemes of relative dimension $2n$ with a principal level N structure). In the following, sheaves over Y_H and $(\mathrm{Lie}\,H, Q_H)$-modules are denoted with a superscript ^{H}.

Let $\iota : Y_G \times Y_G \hookrightarrow Y_H$ be the embedding corresponding to

$$\begin{array}{ccccc} G & \times & G & \hookrightarrow & H\\ \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix} & \times & \begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix} & \longmapsto & \begin{pmatrix}a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ c_1 & 0 & d_1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix}.\end{array}$$

Denote by $p_1, p_2 : Y_G \times Y_G \to Y_G$ the projection to the first and second factor.

Proposition 2.7.1. *Let k be an positive integer (viewed as a scalar weight) and $\kappa \in X(\mathbf{T}^\circ)^+$ be a generic weight such that the holomorphic projection $\mathscr{A}_{k+\kappa} : \mathcal{V}^e_{k+\kappa} \to \omega_{k+\kappa}$ (cf. [Shi00, Proposition 14.2], Proposition 3.7.1, Corollary 3.7.5) exists for $e = |\kappa| = \sum_{i=1}^n \kappa_i$. Then there exists a nonzero morphism*

$$D_{k,k+\kappa} : \iota^{-1}\omega^H_k \longrightarrow p_1^*\omega_{k+\kappa} \otimes p_2^*\omega_{k+\kappa}.$$

By taking global sections, $D_{k,k+\kappa}$ induces a holomorphic differential operator sending Siegel modular forms on $\mathrm{Sp}(4n)$ of scalar weight k to Siegel modular forms on $\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$ of weight $(k+\kappa, k+\kappa)$.

Proof. First, by our construction of differential operators, there is the map

$$D^e_k : \iota^{-1}\omega^H_k \longrightarrow \iota^*\mathcal{V}^{H,e}_{k\otimes\mathrm{Sym}^e\tau^H},$$

so we consider the decomposition of the sheaf $\iota^*\mathcal{V}^{H,e}_{k\otimes\mathrm{Sym}^e\tau^H}$, especially how $\omega_{k+\kappa}$ appears in the decomposition. Equivalently, we consider the decomposition of $V^{H,e}_{k\otimes\mathrm{Sym}^e\tau^H}$ as a $(\mathfrak{g}^\circ\times\mathfrak{g}^\circ, Q_G\times Q_G)$-module.

Write

$$\underline{X}^H = \begin{pmatrix}\underline{X}_1 & \underline{X}_0\\ {}^{\mathrm{t}}\underline{X}_0 & \underline{X}_2\end{pmatrix}, \qquad \underline{Y}^H = \begin{pmatrix}\underline{Y}_1 & \underline{Y}_0\\ {}^{\mathrm{t}}\underline{Y}_0 & \underline{Y}_2\end{pmatrix}$$

in $n\times n$ blocks. The subspace

$$(\underline{X}_1,\,\underline{X}_2,\,\underline{Y}_0)W_k^H[\underline{X}^H,\,\underline{Y}^H]\cap W_k^H[\underline{X}^H]_e[\underline{Y}^H]\subset W_k^H[\underline{X}^H]_e[\underline{Y}^H]$$

is stable under the action of $\mathfrak{g}^\circ\times\mathfrak{g}^\circ$ and $Q_G\times Q_G$. Here the subscript $_e$ means polynomials of degree equal to e. The quotient of $W_k^H[\underline{X}^H]_e[\underline{Y}^H]$ by this submodule is canonically isomorphic to

$$W_k[\underline{X}_0]_e[\underline{Y}_1,\,\underline{Y}_2] \tag{2.7.1}$$

with the induced $(\mathfrak{g}^\circ\times\mathfrak{g}^\circ,Q_G\times Q_G)$-action. Instead of looking at the decomposition of the whole $V^{H,e}_{k\otimes\mathrm{Sym}^e\,\tau^H}\Big|_{(\mathfrak{g}^\circ\times\mathfrak{g}^\circ,Q_G\times Q_G)}$, we consider the decomposition of the quotient (2.7.1). First it is easy to check that $\begin{pmatrix}a_1 & b_1\\ 0 & {}^{\mathrm{t}}a_1^{-1}\end{pmatrix}\times\begin{pmatrix}a_2 & b_2\\ 0 & {}^{\mathrm{t}}a_2^{-1}\end{pmatrix}\in Q_G\times Q_G$ acts on $P(\underline{X}_0,\,\underline{Y}_1,\,\underline{Y}_2)\in W_k[\underline{X}_0]_e[\underline{Y}_1,\,\underline{Y}_2]$ as

$$\begin{pmatrix}a_1 & b_1\\ 0 & {}^{\mathrm{t}}a_1^{-1}\end{pmatrix}\times\begin{pmatrix}a_2 & b_2\\ 0 & {}^{\mathrm{t}}a_2^{-1}\end{pmatrix}\cdot P(\underline{X}_0,\,\underline{Y}_1,\,\underline{Y}_2)=P({}^{\mathrm{t}}a_1\,\underline{X}_0\,a_2,\,a_1^{-1}\,\underline{Y}_1\,{}^{\mathrm{t}}a_1^{-1}+a_1^{-1}b_1,\,a_2^{-1}\,\underline{Y}_2\,{}^{\mathrm{t}}a_2^{-1}+a_2^{-1}b_2).$$

By [Shi00, Theorem 12.7] we know that as representations of $Q_G\times Q_G$,

$$W_k[\underline{X}_0]_e[\underline{Y}_1,\,\underline{Y}_2]=\bigoplus_{\kappa\in X(\mathbf{T}^\circ)^+,\ |\kappa|=e}V_{k+\kappa}\boxtimes V_{k+\kappa}. \tag{2.7.2}$$

By checking the formulas defining the $\mathfrak{g}$-actions, we see that this decomposition actually holds as modules of $(\mathfrak{g}^\circ\times\mathfrak{g}^\circ,Q_G\times Q_G)$. Moreover, for each κ appearing in the decomposition, the highest weight vector inside $V^0_{k+\kappa}\boxtimes V^0_{k+\kappa}$ is given by $\prod_{i=1}^n\det_i(\underline{X}_0)^{\kappa_i-\kappa_{i+1}}$, where $\det_i$ is the determinant of the upper left $i\times i$ minor. Therefore, for $\kappa\in X(\mathbf{T}^\circ)^+$, $|\kappa|$=e, the $(\mathfrak{g}^\circ\times\mathfrak{g}^\circ,Q_G\times Q_G)$-module $V_{k+\kappa}\boxtimes V_{k+\kappa}$ appears as a quotient of $V^H_{k\otimes\mathrm{Sym}^e\,\tau^H}\Big|_{(\mathfrak{g}^\circ\times\mathfrak{g}^\circ,Q_G\times Q_G)}$ and one can write down an explicit map

$$V^H_{k\otimes\mathrm{Sym}^e\,\tau'}\Big|_{(\mathfrak{g}^\circ\times\mathfrak{g}^\circ,Q_G\times Q_G}\xrightarrow{\text{mod }\underline{X}_1,\underline{X}_2,\underline{Y}_0}W'_k[\underline{X}_0]_e[\underline{Y}_1,\,\underline{Y}_2]\longrightarrow V_{k+\kappa}\boxtimes V_{k+\kappa},$$

which induces a morphism of sheaves over $Y_G\times Y_G$,

$$\varrho_{k,\kappa}:\iota^*\mathcal{V}^H_{k\otimes\mathrm{Sym}^e\,\tau^H}\longrightarrow p_1^*\mathcal{V}_{k+\kappa}\otimes p_2^*\mathcal{V}_{k+\kappa}.$$

When the holomorphic projection $\mathscr{A}_{k+\kappa}:\mathcal{V}^e_{k+\kappa}\to\omega_{k+\kappa}$ exists, We define the operator $D_{k,k+\kappa}$ as the composition

$$D_{k,k+\kappa}:\iota^{-1}\omega_k^H\xrightarrow{D_k^e}\iota^*\mathcal{V}^{H,e}_{k\otimes\mathrm{Sym}^e\,\tau^H}\xrightarrow{\varrho_{k,\kappa}}p_1^*\mathcal{V}^e_{k+\kappa}\otimes p_2^*\mathcal{V}^e_{k+\kappa}\xrightarrow{\mathscr{A}_{k+\kappa}}p_1^*\omega_{k+\kappa}\otimes p_2^*\omega_{k+\kappa}.$$

It remains to show that such defined $D_{k,k+\kappa}$ is nonzero. This can be done by some computation in local coordinates.

Take an affine open subset $U^H=\mathrm{Spec}(R')\subset Y_H$ such that $U^H\times_{Y_H}(Y_G\times Y_G)$ is of the form $U\times U$ with $U=\mathrm{Spec}(R)$. Also we pick an ordered basis $\alpha=(\alpha_1,\dots,\alpha_{4n})$ of $\mathcal{H}^1_{dR}\left(\mathcal{A}/U^H\right)$ respecting both the Hodge filtration and the symplectic pairing such that $\alpha^{(1)}\times\alpha^{(2)}\in T^\times_{\mathcal{H}}(U)\times T^\times_{\mathcal{H}}(U)$ with $\alpha^{(1)}=(\iota^*\alpha_1,\dots,\iota^*\alpha_n,\iota^*\alpha_{2n+1},\dots,\iota^*\alpha_{3n})$ and $\alpha^{(2)}=(\iota^*\alpha_{n+1},\dots,\iota^*\alpha_{2n},\iota^*\alpha_{3n+1},\dots,\iota^*\alpha_{4n})$. Then $\omega_k^H(U^H)\simeq R'$, $\Omega^1_{Y_H}(U^H)\simeq R'[\underline{X}^H]_1$, $\Omega^1_{Y_G\times Y_G}(U\times U)\simeq R[\underline{X}_1]_1\otimes R[\underline{X}_2]_1$, and $\Omega^1_{Y_H/Y_G\times Y_G}(U^H)\simeq R'[\underline{X}_0]_1$.

Let $\partial^H_{ij}\in\mathrm{Der}_{\mathbb{Q}(\zeta_N)}(R',R')$, $1\le i\le j\le 2n$, be the dual basis of X^H_{ij} and write $\partial^H=(\partial^H_{ij})$ in $n\times n$ blocks as $\partial^H=\begin{pmatrix}\partial_1 & \partial_0\\ {}^{\mathrm{t}}\partial_0 & \partial_2\end{pmatrix}$. According to (2.2.2) there is a nonzero degree e homogeneous polynomial $P_\kappa(T_{ij})\in\mathbb{Q}[T_{ij}]_{1\le i\le j\le n}$ and a polynomial Q in $\underline{X}_0,\partial_0,\underline{Y}_1,\underline{Y}_2$ whose degree in ∂_0 is strictly less than e such that

$$\varrho_{k,\kappa}\circ D_k^e=u_\iota\left(P_\kappa(X_{0,ij}\partial_{0,ij})+Q(\underline{X}_0,\partial_0,\underline{Y}_1,\underline{Y}_2)\right),$$

where $u_\iota : R' \to R \otimes R$ is the quotient map corresponding to the embedding $Y_G \times Y_G \hookrightarrow Y_H$. The holomorphic projection $\mathscr{A}_{k+\kappa}$ is purely defined on $Y_G \times Y_G$, so does not involve any ∂_0. Thus,

$$\begin{aligned} D_{k,k+\kappa} &= \mathscr{A}_{k+\kappa} \circ u_\iota \left(P_\kappa(X_{0,ij}\partial_{0,ij}) + Q(\underline{X}_0, \partial_0, \underline{Y}_1, \underline{Y}_2)\right) \\ &= u_\iota \left(P_\kappa(X_{0,ij}\partial_{2,ij}) + \tilde{Q}(\underline{X}_0, \partial_0, \partial_1, \partial_2)\right) \end{aligned}$$

with $\tilde{Q}$ being a polynomial whose degree in ∂_0 is still strictly less than e. This implies that the differential operator $D_{k,k+\kappa} \neq 0$, and the coefficient for the highest weight vector $\prod_{i=1}^n \det_i(\underline{X}_0)$ is $\prod_{i=1}^n \det_i(\partial_0)^{\kappa_i - \kappa_{i+1}} + c(\partial_0, \partial_1, \partial_2)$, where the total degree of the homogeneous polynomial of c is e and every term involves either ∂_1 or ∂_2.

□

Remark 2.7.2. In the construction of $D_{k,k+\kappa}$, the increase of the weight is contributed by the conormal differential sheaf $\Omega^1_{Y_H/Y_G \times Y_G}$. We do not consider the part $\Omega^1_{Y_G \times Y_G}$ because its contribution will be killed by the holomophic projection.

Remark 2.7.3. Besides Shimura, holomorphic differential operators are also studied by Böcherer [Böc85], Ibukiyama [Ibu99] uisng invariant pluri-harmonic polynomials and Harris [Har86] using Grothendieck's sheaves of differentials. Harris' approach shows the uniqueness (up to scalars) of holomorphic differential operators in many cases (including the case considered above). Therefore, all the holomorphic differential operators constructed in different approaches must be the same up to scalars. On the other hand, different approaches yield different descriptions of the holomorphic differential operators, and have their own advantages in applications.

3. Overconvergent nearly holomorphic forms and their p-adic families

3.1. **The weight space.** Let p be an odd prime number. The weight space $\mathcal{W}$ is the rigid analytic space defined over $\mathbb{Q}_p$ associated to the noetherian complete algebra $\mathbb{Z}_p[[\mathbf{T}^\circ(\mathbb{Z}_p)]]$. Its $\mathbb{C}_p$-points parametrize continuous homomorphisms from $\mathbf{T}^\circ(\mathbb{Z}_p)$ to $\mathbb{C}_p^\times$, i.e. $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}^\circ(\mathbb{Z}_p), \mathbb{C}_p^\times)$. For $\kappa \in \mathcal{W}$ we can write it as $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ with κ_i being a continuous character of $\mathbb{G}_m(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$ such that $\kappa(\mathrm{diag}(a_1, \ldots, a_n)) = \prod_{i=1}^n \kappa_i(a_i)$. If we fix a topological generator of $1 + p\mathbb{Z}_p$, say $1 + p$, then $\mathcal{W}$ can be identified with the disjoint union of n-dimensional open unit balls indexed by $\widehat{\mathbf{T}^\circ(\mathbb{Z}/p\mathbb{Z})}$, the character group of the torsion part $\mathbf{T}^\circ(\mathbb{Z}/p\mathbb{Z})$ of the group $\mathbf{T}^\circ(\mathbb{Z}_p)$. Explicitly we can write the isomorphism as

$$\begin{aligned} \mathcal{W} &\longrightarrow \widehat{\mathbf{T}^\circ(\mathbb{Z}/p\mathbb{Z})} \times \prod_{i=1}^n \mathcal{B}(1, 1^-) \\ \kappa &\longmapsto (\kappa|_{\mathbf{T}^\circ(\mathbb{Z}/p\mathbb{Z})}, \kappa_1(1+p), \kappa_2(1+p), \ldots, \kappa_n(1+p)). \end{aligned}$$

Here $\mathcal{B}(1, 1^-)$ is the 1-dimensional rigid open unit ball centered at 1. If $\mathcal{U} \subset \mathcal{W}$ is an affinoid subdomain we use $\mathcal{A}(\mathcal{U})$ to denote the affinoid algebra of analytic functions on $\mathcal{U}$ and $\mathcal{A}(\mathcal{U})^\circ$ to denote the subset of $\mathcal{A}(\mathcal{U})$ consisting of power bounded elements.

Given κ we say it is algebraic if $\kappa(\mathrm{diag}(a_1, \ldots, a_n)) = a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n}$ with k_i, $1 \le i \le n$, being integers, and it is arithmetic if it can be written as the product of an algebraic weight with a locally constant character. If κ is arithmetic, we denote by κ_{alg} (resp. κ_{f}) its algebraic part (resp. locally constant part).

Let μ_{p-1} be the group of $(p-1)$-th roots of unity. There is a universal character

$$\kappa^{\mathrm{un}} : \mathcal{W} \times \mathbf{T}^\circ(\mathbb{Z}_p) \longrightarrow \mu_{p-1} \times \mathcal{B}(1, 1^-).$$

Take L to be an extension of $\mathbb{Q}_p$ inside $\mathbb{C}_p$, complete with a valuation v such that $v(p) = 1$. Denote by m_L the maximal ideal of $\mathcal{O}_L$. For each $w \in v(m_L)$ we can define over L the rigid analytic group

$\mathcal{T}^\circ_{1,w} \cong \prod_{i=1}^n \mathcal{B}(1,p^w)$ with $\mathcal{B}(1,p^w)$ being the 1-dimensional closed ball of radius p^w centered at 1 and the rigid analytic group $\mathcal{T}^\circ_w = T^\circ(\mathbb{Z}_p)\mathcal{T}^\circ_{1,w}$. For any affinoid subdomain $\mathcal{U} \subset \mathcal{W}$ there exists some $w \in v(m_L)$ such that the universal character $\kappa^{\mathrm{un}}|_{\mathcal{U}\times \mathbf{T}^\circ(\mathbb{Z}_p)}$ extends to a map between rigid analytic spaces

$$\kappa^{\mathrm{un}} : \mathcal{U} \times \mathcal{T}^\circ_w \longrightarrow \mu_{p-1} \times \mathcal{B}(1,1^-).$$

For such $\mathcal{U}$ and w we say that the universal character κ^{un} over $\mathcal{U}$ is w-analytic. In order to see the existence of such a w it suffices to look at the case where $\mathcal{U}$ is a closed ball inside the identity connected component $\mathcal{W}^\circ$ of the weight space, i.e. $\mathcal{U} = \mathcal{W}(t)^\circ = \prod_{i=1}^n \mathcal{B}(1,p^t)$ for some $t \in v(m_L)$. Let $Y_1,\dots,Y_n$ (resp. $S_1,\dots,S_n$) be the coordinates of $\mathcal{W}(t)^\circ$ (resp. the neighborhood $a \cdot \prod_{i=1}^n \mathcal{B}(1,p^w) = \prod_{i=1}^n \mathcal{B}(a_i,p^w)$ of $a = \mathrm{diag}(a_1,\dots,a_n) \in \mathbf{T}^\circ(\mathbb{Z}_p)$) with coordinate ring $\mathcal{A}(\mathcal{W}(t)^\circ) = L\langle Y_1,\dots,Y_n\rangle$ (resp. $L\langle S_1,\dots,S_n\rangle$). The universal character can be extended to $\mathcal{W}(t)^\circ \times a \cdot \prod_{i+1}^n \mathcal{B}(1,p^w)$ as long as $(1+p^tY_i)^{a_i}(1+p^tY_i)^{\frac{\log(1+p^wS_i)}{\log(1+p)}}$ belongs to $L\langle Y_i,S_i\rangle$ for all $1 \le i \le n$. The factor $(1+p^tY_i)^{a_i} = \sum_{j=1}^\infty \binom{a_i}{j} p^{tj}Y_i^j$ is always inside $1+p^t\mathcal{O}_L\langle Y_i\rangle$, and the factor $(1+p^tY_i)^{\frac{\log(1+p^wS_i)}{\log(1+p)}} = \exp\left(\log(1+p^tY_i)\cdot \frac{\log(1+p^wS_i)}{\log(1+p)}\right)$ lies inside $L\langle Y_i,S_i\rangle$ if we choose w large enough such that the supreme norm of the function $\log(1+X)$ over $\mathcal{B}(0,p^t)$ satisfies $|\log(1+X)|_{\mathcal{B}(o,p^t)} < p^{w-\frac{1}{p-1}}$. If the universal weight κ^{un} is w-analytic over $\mathcal{U}$, then it is obvious that any point $\kappa \in \mathcal{U}(L)$ is a w-analytic weight, i.e. the character $\kappa : \mathbf{T}^\circ(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ extends to an analytic map $\kappa : \mathcal{T}^\circ_w \to \mu_{p-1} \times \mathcal{B}(1,1^-)$.

Let $\mathfrak{T}^\circ_{1,w}$ be the formal group defined by

$$\mathfrak{T}^\circ_{1,w}(R) = \mathrm{Ker}\left(\mathbf{T}^\circ(R) \longrightarrow \mathbf{T}^\circ(R/p^wR)\right) \tag{3.1.1}$$

for all flat, p-adically complete $\mathcal{O}_L$-algebras R. As a formal scheme $\mathfrak{T}^\circ_{1,w}$ is isomorphic to $\mathrm{Spf}(\mathcal{O}_L\langle S_1,\dots,S_n\rangle)$. The identity component $\mathcal{W}(t)^\circ$ of $\mathcal{W}(t)$ has a natural formal model $\mathfrak{W}(t)^\circ$ isomorphic to $\mathrm{Spf}(\mathcal{O}_L\langle Y_1,\dots,Y_n\rangle)$. Given an affinoid subdomain $\mathcal{U} \subset \mathcal{W}(t)^\circ$ and an open formal subscheme $\mathfrak{U}$ of an admissible blow-up of $\mathfrak{W}(t)^\circ$ such that $\mathcal{U}$ is the rigid fibre of $\mathfrak{U}$, the above discussion shows that for $w \in v(m_L)$ big enough the formal universal character

$$\kappa^{\mathrm{un}} : \mathfrak{U} \times \mathfrak{T}^\circ_{1,w} \longrightarrow \widehat{\mathbb{G}}_m$$

can be defined and it specializes to a formal character $\kappa : \mathfrak{T}^\circ_{1,w} \to \widehat{\mathbb{G}}_m$ for each $\kappa \in \mathcal{U}(L)$.

3.2. **The analytic $(\mathfrak{g},\mathcal{Q}_w)$-modules $V_{\kappa,w}$ and $V_{\kappa^{\mathrm{un}},w}$.** This section is an analogue of §2.3 in the p-adic analytic and formal setting. Fix the p-adic field L and $w \in v(m_L)$ as in the previous section. Let $\mathfrak{A}_L$ be the category of L-affinoid algebras and $\mathbf{Adm}_{\mathcal{O}_L}$ be the category of admissible $\mathcal{O}_L$-algebras, i.e. the flat $\mathcal{O}_L$-algebras that are quotients of $\mathcal{O}_L\langle X_1,\dots,X_s\rangle$ for some $s \in \mathbb{N}$. First we define several rigid analytic groups and formal groups. Like the formal torus $\mathfrak{T}^\circ_{1,w}$ we define the formal groups $\mathfrak{M}^\circ_{1,w}$ and $\mathfrak{B}_{1,w}$ over $\mathcal{O}_L$ by

$$\mathfrak{M}^\circ_{1,w}(R) = \mathrm{Ker}\left(\mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n,R/p^wR)\right),$$
$$\mathfrak{B}_{1,w}(R) = \mathrm{Ker}\left(\mathbf{B}(R) \longrightarrow \mathbf{B}(R/p^wR)\right)$$

for all $R \in \mathbf{Adm}_{\mathcal{O}_L}$. Define $\mathfrak{N}_{1,w}$ to be the unipotent part of $\mathfrak{B}_{1,w}$. Let $\mathrm{GL}(n)_{\mathrm{an}}$, $\mathbf{B}_{\mathrm{an}}$, $\mathbf{N}_{\mathrm{an}}$, $\mathbf{T}^\circ_{\mathrm{an}}$, $\mathbf{Q}_{\mathrm{an}}$, $\mathbf{U}_{\mathrm{an}}$ be the rigid analytic groups associated to the groups schemes $\mathrm{GL}(n)$, $\mathbf{B}$, $\mathbf{N}$, $\mathbf{T}^\circ$, $\mathbf{Q}$, $\mathbf{U}$, and $\mathrm{GL}(n)_{\mathrm{rig}}$, $\mathbf{B}_{\mathrm{rig}}$, $\mathbf{T}^\circ_{\mathrm{rig}}$, $\mathbf{Q}_{\mathrm{rig}}$ be the generic fibre of the formal completion of $\mathrm{GL}(n)$, $\mathbf{B}$, $\mathbf{T}^\circ$, $\mathbf{Q}$ along p. The rigid fibre $\mathcal{M}^\circ_{1,w}$, $\mathcal{B}_{1,w}$, $\mathcal{T}^\circ_{1,w}$ of the formal groups $\mathfrak{M}^\circ_{1,w}$, $\mathfrak{B}_{1,w}$, $\mathfrak{T}^\circ_{1,w}$ can be naturally regarded as rigid analytic subgroups of $\mathrm{GL}(n)_{\mathrm{rig}}$, $\mathbf{B}_{\mathrm{rig}}$, $\mathbf{T}^\circ_{\mathrm{rig}}$. Set $I(\mathbb{Z}_p) = \{g \in \mathrm{GL}(n,\mathbb{Z}_p) : g \mod p \in \mathbf{B}(\mathbb{Z}/p\mathbb{Z})\}$

to be the Iwahori subgroup of $\mathrm{GL}(n, \mathbb{Z}_p)$ and $N^-(\mathbb{Z}_p)$ to be the unipotent subgroup of $I(\mathbb{Z}_p)$ consisting of lower triangular matrices with 1 on the diagonal. $I(\mathbb{Z}_p) = N^-(\mathbb{Z}_p)\,\mathbf{B}(\mathbb{Z}_p)$ is the Iwahori decomposition. We define the rigid analytic subgroup $\mathcal{I}_w$ of $\mathrm{GL}(n)_{\mathrm{rig}}$ by $\mathcal{I}_w = I(\mathbb{Z}_p)\cdot\mathcal{M}^\circ_{1,w}$. Fixing a set S of representatives in $I(\mathbb{Z}_p)$ of $I(\mathbb{Z}/p^{[w]}\mathbb{Z})$, the group $\mathcal{I}_w$ can be written as the disjoint union $\coprod_{\gamma\in S}\gamma\cdot\mathcal{M}^\circ_{1,w}$. Similarly we define $\mathcal{B}_w = \mathbf{B}(\mathbb{Z}_p)\cdot\mathcal{B}_{1,w}\subset\mathbf{B}_{\mathrm{rig}}$. The group $\mathcal{T}^\circ_w = \mathbf{T}^\circ(\mathbb{Z}_p)\cdot\mathcal{T}^\circ_{1,w}\subset\mathbf{T}^\circ_{\mathrm{rig}}$ is already defined in last section. There is a projection $\pi : \mathbf{Q}_{\mathrm{an}}\to\mathrm{GL}(n)_{\mathrm{an}}$ sending $\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$ to a. We define the rigid analytic subgroup $\mathcal{Q}_w\subset\mathbf{Q}_{an}$ as

$$\mathcal{Q}_w = \pi^{-1}(\mathcal{I}_w) \tag{3.2.1}$$

Note that $\mathcal{Q}_w$ is not contained inside $\mathbf{Q}_{\mathrm{rig}}$.

Now take $\kappa = (\kappa_1,\dots,\kappa_n)\in\mathcal{W}(\mathbb{C}_p)$ to be a w-analytic weight and set $\kappa' = (-\kappa_n,\dots,-\kappa_1)$ which is also w-analytic. Extend κ' to a character of $\mathcal{B}_w$ through the quotient map $\mathcal{B}_w\to\mathcal{T}^\circ_w$. Define the w-analytic left $\mathcal{I}_w$-module $W_{\kappa,w}$ by

$$W_{\kappa,w}(R) = \{f : \mathcal{I}_w(R)\to R : f(xb) = \kappa'(b)f(x),\ \text{for all } b\in\mathcal{B}_w(R),\ x\in\mathcal{I}_w(R) \text{ and } f \text{ is analytic}\}$$

for all $R\in\mathfrak{A}_L$, with $\mathcal{I}_w$ acting through the left inverse translation. Because of the Iwahori decomposition, $W_{\kappa,w}$ consists of analytic functions on

$$N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})\times\begin{pmatrix} 1 & 0 & \cdots & 0\\ \mathcal{B}(0,p^w) & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{B}(0,p^w) & \mathcal{B}(0,p^w) & \cdots & 1\end{pmatrix}.$$

Therefore as a module over R we see $W_{\kappa,w}(R) = \oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}R\langle T_{ij}\rangle_{1\le j<i\le n}$, i.e. $|N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})|$ copies of strictly convergent power series in $n(n-1)/2$ variables.

From this description we see that there is a natural formal model of $W_{\kappa,w}$, whose R-points are $\oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}R\langle T_{ij}\rangle_{1\le j<i\le n}$ for $R\in\mathbf{Adm}_{\mathcal{O}_L}$, equipped with a functorial action of $I(\mathbb{Z}_p)$ and $\mathfrak{M}^\circ_{1,w}$. We denote the formal model still by $W_{\kappa,w}$.

With $W_{\kappa,w}$ we define the w-analytic $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa,w}$ in the same way as we define the $(\mathfrak{g},\mathbf{Q})$-module V_κ from the algebraic representation W_κ of $\mathrm{GL}(n)$ in §2.3. For all $R\in\mathfrak{A}_L$

$$V_{\kappa,w}(R) = W_{\kappa,w}(R)\otimes_R R[\underline{Y}] = W_{\kappa,w}(R)\otimes_R R[Y_{ij}]_{1\le i\le j\le n}.$$

The action of $g = \begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in\mathcal{Q}_w$ and $\mu^-_{ij}\in\mathfrak{u}^-$ on $P(Y)\in V_{\kappa,w}$ is given by the formulas

$$(g\cdot P)(\underline{Y}) = a\cdot P(a^{-1}b + a^{-1}\underline{Y}d), \tag{3.2.2}$$

(3.2.3)

$$(\mu^-_{ij}\cdot P)(\underline{Y}) = \sum_{1\le k\le n}(Y_{ki}\varepsilon_{kj} + Y_{kj}\varepsilon_{ki})\cdot P(Y) - \sum_{1\le k\le l\le n}(Y_{ki}Y_{jl} + Y_{kj}Y_{il})\frac{\partial}{\partial Y_{kl}}P(Y),\quad i\ne j,$$

$$(\mu^-_{ii}\cdot P)(\underline{Y}) = \sum_{1\le k\le n}Y_{ki}\varepsilon_{ki}\cdot P(\underline{Y}) - \sum_{1\le k\le l\le n}Y_{ki}Y_{il}\frac{\partial}{\partial Y_{kl}}P(\underline{Y}).$$

The compatibility is checked in the same way as in §2.3 and as $\mathcal{Q}_w$-representations there is the filtration

$$\mathrm{Fil}^r V_{\kappa,w}(R) = V^r_{\kappa,w}(R) = W_{\kappa,w}(R)\otimes_R R[\underline{Y}]_{\le r}$$

satisfying $\mathfrak{g}\cdot V^r_{\kappa,w}\subset V^{r+1}_{\kappa,w}$. By definition $V^0_{\kappa,w}=W_{\kappa,w}$ as $\mathcal{Q}_w$-representations if we regard the $\mathcal{I}_w$-representation $W_{\kappa,w}$ as a $\mathcal{Q}_w$-representation via the projection $\mathcal{Q}_w\to\mathcal{I}_w$. For $i\in\mathbb{Z}$ we can twist $V_{\kappa,w}$ by the i-th power of the multiplier ν and get the w-analytic $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa,w}(i)$.

Recall that J is defined to be the algebraic representation V^1_{triv} of $\mathbf{Q}$. It restricts to an analytic $\mathcal{Q}_w$-representation and parallel to (2.3.6) we have

$$V^r_{\kappa,w}\cong V^0_{\kappa,w}\otimes\mathrm{Sym}^r J=W_{\kappa,w}\otimes\mathrm{Sym}^r J$$

as analytic $\mathcal{Q}_w$-representations.

A little more generally, given $(\rho,W_\rho)\in\mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$, an algebraic representation of $\mathrm{GL}(n)$ free of finite rank, the tensor product $W_{\kappa\otimes\rho,w}=W_{\kappa,w}\otimes W_\rho$ is an analytic $\mathcal{I}_w$-representation, and we can define the corresponding analytic $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa\otimes\rho,w}$ and $\mathcal{Q}_w$-representation $V^r_{\kappa\otimes\rho,w}$ for $r\geq 0$.

All of the above constructions carry over to the universal w-analytic weight κ^{un} over an affinoid subdomain $\mathcal{U}\subset\mathcal{W}$.

3.3. **The Andreatta–Iovita–Pilloni construction.** We briefly recall the constructions in [AIP15, Chapter 3,4,5]. Let σ be the Frobenious endomorphism of $\mathcal{O}_L/p\mathcal{O}_L$. For any finite group scheme H over $\mathcal{O}_L$ we denote by H^D its Cartier dual and ω_H its sheaf of invariant differentials. Given a Barsotti–Tate group G over $\mathcal{O}_L$ of dimension n, the Hasse invariant $\mathrm{Ha}(G)\in\det(\omega_{G[p]^D})^{\otimes p-1}$ is defined to be the determinant of the σ-linear endomorphism on $\omega_{G[p]^D}$ induced by the relative Frobenius. The Hodge height $\mathrm{Hdg}(G)\in[0,1]$ is defined as the truncated valuation of $\mathrm{Ha}(G)$.

Let $\mathbf{NAdm}_{\mathcal{O}_L}$ be subcategory of $\mathbf{Adm}_{\mathcal{O}_L}$ consisting of those objects that are normal. Fix $R\in\mathbf{NAdm}_{\mathcal{O}_L}$ and suppose that G is a rank n semi-abelian scheme over $S=\mathrm{Spec}(R)$ whose restriction to an open dense subscheme of S is abelian. Take a positive integer $m\in\mathbb{N}_{>0}$ and $v<\frac{1}{2p^{m-1}}$ (resp. $v<\frac{1}{3p^{m-1}}$ if $p=3$) such that for any $x\in S_{\mathrm{rig}}$ the Hodge height $\mathrm{Hdg}(x):=\mathrm{Hdg}(G_x[p^\infty])\leq v$. Write R_w to denote R/p^wR. We summarize, in the following theorem, some results about canonical subgroups in families used in [AIP15].

Theorem 3.3.1. *([AIP15, Proposition 4.1.3, Proposition 4.3.1]) There is a finite flat canonical subgroup $H_m\subset G[p^m]$ of level m over S, which, at each point $x\in S_{\mathrm{rig}}$, specializes to the canonical subgroup $H_{m,x}\subset G_x[p^m]$ as constructed in [Far11, Theorem 6] . Moreover, assuming $H^D_m(R[1/p])\simeq(\mathbb{Z}/p^m\mathbb{Z})^n$, then there is a free sub-sheaf of R-modules $\mathcal{F}\subset\omega_G$ of rank n containing $p^{\frac{v}{p-1}}\omega_G$, equipped with an isomorphism*

$$\mathrm{HT}_w: H^D_m(R[1/p])\otimes_{\mathbb{Z}}R_w\xrightarrow{\sim}\mathcal{F}\otimes_R R_w,$$

induced from the Hodge–Tate map on H^D_m, for all $w\in(0,m-v\frac{p^m}{p-1}]\cap v_p(\mathcal{O}_L)$.

Fix $N\geq 3$ prime to p. Let K be a finite extension of $\mathbb{Q}_p$ with valuation v such that $v(p)=1$ and a uniformizer ϖ. Denote by Y the Siegel variety defined over $\mathcal{O}_K$ parametrizing principally polarized abelian schemes of dimension n with principal level N structure. Let X be a smooth toroidal compactification. The universal abelian scheme $A\to Y$ extends to a semi-abelian scheme $G\to X$. Set $\mathfrak{X}$ to be the formal scheme obtained by completing X along its special fibre. On the associated rigid analytic space $X_{\mathrm{rig}}=X_{\mathrm{an}}$, we have the Hodge height function $\mathrm{Hdg}: X_{\mathrm{rig}}\to[0,1]$. For $v\in v(\mathcal{O}_K)$ we define the open subset $\mathcal{X}(v)=\{x\in X_{\mathrm{rig}}:\mathrm{Hdg}(x)\leq v\}$. Let $\tilde{\mathfrak{X}}(v)$ be the admissible blow-up of $\mathfrak{X}$ along the ideal (Ha,p^v), and $\mathfrak{X}(v)$ be the p-adic completion of the normalization of the largest open formal sub-scheme of $\tilde{\mathfrak{X}}(v)$ where the ideal (Ha,p^v) is generated by Ha. This $\mathfrak{X}(v)$ is a formal model of $\mathcal{X}(v)$. By construction the semi-abelian scheme $G\to X$ gives rise to semi-abelian schemes over $\mathcal{X}(v)$ and $\mathfrak{X}(v)$, which we still denote by G. For $m\in\mathbb{N}_{>0}$ and $v<\frac{1}{2p^{m-1}}$ (resp. $v<\frac{1}{3p^{m-1}}$ if $p=3$), there is the level m canonical subgroup $H_m\subset G[p^m]$. Define $\mathcal{X}_1(p^m)(v)=\underline{\mathrm{Isom}}_{\mathcal{X}(v)}((\mathbb{Z}/p^m\mathbb{Z})^n,H^D_m)$ to be the finite étale cover of $\mathcal{X}(v)$ parametrizing the trivializations ψ of the Cartier dual of H_m. The group $\mathrm{GL}(n,\mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{X}_1(p)(v)$. The quotient $\mathcal{X}_{\rm Iw}(v) = \mathcal{X}_1(p)(v))/\,\mathbf{B}(\mathbb{Z}/p\mathbb{Z})$ by the Borel subgroup is still finite étale over $\mathcal{X}(v)$. As formal models of $\mathcal{X}_1(p^m)(v)$, $\mathcal{X}_{\rm Iw}(v)$, we take $\mathfrak{X}_1(p^m)(v)$, $\mathfrak{X}_{\rm Iw}(v)$ to be the normalizations of $\mathfrak{X}(v)$ inside the corresponding rigid spaces. There is the chain of formal schemes

$$\mathfrak{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathfrak{X}_{\rm Iw}(v) \xrightarrow{\pi_0} \mathfrak{X}(v).$$

Let $\mathfrak{Y}, \mathfrak{Y}(v), \mathfrak{Y}_1(p^m)(v), \mathfrak{Y}_{\rm Iw}(v)$ be the open formal subschemes of $\mathfrak{X}, \mathfrak{X}(v), \mathfrak{X}_1(p^m)(v), \mathfrak{X}_{\rm Iw}(v)$ that are the complements of the boundary C. Although $\mathfrak{Y}(v), \mathfrak{Y}_1(p^m)(v), \mathfrak{Y}_{\rm Iw}(v)$ are not moduli spaces, they admit modular interpretations for $R \in \mathbf{NAdm}$ (cf. [AIP15, Proposition 5.2.1.1]). Let $Y_{\rm an}$ be the analytification of Y with the natural open immersion $Y_{\rm an} \hookrightarrow X_{\rm an}$. Set $\mathcal{Y}(v), \mathcal{Y}_1(p^m)(v), \mathcal{Y}_{\rm Iw}(v)$ to be the fibre products of $\mathcal{X}(v), \mathcal{X}_1(p^m)(v), \mathcal{X}_{\rm Iw}(v)$ with $Y_{\rm an}$ over $X_{\rm an}$.

By the construction of $\mathfrak{X}_1(p^m)(v)$, we can apply Theorem 3.3.1 to construct a locally free subsheaf $\mathcal{F} \subset \omega\big(\mathcal{G}/\mathfrak{X}_1(p^m)(v)\big)$ of rank n, equipped with the isomorphism

$$\mathrm{HT}_w \circ \psi : (\mathbb{Z}/p^m\mathbb{Z})^n \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}_1(p^m)(v),w} \xrightarrow{\sim} \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w} \tag{3.3.1}$$

for $w \in (0, m - v\frac{p^m}{p-1}] \cap v(\mathcal{O}_K)$.

From now on we assume $w \in (m - 1 + \frac{v}{p-1}, m - \frac{p^m v}{p-1}] \cap v(\mathcal{O}_K)$, so m is determined by w. Define the $\mathfrak{M}^\circ_{1,w}$-torsor $\mathfrak{T}^\times_{\mathcal{F},w}(v)$ over $\mathfrak{X}_1(p^m)(v)$ by

$$\mathfrak{T}^\times_{\mathcal{F},w}(v) = \underline{\mathrm{Isom}}_{\mathfrak{X}_1(p^m)(v),\psi,w}(\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}, \mathcal{F}),$$

where the subscript ψ, w means that we require the isomorphism to be w-compatible with (3.3.1) as explained below. We always fix the canonical global basis of the n copies of the structure sheaf $\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}$ and the canonical basis of the $\mathbb{Z}/p^m\mathbb{Z}$-module $(\mathbb{Z}/p^m\mathbb{Z})^n$. Then locally over $\mathfrak{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(p^m)(v)$, an isomorphism α from R^n to $\mathcal{F}(\mathfrak{U})$ corresponds to an ordered basis $\alpha_1, \dots \alpha_n$ of the free R-module $\mathcal{F}(\mathfrak{U})$ and ψ gives rise to an ordered basis $x_1, \dots, x_n$ of $H^D_m(R[1/p])$. We say that α is w-compatible with (3.3.1) if $\alpha_i \equiv \mathrm{HT}_w(x_i) \mod p^w R$ for all $1 \le i \le n$. An element $a \in \mathfrak{M}^\circ_{1,w}(R)$ acts on α by sending it to $\alpha \circ a$, or equivalently sending the corresponding basis $(\alpha_1, \dots, \alpha_n)$ to $(\alpha_1, \dots, \alpha_n) \cdot a$. This action makes $\mathfrak{T}^\times_{\mathcal{F},w}(v)$ a $\mathfrak{M}^\circ_{1,w}$-torsor over $\mathfrak{X}_1(p^m)(v)$.

For a w-analytic weight $\kappa \in \mathcal{W}(K)$ we can form the contracted product and get a locally free formal sheaf

$$\tilde{\mathfrak{w}}^\dagger_{\kappa,w} := \mathfrak{T}^\times_{\mathcal{F},w}(v) \times^{\mathfrak{M}^\circ_{1,w}} W_{\kappa,w}$$

over $\mathfrak{X}_1(p^m)(v)$. In particular this $\tilde{\mathfrak{w}}^\dagger_{\kappa,w}$ is a flat formal Banach sheaf in the sense of [AIP15, Definition A.1.1.1]. Therefore we can apply the procedure worked out in [AIP15, A.2.2] to get the associated Banach sheaf $\tilde{\omega}^\dagger_{\kappa,w}$ over the rigid analytic fibre $\mathcal{X}_1(p^m)(v)$ (see [AIP15, Definition A.2.1.2] for the definition of a Banach sheaf). For any affinoid subdomain $\mathcal{U} \subset \mathcal{X}_1(p^m)(v)$ and an admissible blow-up $h : \mathfrak{X}' \to \mathfrak{X}_1(p^m)(v)$ such that $\mathcal{U}$ is the rigid fibre of an open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}'$, the local sections of $\tilde{\omega}^\dagger_{\kappa,w}$ over $\mathcal{U}$ are

$$\tilde{\omega}^\dagger_{\kappa,w}(\mathcal{U}) = h^*\tilde{\mathfrak{w}}^\dagger_{\kappa,w}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K,$$

which is naturally equipped with a complete norm (independent of h up to equivalence) with $\tilde{\mathfrak{w}}^\dagger_{\kappa,w}(\mathfrak{U})$ being the unit ball.

The group $I(\mathbb{Z}/p^m\mathbb{Z})$ acts on $\mathfrak{X}_1(p^m)(v)$ with $\mathfrak{X}_{\rm Iw}(v)$ as the quotient. Under this action the sheaf $\tilde{\omega}^\dagger_{\kappa,w}$ is $I(\mathbb{Z}/p^m\mathbb{Z})$-equivariant. In order to see this we need to construct the isomorphism

$\varphi_\omega(\bar\gamma) : \omega^\dagger_{\kappa,w} \to \bar\gamma^*\omega^\dagger_{\kappa,w}$ for each $\bar\gamma \in I(\mathbb{Z}/p^m\mathbb{Z})$. Let $\bar\gamma^*\mathfrak{T}^\times_{\mathcal{F},w}(v)$ be the fibre product

$$\begin{array}{ccc} \bar\gamma^*\mathfrak{T}^\times_{\mathcal{F},w}(v) & \longrightarrow & \mathfrak{T}^\times_{\mathcal{F},w}(v) \\ \downarrow & \square & \downarrow \\ \mathfrak{X}_1(p^m)(v) & \xrightarrow{\bar\gamma} & \mathfrak{X}_1(p^m)(v). \end{array}$$

We have $\bar\gamma^*\mathfrak{T}^\times_{\mathcal{F},w}(v) \cong \underline{\mathrm{Isom}}_{\mathfrak{X}_1(p^m)(v),\psi\circ\bar\gamma,w}(\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}, \bar\gamma^*\mathcal{F})$ and $\bar\gamma^*\tilde{\mathfrak{w}}^\dagger_{\kappa,w} \cong \bar\gamma^*\mathfrak{T}^\times_{\mathcal{F},w}(v) \times^{\mathfrak{M}^\circ_{1,w}} W_{\kappa,w}$. The sheaf $\omega\big(\mathcal{G}/\mathfrak{X}_1(p^m)(v)\big)$ is the pullback of $\omega\big(\mathcal{G}/\mathfrak{X}_{\mathrm{Iw}}(v)\big)$ and hence is naturally $I(\mathbb{Z}/p^m\mathbb{Z})$-equivariant. By construction the sub-sheaf $\mathcal{F} \subset \omega\big(\mathcal{G}/\mathfrak{X}_{\mathrm{Iw}}(v)\big)$ is an $I(\mathbb{Z}/p^m\mathbb{Z})$-equivariant subsheaf of $\omega\big(\mathcal{G}/\mathfrak{X}_1(p^m)(v)\big)$ so there is the isomorphism $\varphi_{\mathcal{F}}(\bar\gamma) : \mathcal{F} \to \bar\gamma^*\mathcal{F}$. Take an open formal subscheme $\mathfrak{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(p^m)(v)$ over which the sheaf $\mathcal{F}$ can be trivialized. Local sections of $\tilde{\mathfrak{w}}^\dagger_{\kappa,w}$ over $\mathfrak{U}$ are pairs (α, f) with $\alpha \in \mathfrak{T}^\times_{\mathcal{F},w}(v)(\mathfrak{U})$ and $f \in W_{\kappa,w}\hat\otimes R$ modulo the equivalence relations $(\alpha \circ a, f) \sim (\alpha, a \cdot f)$, $a \in \mathfrak{M}^\circ_{1,w}(R)$. Pick a lift $\gamma \in I(\mathbb{Z}_p)$ of $\bar\gamma$ and define

$$\begin{aligned} \varphi_{\mathfrak{w}}(\bar\gamma) : \mathfrak{T}^\times_{\mathcal{F},w}(v) \times^{\mathfrak{M}^\circ_{1,w}} W_{\kappa,w}(\mathfrak{U}) &\longrightarrow \bar\gamma^*\mathfrak{T}^\times_{\mathcal{F},w}(v) \times^{\mathfrak{M}^\circ_{1,w}} W_{\kappa,w}(\mathfrak{U}) \\ (\alpha, f) &\longmapsto (\varphi_{\mathcal{F}}(\bar\gamma) \circ \alpha \circ \gamma, \gamma^{-1} \cdot f). \end{aligned}$$

The map $\varphi_{\mathfrak{w}}(\bar\gamma)$ is well defined, independent of the choice of the lift γ, and patches to an isomorphism $\varphi_{\mathfrak{w}}(\bar\gamma) : \tilde{\mathfrak{w}}^\dagger_{\kappa,w} \to \bar\gamma^*\tilde{\mathfrak{w}}^\dagger_{\kappa,w}$. Inverting p we get $\varphi_\omega(\bar\gamma) : \tilde\omega^\dagger_{\kappa,w} \to \bar\gamma^*\tilde\omega^\dagger_{\kappa,w}$. Since $I(\mathbb{Z}/p^m\mathbb{Z})$ is a finite group, the $I(\mathbb{Z}/p^m\mathbb{Z})$-invariant of the pushforward $\pi_{1,*}\tilde\omega^\dagger_{\kappa,w}$ is a Banach sheaf over $\mathcal{X}_{\mathrm{Iw}}(v)$.

Definition 3.3.2. The Banach sheaf of w-analytic, v-overconvergent, weight κ Siegel modular forms of principal level N is defined as

$$\omega^\dagger_{\kappa,w} := \big(\pi_{1,*}\tilde\omega^\dagger_{\kappa,w}\big)^{I(\mathbb{Z}/p^m\mathbb{Z})}.$$

We also want to associate to the Banach sheaf $\omega^\dagger_{\kappa,w}$ a contracted product interpretation, which will bring us some convenience when defining some morphisms. By taking the rigid fibre of the $\mathfrak{M}^\circ_{1,w}$-torsor $\mathfrak{T}^\times_{\mathcal{F},w}(v)$ over $\mathfrak{X}_1(p^m)(v)$, we get

$$\mathcal{T}^\times_{\mathcal{F},w}(v) \xrightarrow{\pi_2} \mathcal{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathcal{X}_{\mathrm{Iw}}(v).$$

The rigid analytic space $\mathcal{T}^\times_{\mathcal{F},w}(v)$ is a $\mathcal{M}^\circ_{1,w}$-torsor over $\mathcal{X}_1(p^m)(v)$ and the cover $\pi_1 : \mathcal{X}_1(p^m)(v) \to \mathcal{X}_{\mathrm{Iw}}(v)$ is finite étale. The group $I(\mathbb{Z}_p)$ acts on $\mathcal{T}^\times_{\mathcal{F},w}(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ by sending α to $\varphi_{\mathcal{F}}(\bar\gamma) \circ \alpha \circ \gamma$. This $I(\mathbb{Z}_p)$-action together with the $\mathcal{M}^\circ_{1,w}$-torsor structure on $\mathcal{T}^\times_{\mathcal{F},w}(v)$ makes it an $\mathcal{I}_w$-torsor over $\mathcal{X}_{\mathrm{Iw}}(v)$. Let $\mathcal{S}$ be the category whose objects are affinoid subdomains of $\mathcal{X}_{\mathrm{Iw}}(v)$ admitting local sections of the projection $\pi_1 \circ \pi_2$ with inclusions as morphisms. We can define a presheaf on $\mathcal{S}$ by the contracted product

$$\mathcal{T}^\times_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}. \tag{3.3.2}$$

It is isomorphic to the restriction of the sheaf $\omega^\dagger_{\kappa,w}$ to $\mathcal{S}$. We call (3.3.2) a contracted product interpretation of $\omega^\dagger_{\kappa,w}$. Since the objects of $\mathcal{S}$ form a basis of the Grothendieck topology on $\mathcal{X}_1(p^m)(v)$, in order to construct morphisms between the sheaves over $\mathcal{X}_{\mathrm{Iw}}(v)$, it suffices to construct morphisms between their restrictions to $\mathcal{S}$. Therefore the contracted product interpretation $\mathcal{T}^\times_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}$ will be useful in constructing morphisms between sheaves that are related to $\omega^\dagger_{\kappa,w}$. By abuse of notation we will write $\omega^\dagger_{\kappa,w} = \mathcal{T}^\times_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}$.

Define the $\mathcal{O}_K$ schemes $T_\omega = \underline{\mathrm{Hom}}_X\left(\mathcal{O}^n_X, \omega(\mathcal{G}/X)\right)$ and $T^\times_\omega = \underline{\mathrm{Isom}}_X\left(\mathcal{O}^n_X, \omega(\mathcal{G}/X)\right)$ over X. Let $T_{\omega,\mathrm{an}}$, $T^\times_{\omega,an}$ be their rigid analytifications, and $\mathfrak{T}_\omega$, $\mathfrak{T}^\times_\omega$ be their formal completions along the special fibres. Also take $T_{\omega,\mathrm{rig}}, T^{\times}_{\omega,\mathrm{rig}}$ to be the rigid fibre of $\mathfrak{T}_\omega, \mathfrak{T}^{\times}_\omega$. Set $\mathcal{T}_{\omega,\mathrm{an}}(v), \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v), \mathcal{T}_{\omega,\mathrm{rig}}(v), \mathcal{T}^{\times}_{\omega,\mathrm{rig}}(v)$ to be the corresponding base changes to $\mathcal{X}_{\mathrm{Iw}}(v)$. Due to the requirement $w \in (m-1+\frac{v}{p-1}, m - v\frac{p^m}{p-1}] \cap v(\mathcal{O}_K)$, the argument of [AIP15, Proposition 5.3.1] shows that there is a natural open immersion

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v) \longrightarrow \mathcal{T}_{\omega,\mathrm{rig}}(v) \cap \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v).$$

Therefore local sections of the projection $\mathcal{T}^{\times}_{\mathcal{F},w}(v) \to \mathcal{X}_{\mathrm{Iw}}(v)$ correspond to local basis of the sheaf $\omega(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))$ satisfying w-compatibility conditions defined by the Hodge–Tate map HT_w. Note that $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ does not lie inside $\mathcal{T}^{\times}_{\omega,\mathrm{rig}}(v)$. When κ is algebraic this open immersion induces a canonical inclusion of $\omega_\kappa|_{\mathcal{X}_{\mathrm{Iw}}(v)}$ into $\omega^{\dagger}_{\kappa,w}$.

In [AIP15] another two formal schemes are introduced. They are defined as

$$\mathfrak{IW}_w(v) = \mathfrak{T}^{\times}_{\mathcal{F},w}/\mathfrak{B}_{1,w}, \qquad \text{and} \qquad \mathfrak{IW}^{+}_w(v) = \mathfrak{T}^{\times}_{\mathcal{F},w}/\mathfrak{N}_{1,w},$$

with maps

$$\mathfrak{IW}^{+}_w(v) \xrightarrow{g} \mathfrak{IW}_w(v) \xrightarrow{\pi_3} \mathfrak{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathfrak{X}_{\mathrm{Iw}}(v).$$

The group $\mathfrak{T}^{\circ}_{1,w}$ acts on $\mathfrak{IW}^{+}_w(v)$ over $\mathfrak{IW}_w(v)$, and so on the pushforward of the structure sheaf $g_*\mathcal{O}_{\mathfrak{IW}^{+}_w(v)}$. Define the invertible sheaf $\mathfrak{L}_\kappa = g_*\mathcal{O}_{\mathfrak{IW}^{+}_w(v)}[\kappa']$ to be the κ'-invariant of the $\mathfrak{T}^{\circ}_{1,w}$-action on $g_*\mathcal{O}_{\mathfrak{IW}^{+}_w(v)}$. Take the rigid fibres $\mathcal{IW}^{+}_w(v)$, $\mathcal{IW}^{+}_w(v)$, $\mathcal{L}_\kappa$. There is a $\mathbf{B}(\mathbb{Z}_p)$-action on $\mathcal{IW}^{+}_w(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ which, together with κ, makes $\pi_{3,*}\mathcal{L}_\kappa$ a $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$-equivariant Banach sheaf with respect to the natural $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$-action on $\mathcal{X}_1(p^m)(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$. In [AIP15] the invariant $(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{B(\mathbb{Z}/p^m\mathbb{Z})}$ is defined to be the Banach sheaf of w-analytic, v-overconvergent, weight κ Siegel modular forms. It is easy to see that the map $W_{\kappa,w} \to \mathbb{A}^1_K$ by evaluation at identity induces an isomorphism between $(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{B(\mathbb{Z}/p^m\mathbb{Z})}$ and the sheaf $\omega^{\dagger}_{\kappa,w}$ in Definition 3.3.2 .

All the above constructions run parallelly for the w-analytic universal weight κ^{un} corresponding to $\mathcal{U} \subset \mathcal{W}$, so that we can define the Banach sheaf $\omega^{\dagger}_{\kappa^{\mathrm{un}},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}$ and the flat formal Banach sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}$ over $\mathfrak{X}_1(p^m)(v) \times \mathfrak{U}$.

3.4. **Nearly overconvergent Siegel modular forms.**

3.4.1. *The Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ and its global sections.* Recall that in §2.4 we defined the locally free sheaf of finite rank $\mathcal{J}$ over X, and for $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ we have $\mathcal{V}^r_\rho \cong \omega_\rho \otimes \mathrm{Sym}^r\,\mathcal{J}$ as locally free sheaves with Hecke actions. Take the rigid analytification of $\mathcal{J}$ and pull it back to $\mathcal{X}_{\mathrm{Iw}}(v)$. We denote the resulting coherent sheaf over $\mathcal{X}_{\mathrm{Iw}}(v)$ still by $\mathcal{J}$. Similarly let $\mathfrak{J}$ be the locally free formal sheaf of finite rank over $\mathfrak{X}_{\mathrm{Iw}}(v)$ obtained by completing $\mathcal{J}$ along the special fibre of X and pulling it back. $\mathrm{Sym}^r\,\mathcal{J}$ is the rigid fibre of $\mathrm{Sym}^r\,\mathfrak{J}$. Since $\mathrm{Sym}^r\,\mathcal{J}$ is locally free of finite rank and $\mathcal{X}_{\mathrm{Iw}}(v)$ is quasi-compact, it can be equipped with a Banach sheaf structure by choosing a cover and local basis. All such structures are equivalent to the one given by the formal model $\mathrm{Sym}^r\,\mathfrak{J}$. The tensor product of $\mathrm{Sym}^r\,\mathcal{J}$ with a Banach sheaf is still a Banach sheaf under the tensor product semi-norm. The flatness of $\mathrm{Sym}^r\,\mathcal{J}$ guarantees that the sheaf conditions of the Banach sheaf are preserved under the operation of tensoring with $\mathrm{Sym}^r\,\mathcal{J}$. Also the spaces of local sections of the tensor product sheaf is complete with respect to the tensor product semi-norm (i.e. there is no need to take completed tensor product).

Definition 3.4.1. The Banach sheaf of w-analytic, v-overconvergent nearly holomorphic forms of principal level N, weight κ (resp. universal weight κ^{un} over $\mathcal{U} \subset \mathcal{W}$) and (non-holomorphy) degree r is defined as

$$\mathcal{V}^{\dagger,r}_{\kappa,w} := \omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r\,\mathcal{J} \qquad (\text{resp. } \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w} := \omega^{\dagger}_{\kappa^{\mathrm{un}},w} \otimes \mathrm{Sym}^r\,\mathcal{J}).$$

The space of global sections of a Banach sheaf over a quasi-compact rigid analytic space can be equipped with a norm by choosing a suitable admissible covering by affinoids. All such norms are equivalent and the space of global sections are complete under these norms.

Definition 3.4.2. The K-Banach space (resp. $\mathcal{A}(\mathcal{U})$-Banach module) of w-analytic, v-overconvergent nearly holomorphic forms of principal level N, weight κ (resp. universal weight κ^{un} over $\mathcal{U} \subset \mathcal{W}$) and (non-holomorphy) degree r and the corresponding cuspidal part is defined as

$$N^{\dagger,r}_{\kappa,w,v} := H^0(\mathcal{X}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}) \qquad (\text{resp. } N^{\dagger,r}_{\mathcal{U},w,v} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}),$$
$$N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}} := H^0(\mathcal{X}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}(-C)) \qquad (\text{resp. } N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}(-C)).$$

Following [Urb14] we also call overconvergent nearly holomorphic forms nearly overconvergent forms.

For later use we also define a locally free formal Banach sheaf $\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa,w}$ over $\mathfrak{X}_1(p^m)(v)$ by the tensor product $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathfrak{J}$. Let $\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w}$ be its rigid fibre which is an $I(\mathbb{Z}/p^m\mathbb{Z})$-equivariant Banach sheaf. Then we have $\mathcal{V}^{\dagger,r}_{\kappa,w} = \left(\pi_{1,*}\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w}\right)^{I(\mathbb{Z}/p^m\mathbb{Z})}$.

3.4.2. *The $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ and contracted product interpretation of $\mathcal{V}^{\dagger,r}_{\kappa,w}$.* The definition of the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ as $\omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}$ is already convenient for constructing unramified Hecke operators and $\mathbb{U}_p$-operators. However, for the construction of differential operators and holomorphic projections, it is preferable to have a contracted product interpretation involving a $\mathcal{Q}_w$-torsor and the $\mathcal{Q}_w$-submodule $V^r_{\kappa,w}$ of the $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa,w}$ defined in §3.2.

The $\mathcal{O}_K$-scheme $T^{\times}_{\mathcal{H}} = \underline{\mathrm{Isom}}_X\left(\mathcal{O}^{2n}_X, \mathcal{H}^1_{dR}(\mathcal{A}/Y)^{\mathrm{can}}\right)$ is defined as in §2.1. Let $T^{\times}_{\mathcal{H},\mathrm{an}}$ be its analytification and $\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)$ be the base change to $\mathcal{X}_{\mathrm{Iw}}(v)$. There is a natural projection

$$\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v) \longrightarrow \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v).$$

We define the $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ as

$$\mathcal{T}^{\times}_{\mathcal{H},w}(v) := \mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v) \times_{\mathcal{T}^{\times}_{\omega,\mathrm{an}}(v)} \mathcal{T}^{\times}_{\mathcal{F},w}(v).$$

It is not difficult to see that the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ admits the following contracted product interpretation

$$\mathcal{V}^{\dagger,r}_{\kappa,w} = \mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,w}.$$

3.4.3. *Summary.* We record below several interpretations of the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ and its global sections, which we will use later for convenience according to different purposes.

(i) $\mathcal{V}^{\dagger,r}_{\kappa,w} = \omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}$,

(ii) $\mathcal{V}^{\dagger,r}_{\kappa,w} = (\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})} \otimes \mathrm{Sym}^r \mathcal{J}$, and for global sections

$$N^{\dagger,r}_{\kappa,w,v} = H^0\left(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \mathrm{Sym}^r \mathcal{J}\right)^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})},$$

(iii) $\mathcal{V}^{\dagger,r}_{\kappa,w} = (\pi_{1,*}\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})}$, and for global sections

$$\begin{aligned} N^{\dagger,r}_{\kappa,w,v} &= H^0(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})} \\ &= \left(H^0(\mathfrak{X}_1(p^m)(v), \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w} \otimes \pi_1^* \mathrm{Sym}^r \mathfrak{J})[1/p]\right)^{I(\mathbb{Z}/p^m\mathbb{Z})}, \end{aligned}$$

(iv) $\mathcal{V}^{\dagger,r}_{\kappa,w} = \mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,w}$.

It is easy to see that in all the above constructions we can replace κ by the w-analytic universal weight κ^{un} corresponding to $\mathcal{U} \subset \mathcal{W}$, and consider the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}$ as well as the $\mathcal{A}(\mathcal{U})$-Banach module $N^{\dagger,r}_{\mathcal{U},w,v} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w})$.

In the following we need also to consider the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} := \omega^{\dagger}_{\kappa,w} \otimes \omega_\rho \otimes \mathrm{Sym}^r \mathcal{J}$ and its global sections $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ for some $(\rho, W_\rho) \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$. Here ω_ρ is the base change to $\mathcal{X}_{\mathrm{Iw}}(v)$ of the analytification of the automorphic sheaf $\mathcal{E}(W_\rho)$. From $\mathcal{E}(W_\rho)$ one also gets the locally free formal sheaf of finite rank $\mathfrak{w}_\rho$ over $\mathfrak{X}_{\mathrm{Iw}}(v)$ whose rigid fibre is ω_ρ. When working with $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w}$ and $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$, we can replace $\mathrm{Sym}^r \mathcal{J}$ and $\mathrm{Sym}^r \mathfrak{J}$ in (ii)(iii) by $\omega_\rho \otimes \mathrm{Sym}^r \mathcal{J}, \mathfrak{w}_\rho \otimes \mathrm{Sym}^r \mathfrak{J}$, and $V^r_{\kappa,w}$ in (iv) by $V^r_{\kappa\otimes\rho,w}$.

3.5. **The Banach $\mathcal{A}(\mathcal{U})$-module $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is projective.** The goal of this section is to prove the proposition below following the arguments in [AIP15, §8].

Proposition 3.5.1. *$N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is a projective Banach $\mathcal{A}(\mathcal{U})$-module. For every $\kappa \in \mathcal{U}$ with the corresponding maximal ideal $\mathfrak{m}_\kappa \subset \mathcal{A}(\mathcal{U})$ we have $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} \otimes \mathcal{A}(\mathcal{U})/\mathfrak{m}_\kappa\mathcal{A}(\mathcal{U}) \xrightarrow{\sim} N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}}$.*

Proof. We use the interpretation (iii) in §3.4.3 and the same proof works if we replace κ^{un} by $\kappa^{\mathrm{un}}\otimes\rho$ with $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$. Our case differs very little from that in [AIP15, §8]. Instead of repeating the whole proof here, we just point out the main ingredients there and explain that their arguments for the formal Banach sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ are applicable to $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C) = \tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w} \otimes \pi_1^* \mathrm{Sym}^r \mathfrak{J}(-C)$. Below for simplicity we write $\pi_1^* \mathrm{Sym}^r \mathfrak{J}$ as $\mathrm{Sym}^r \mathfrak{J}$.

We use the notation in [AIP15, §8.2]. Let $X^\star$ be the minimal compactification of Y. There is a proper morphism $X \to X^\star$. Like $\mathfrak{X}(v)$ one can define $\mathfrak{X}^\star(v)$ to be the p-adic completion of the normalization of the largest open formal subscheme of the blow-up of $\mathfrak{X}^\star$ along the ideal (Ha, p^v) where it is generated by Ha. We have the projection $\eta : \mathfrak{X}_1(p^m)(v) \to \mathfrak{X}^\star(v)$. We may assume that $\mathcal{U}$ lies inside the identity component $\mathcal{W}^\circ$ and take $\mathfrak{U}$ to be the open formal subscheme of an admissible blow-up of $\mathfrak{W}^\circ$ whose rigid fibre is $\mathcal{U}$. We use the subscript l to mean reduction modulo ϖ^l. [AIP15, Corollary 8.1.6.2] shows that $\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa^{\mathrm{un}},w}$ is a small formal Banach sheaf over $\mathfrak{X}_1(p^m)(v)$ with $\mathrm{Sym}^r \mathfrak{J}_1$ as the required coherent sheaf in the definition of small formal Banach sheaves (cf. [AIP15, Definition A.1.2.1]).

First we claim that the proposition follows from the following base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$. For all $l \in \mathbb{N}$, considering the the diagram

$$\begin{array}{ccc} X_1(p^m)(v)_l \times \mathfrak{U}_l & \xrightarrow{i} & X_1(p^m)(v)_{l+1} \times \mathfrak{U}_{l+1} \\ \downarrow{\scriptstyle \eta_l \times 1} & & \downarrow{\scriptstyle \eta_{l+1} \times 1} \\ X^\star(v)_l \times \mathfrak{U}_l & \xrightarrow{i'} & X^\star(v)_{l+1} \times \mathfrak{U}_{l+1} \end{array},$$

the base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ is

$$i'^*(\eta_{l+1} \times 1)_* \tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w,l+1}(-C) = (\eta_l \times 1)_* \tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w,l}(-C). \tag{3.5.1}$$

Once this base change property is proved, we deduce that $(\eta \times 1)_* \tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ is a small formal Banach sheaf with $(\eta \times 1)_* \mathrm{Sym}^r \mathfrak{J}_1$ as the required coherent sheaf. Then applying [AIP15, Theorem A.1.2.2] and the arguments in [AIP15, Corollary 8.2.3.1, 8.2.3.2], we conclude that the module $H^0\big(\mathcal{X}_1(p^m)(v) \times \mathcal{U}, \tilde{\mathcal{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)\big)$ is a projective Banach $\mathcal{A}(\mathcal{U})$-module and the map

$$H^0\big(\mathcal{X}_1(p^m)(v) \times \mathcal{U}, \tilde{\mathcal{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)\big) \otimes \mathcal{A}(\mathcal{U})/\mathfrak{m}_\kappa\mathcal{A}(\mathcal{U}) \longrightarrow H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger}_{\kappa,w}(-C)\big)$$

is an isomorphism. The statement of the proposition follows by taking the invariant of the finite group $I_n(\mathbb{Z}/p^m\mathbb{Z})$.

We are left to show the base change property (3.5.1). Let $V' \subset V = \mathbb{Z}^{\oplus 2n}$ be an isotropic direct factor of rank r' and $Y_{V'}$ be the V'-stratum of $X^\star$ with universal abelian scheme $A_{V'} \to Y_{V'}$. We start by recalling the description of the localization of the projection from the toroidal compactification to the minimal compactification at a point belonging to the stratum $Y_{V'}$ of $X^\star$ given in [AIP15, §8.2]. There are the abelian scheme $\mathcal{B}_{V'} \to Y_{V'}$ parametrizing the extensions of $A_{V'}$ by $V' \otimes \mathbb{G}_m$ and an isogeny $\mathcal{B}_{V'} \to A_{V'}^{r'}$ of degree a power of N. Over $\mathcal{B}_{V'}$ lies $\mathcal{M}_{V'}$ which is a torsor under the torus with character group $S_{V'}^\vee$, isogeneous to $\mathrm{Hom}(\mathrm{Sym}^2 V/V'^\perp, \mathbb{G}_m)$. Set $\mathcal{M}_{V'} \to \mathcal{M}_{V',\mathcal{S}}$ to be torus embedding associated to a polyhedral decomposition $\mathcal{S}$ of the cone $C(V/V'^\perp)$ of symmetric semi-definite bilinear forms on $V/V'^\perp$. In the same manner as in §3.3 one defines $\mathfrak{Y}_{V'}(v), \mathfrak{Y}_1(p^m)_{V'}(v), \mathfrak{B}_{V'}(v), \mathfrak{M}_{V',\mathcal{S}}(v)$. Put $\mathfrak{B}_1(p^m)_{V'}(v) = \mathfrak{B}_{V'}(v) \times_{\mathfrak{A}^r_{V'}} (\mathfrak{A}_{V'}/H_{m,V'})^r$ and $\mathfrak{M}_1(p^m)_{V',\mathcal{S}}(v) = \mathfrak{M}_{V',\mathcal{S}}(v) \times_{\mathfrak{B}_{V'}(v)} \mathfrak{B}_1(p^m)_{V'}(v)$. Take a geometric point $\bar{x} \in X^\star(v)_l$ and consider the projection $X_1(p^m)(v)_l \to X^\star(v)_l$ localized at $\bar{x}$. The completion $X_1\widehat{(p^m)}(v)_{l,\bar{x}}$ is isomorphic to a disjoint union of spaces $\mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{l,\bar{y}}/\Gamma_1(p^m)_{V'}$ with some geometric point $\bar{y} \in Y_1(p^m)_{V'}(v)_l$. The spaces fit into the diagram

$$\text{(3.5.2)}\qquad \begin{array}{ccccc} \mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{l,\bar{y}} & \xrightarrow{h_2} & \mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{l,\bar{y}}/\Gamma_1(p^m)_{V'} & \longrightarrow & X_1\widehat{(p^m)}(v)_{l,\bar{x}} . \\ \downarrow h_1 & & \downarrow & & \\ \mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}} & \longrightarrow & Y_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}} & & \end{array}$$

Because of the exact sequence

$$0 \to \tilde{\mathfrak{w}}^\dagger_{\kappa,w,1} \otimes \mathrm{Sym}^r \mathfrak{J}_1(-C) \stackrel{\varpi^{l-1}}{\to} \tilde{\mathfrak{w}}^\dagger_{\kappa,w,l} \otimes \mathrm{Sym}^r \mathfrak{J}_l(-C) \to \tilde{\mathfrak{w}}^\dagger_{\kappa,w,l-1} \otimes \mathrm{Sym}^r \mathfrak{J}_{l-1}(-C) \to 0,$$

the base change property for $\tilde{\mathfrak{V}}^\dagger_{\kappa^{\mathrm{un}},w}(-C)$ will follow from the vanishing result

$$\text{(3.5.3)}\qquad H^1\big(\mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}, \tilde{\mathfrak{w}}^\dagger_{\kappa,w,1} \otimes \mathrm{Sym}^r \mathfrak{J}_1(-C)\big) = 0$$

for all $\kappa \in \mathcal{U}$. The coherent $\mathrm{Sym}^r \mathfrak{J}$ has a filtration with graded pieces being automorphic sheaves attached to algebraic $\mathrm{GL}(n)$-representations that are free of finite rank,and the sheaf $\tilde{\mathfrak{w}}^\dagger_{\kappa,w,1}$ is an inductive limit of iterated extensions of the trivial sheaf [AIP15, Corollary 8.1.6.2]. Therefore (3.5.3) is a corollary of the general vanishing result: for all $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ and $i > 0$,

$$\text{(3.5.4)}\qquad H^i\big(\mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}, \mathfrak{w}_{\rho,1}(-C)\big) = 0,$$

where $\mathfrak{w}_{\rho,1}$ is the pullback to $\mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}$ of the automorphic sheaf ω_ρ on X. The proof of (3.5.4) is an adaption of [Lan17, §8.2] in the situation (3.5.2). It is enough to show

$$H^i\Big(\Gamma_1(p^m)_{V'}, H^j\big(\mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{1,\bar{y}}, h_2^*\mathfrak{w}_{\rho,1}(-C)\big)\Big) = 0 \quad \text{if} \quad i + j > 0.$$

Over $\mathcal{B}_{V'}$ there is the universal semi-abelian scheme

$$0 \longrightarrow V' \otimes \mathbb{G}_m \longrightarrow G_{V'} \longrightarrow A_{V'} \longrightarrow 0,$$

so using the $\mathrm{GL}(n)$-torsor $\underline{\mathrm{Isom}}_{\mathcal{B}_{V'}}\big(\mathcal{O}^n_{\mathcal{B}_{V'}}, \omega(G_{V'}/\mathcal{B}_{V'})\big)$ one constructs a locally free sheaf of finite rank $\underline{\omega}_\rho$ over $\mathcal{B}_{V'}$. Its pullback $\underline{\mathfrak{w}}_{\rho,1}$ to $\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}$ satisfies

$$h_1^*\underline{\mathfrak{w}}_{\rho,1} = h_2^*\mathfrak{w}_{\rho,1}.$$

The action of $\Gamma_1(p^m)_{V'}$ on $S_{V'}$ factors through a quotient $\Gamma'_1(p^m)_{V'}$ whose action on the set $\{\lambda \in S_{V'} \cap C(V/V'^{\perp})^\vee : \lambda > 0\}$ is free. Take S_0 to be a set of representatives of the orbits. Applying [Lan17, Lemma 8.2.3.12], [FC90, Theorem V.2.7] we get

$$\begin{aligned}
&H^i\Big(\Gamma_1(p^m)_{V'}, H^j\big(\mathcal{M}_1\widehat{(p^m)}_{V',\mathcal{S}}(v)_{1,\bar{y}}, h_2^*\mathfrak{w}_{\rho,1}(-C)\big)\Big)\\
&= H^i\Big(\Gamma_1(p^m)_{V'}, \prod_{\lambda\in S_{V'}\cap C(V/V'^{\perp})^\vee,\lambda>0} H^j\big(\mathcal{B}_1\widehat{(p^m)}_{V'}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes \underline{\mathfrak{w}}_{\rho,1}\big)\Big)\\
&= \begin{cases} \prod_{\lambda\in S_0} H^j\big(\mathcal{B}_1\widehat{(p^m)}_{V'}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes \underline{\mathfrak{w}}_{\rho,1}\big) & i=0\\ 0 & i>0\end{cases}.
\end{aligned}$$

Here $\mathcal{L}(\lambda)$ is an ample invertible sheaf over the abelian scheme $\mathfrak{B}_1(p^m)_{V'}(v)$ for $\lambda \in S_0$ [FC90, p. 143]. We reduce to show

$$H^j\big(\mathcal{B}_1\widehat{(p^m)}_{V'}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes \underline{\mathfrak{w}}_{\rho,1}\big) = 0 \quad \text{if} \quad j>0. \tag{3.5.5}$$

One observation is that, over $\mathcal{B}_{V'}$, the sheaf of invariant differentials of the torus part and the quotient abelian part of the semi-abelian scheme $G_{V'}$ can be trivialized. Hence the sheaf $\underline{\omega}_\rho$ can be constructed using a torsor of a unipotent subgroup $N_{V'} \subset \mathrm{GL}(n)$ with the $N_{V'}$-representation $\rho|_{N_{V'}}$. Then [Lan17, Lemma 8.2.4.16] says that $\rho|_{N_{V'}}$ admits a filtration with $N_{V'}$ acting trivially on each graded piece. Thus $\underline{\omega}_\rho$ is an iterated extension of the trivial sheaf, and (3.5.5) follows from the vanishing results for $H^j\big(\mathcal{B}_1\widehat{(p^m)}_{V'}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\big)$, $j>0$ [Mum70, §III.16]. □

3.6. **The differential operators.** Let $\Omega^1_{\mathcal{X}_{\mathrm{Iw}}(v)}$ be the sheaf of differentials on $\mathcal{X}_{\mathrm{Iw}}(v)$ defined as in [FvdP04, Ex. 4.4.1]. Over $\mathcal{X}_{\mathrm{Iw}}(v)$ we have the integrable Gauss–Manin connection

$$\nabla : \mathcal{H}^1_{dR}(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))^{\mathrm{can}} \to \mathcal{H}^1_{dR}(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))^{\mathrm{can}} \otimes \Omega^1_{\mathcal{X}_{\mathrm{Iw}}(v)}(\log C).$$

For a w-analytic weight $\kappa \in \mathcal{W}(\mathbb{C}_p)$ and $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$, we defined in §3.2 the $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa\otimes\rho,w}$. The Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} = \omega^\dagger_{\kappa,w}\otimes\omega_\rho\otimes\mathrm{Sym}^r\mathcal{J}$ on $\mathcal{X}_{\mathrm{Iw}}(v)$ has the contracted product interpretation $\mathcal{T}^\times_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w} V_{\kappa\otimes\rho,w}$. Using this contracted product interpretation and the construction in §2.2, we obtain a connection

$$\nabla_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \longrightarrow \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}\otimes\Omega_{\mathcal{X}_{\mathrm{Iw}}(v)}(\log C) \cong \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}(-1).$$

Recall that τ is the symmetric square of the standard representation of $\mathrm{GL}(n)$. Composing it with $t^+ : \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}(-1) \to \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}$ we get the following differential operator which can be thought of as an p-adic analytic version of the Maass–Shimura differential operators

$$D_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \longrightarrow \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}.$$

Besides, there is the Shimura's E-operator [Shi00, §12.9], whose construction relies only on the fact that we have the morphism of $\mathcal{Q}_w$-representations

$$V^r_{\kappa\otimes\rho,w}/V^0_{\kappa\otimes\rho,w} \longrightarrow V^{r-1}_{\kappa\otimes\rho,w}\otimes V^0_{\tau^\vee}(1) = V^{r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}(1).$$

To be explicit, let $\underline{Z} = (Z_{ij})_{1\le i,j\le n}$ be the basis of $\tau^\vee$ with $a \in \mathrm{GL}(n)$ acting on $\underline{Z}$ by $a^{-1}\,\underline{Z}{}^{\mathrm{t}}a^{-1}$. Then the morphism is given by $\sum_{1\le i\le j\le n} Z_{ij}\frac{\partial}{\partial Y_{ij}}$. The r-th iteration divided by $r!$ is an isomorphism

$$\frac{1}{r!}\Big(\sum_{1\le i\le j\le n} Z_{ij}\frac{\partial}{\partial Y_{ij}}\Big)^r : V^r_{\kappa\otimes\rho,w}/V^{r-1}_{\kappa\otimes\rho,w} \xrightarrow{\sim} V^0_{\kappa\otimes\rho\otimes\mathrm{Sym}^r\tau^\vee,w}(r). \tag{3.6.1}$$

We write the induced operator on the Banach sheaves as

$$\epsilon_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \longrightarrow \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}(1),$$

and its composition with $t^-: \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}(1) \to \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}$ as

$$E_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \longrightarrow \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}(1) \xrightarrow{t^-} \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}.$$

We can also iterate the operators and obtain

$$D^e_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\rho,w} \longrightarrow \mathcal{V}^{\dagger,r+e}_{\kappa\otimes\rho\otimes\mathrm{Sym}^e\,\tau,w},$$
$$E^e_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\rho,w} \longrightarrow \mathcal{V}^{\dagger,r-e}_{\kappa\otimes\rho\otimes\mathrm{Sym}^e\,\tau^\vee,w},$$

for $e \in \mathbb{N}$. A section of the sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w}$ lies inside $\mathcal{V}^{\dagger,r'}_{\kappa\otimes\rho,w}$ for $0 \leq r' < r$ if and only is it is annihilated by $E^{r'+1}_{\kappa\otimes\rho,w}$.

3.7. **The holomorphic projection.** Besides the definition of the space of nearly holomorphic forms, its algebraic structure and the Maass–Shimura differential operators, another main ingredient in Shimura's theory of nearly holomorphic forms is the holomorphic projection. Shimura's construction [Shi00, Proposition 14.2] can be adapted to our p-adic analytic context.

Define the functions $\mathrm{Log}_1, \dots, \mathrm{Log}_n$ on the weight space $\mathcal{W}$ by

$$\mathrm{Log}_i(\kappa) := \frac{\log_p(\kappa_i(1+p)^t)}{\log_p((1+p)^t)},$$

for $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathcal{W}$ and some $t \in \mathbb{N}$ sufficiently large. Let $K(\mathrm{Log}_1, \dots, \mathrm{Log}_n)$ be the the fraction field of $K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$. For an affinoid subdomain $\mathcal{U} \subset \mathcal{W}$ such that $\kappa^{\mathrm{un}}|_{\mathcal{U}}$ is w-analytic, we prove in this section the following proposition.

Proposition 3.7.1. *There is an $\mathcal{A}(\mathcal{U})$-linear continuous map*

$$\mathscr{A} : N^{\dagger,r}_{\mathcal{U},w,v} \longrightarrow N^{\dagger,0}_{\mathcal{U},w,v} \otimes_K K(\mathrm{Log}_1, \dots, \mathrm{Log}_n)$$

whose restriction to $N^{\dagger,0}_{\mathcal{U},w,v}$ is the identity.

In order to simplify notation for the rest of this section we omit all the subscripts from the differential operators and E-operators as well as the subscript w from $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\,\tau\otimes\mathrm{Sym}^{e'}\,\tau^\vee,w}$.

Suppose $\mathrm{Spm}(R) \subset \mathcal{X}_{\mathrm{Iw}}(v)$ is an affinoid subdomain such that there exists a section $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$, and we regard α as a basis $(\alpha_1, \dots, \alpha_{2n})$ of $\mathcal{H}^1_{dR}(A/R)$ satisfying certain conditions. Given $D \in \mathrm{Der}_K(R,R)$ in order to decide the action of $\nabla(D)$ on sections of $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\rho}$ over $\mathrm{Spm}(R)$, we need to consider the element $X(D,\alpha) \in \mathfrak{g} \otimes R$ defined by

$$\nabla(D)\alpha = \alpha \cdot X(D,\alpha).$$

Let $\overline{X(D,\alpha)}$ denote the image of $X(D,\alpha)$ inside the quotient $\mathfrak{g}/\mathfrak{q} \cong \mathfrak{u}^-$. The Levi subgroup $\mathbf{M}$ acts on $\mathfrak{u}^-$ by conjugation. Hence $a \in \mathrm{GL}(n,R)$ acts on $\overline{X(D,\alpha)}$ by sending it to ${}^{\mathrm{t}}a^{-1}\overline{X(D,\alpha)}a^{-1}$. This $\mathrm{GL}(n)$-action is isomorphic to $\tau^\vee$. Given $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$ and a basis $\{e_i\}_{1\leq i\leq n(n+1)/2}$ of the $\mathrm{GL}(n)$-representation τ, the dual basis $\{e_i^\vee\}_{1\leq i\leq n(n+1)/2}$ gives rise to a basis $\left\{D_{e_i^\vee,\alpha}\right\}_{1\leq i\leq n(n+1)/2}$ of the tangent space $\mathrm{Der}_K(R,R)$. One can check by definition that the element $\overline{X(D_{e_i^\vee,\alpha},\alpha)}$ inside $\mathfrak{u}^- \otimes R$ is independent of the choice of $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$, and we abbreviate it as $\overline{X(e_i^\vee)}$.

Lemma 3.7.2. $\left\{\overline{X(D_{e_i^\vee})}\right\}_{1\leq i\leq n(n+1)/2}$ *form a basis of $\mathfrak{u}^- \otimes R \cong \tau^\vee(R)$, which is dual to the basis* $\{e_i\}_{1\leq i\leq n(n+1)/2}$.

Proof. The statement is an equality statement and does not depend on the choice of $\{e_i\}_{1\le i\le n(n+1)/2}$. Hence it suffices to prove it for the Siegel variety Y, and we can further reduce to the Siegel upper half space $\mathfrak{h}_n$ and take α to be the holomorphic basis (dw,β) of $\mathcal{H}^1_{dR}(A_{\mathfrak{h}_n}/\mathfrak{h}_n)$ constructed in §2.5. Denote by KS the Kodaira–Spencer map. Explicit computation using (2.5.4) shows that

$$\mathrm{KS}(dw_i dw_j) = 2\pi i\cdot dz_{ij}\quad 1\le i\le j\le n. \tag{3.7.1}$$

Put $\underline{X}=(X_{ij})$ as in §2.6. Then $(X_{ij})_{1\le i\le j\le n}$ can be regarded as a basis spanning the representation τ. It is dual to the basis μ^-_{ij} of $\mathfrak{u}^-$. (3.7.1) shows that dz_{ij} corresponds to X_{ij} under the basis (dw,β) so $\partial/\partial z_{ij} = D_{X^\vee_{ij},(dw,\beta)}$. By (2.5.4) we have $\overline{X(X^\vee_{ij})}=\mu^-_{ij}$ and the statement is proved. □

The morphism $\tau\otimes\tau^\vee\to\mathrm{triv}$ of $\mathrm{GL}(n)$-representations induces the contraction operator

$$\theta^e:\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau\otimes\mathrm{Sym}^e\tau^\vee}\longrightarrow\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}}.$$

Lemma 3.7.3. *The composition*

$$E^e\theta^e D^e:\mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}\xrightarrow{D^e}\mathcal{V}^{\dagger,e}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee\otimes\mathrm{Sym}^e\tau}\xrightarrow{\theta^e}\mathcal{V}^{\dagger,e}_{\kappa^{\mathrm{un}}}\xrightarrow{E^e}\mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}$$

is an $\mathcal{O}_{\mathcal{X}_{\mathrm{Iw}(v)}\times\mathcal{U}}$-linear morphism of Banach sheaves over $\mathcal{X}_{\mathrm{Iw}(v)}\times\mathcal{U}$, induced by an endomorphism of the $\mathcal{Q}_w$-representation $V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}$.

Proof. There exists a contraction map $\tilde\theta^e:\mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee\otimes\mathrm{Sym}^e\tau\otimes\mathrm{Sym}^e\tau^\vee}\to\mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}$ induced from a morphism of the corresponding representations such that $E^e\theta^eD^e=\tilde\theta^eE^eD^e$. Therefore it is enough to show that the map $E^eD^e:\mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}\longrightarrow\mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee\otimes\mathrm{Sym}^e\tau\otimes\mathrm{Sym}^e\tau^\vee}$ is induced from a morphism of $\mathcal{I}_w$-representations. Still take $\underline{X}=(X_{ij})$ as a basis of τ and write $V_{\kappa^{\mathrm{un}},w}=W_{\kappa^{\mathrm{un}},w}[\underline{Y}]$ with $\underline{Y}=(Y_{ij})_{1\le i\le j\le n}$ as in §3.2. Locally over $\mathrm{Spm}(R)\subset\mathcal{X}_{\mathrm{Iw}}(v)$, we fix a section $\alpha\in\mathcal{T}^\times_{\mathcal{H},w}(v)(R)$ and let $D_{X^\vee_{ij},\alpha}$ be the basis of $\mathrm{Der}_K(R,R)$ associated to $X^\vee_{ij}$ and α. With these choices of local coordinates the map E^eD^e can be written as

$$E^eD^e:\mathcal{T}^\times_{\mathcal{H},w}(v)(R)\times^{\mathcal{Q}_w(R)}V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}(R)\longrightarrow\mathcal{T}^\times_{\mathcal{H},w}(v)(R)\times^{\mathcal{Q}_w(R)}V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee\otimes\mathrm{Sym}^e\tau\otimes\mathrm{Sym}^e\tau^\vee}(R)$$
$$(\alpha,u)\longmapsto(\alpha,P_{\alpha,u,e}(\underline{X},\underline{Y})),$$

with $P_{\alpha,u,e}(\underline{X},\underline{Y})$ being a homogenous polynomial of degree e in $\underline{X}$ and degree e in $\underline{Y}$ whose coefficients lie in $V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}(R)$. The claim that E^eD^e is induced from a morphism of $\mathcal{I}_w$-representations is equivalent to the equality

$$a\cdot\left(P_{\alpha,u,e}(\underline{X},\underline{Y})\right)=P_{\alpha\cdot a,u,e}(\underline{X},\underline{Y}), \tag{3.7.2}$$

for all $a\in\mathcal{I}_w(R)$ and $u\in V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}(R)$. Note that by (2.2.2) the operator E^e annihilates all terms in $D^e((\alpha,u))$ involving derivations of the base ring R or the action of $\mathfrak{q}\subset\mathfrak{g}$, because they do not increase the degree in $\underline{Y}$. We get

$$P_{\alpha,u,e}(\underline{X},\underline{Y})=\sum_{1\le i\le j\le n}\left(\overline{X(X^\vee_{ij})}\cdot P_{\alpha,u,e-1}(\underline{X},\underline{Y})\right)X_{ij},$$

where $\overline{X(X_{ij}^\vee)}\cdot$ is regarded as an element of $\mathfrak{u}^-$ through $\mathfrak{u}^- \cong \mathfrak{g}/\mathfrak{q}$. We show (3.7.2) by induction. The $e=0$ case is true by definition of the contracted product. Assuming it is true for $e-1$, then

$$\begin{aligned}
a\cdot P_{\alpha,u,e}(\underline{X},\underline{Y}) &= a\cdot \sum_{1\le i\le j\le n}\left(\overline{X(X_{ij}^\vee)}\cdot P_{\alpha,u,e-1}(\underline{X},\underline{Y})\right)X_{ij}\\
&= \sum_{1\le i\le j\le n}\left(\left({}^{\mathrm{t}}a^{-1}\overline{X(X_{ij}^\vee)}a^{-1}\right)\cdot\left(a\cdot P_{\alpha,u,e-1}(\underline{X},\underline{Y})\right)\right)\left(a\cdot X_{ij}\right)\\
&= \sum_{1\le i\le j\le n}\left(\overline{X(X_{ij}^\vee)}\cdot\left(a\cdot P_{\alpha,u,e-1}(\underline{X},\underline{Y})\right)\right)X_{ij}\\
&= \sum_{1\le i\le j\le n}\left(\overline{X(X_{ij}^\vee)}\cdot P_{\alpha\cdot a,u,e-1}(\underline{X},\underline{Y})\right)X_{ij}\\
&= P_{\alpha\cdot a,u,e}(\underline{X},\underline{Y}).
\end{aligned}$$

The second equality uses the compatibility of the action of $\mathfrak{g}$ and $\mathcal{I}_w$ and the third equality follows from Lemma 3.7.2. ☐

Denote by $\varphi(\kappa^{\mathrm{un}},e)$ the endomorphism of $W_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee} = V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}$ giving rise to $E^e\theta^eD^e$.

Lemma 3.7.4. *There exists an element $\widetilde{\varphi}\in\mathrm{End}(W_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee,w})$ and a nonzero $\eta\in K[\mathrm{Log}_1,\dots,\mathrm{Log}_n]$ such that $\widetilde{\varphi}\circ\varphi(\kappa^{\mathrm{un}},e)=\varphi(\kappa^{\mathrm{un}},e)\circ\widetilde{\varphi}=\eta$.*

Proof. As an $\mathcal{A}(\mathcal{U})$-Banach module, we have $W_{\kappa^{\mathrm{un}},w}\cong\oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}\mathcal{A}(\mathcal{U})\left\langle\underline{T}\right\rangle$, the direct sum of $|N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})|$ copies of strictly convergent power series in $\underline{T}$ with $\underline{T}=(T_{ij})_{1\le i<j\le n}$. Let $W^0=\mathcal{A}(\mathcal{U})[\underline{T}]$ be the polynomial part of one copy. Fix a basis $\underline{Z}=(Z_{ij})_{1\le i,j\le n}$, $Z_{ij}=Z_{ji}$ of $\tau^\vee$ with $a\in\mathrm{GL}(n)$ acting by $a\cdot\underline{Z}={}^{\mathrm{t}}a^{-1}\,\underline{Z}a^{-1}$. Then $W_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee,w}\cong\oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}\mathcal{A}(\mathcal{U})[\underline{Z}]_e\left\langle\underline{T}\right\rangle$ where the subscript e means homogenous polynomials of degree e. Like W^0, set $W^0_e=W^0\otimes\mathrm{Sym}^e\tau^\vee=\mathcal{A}(\mathcal{U})[\underline{Z}]_e[\underline{T}]$. Both W^0 and W^0_e are closed under the action of $\mathfrak{gl}(n)$. Only Lie algebra action is involved in the differential operators, so $\varphi(\kappa^{\mathrm{un}},e)$ restricts to an endomorphism of the $\mathfrak{gl}(n)$-module W^0_e. We can write W^0_e as a direct sum of its weight spaces $W^0_e=\oplus_\lambda W^0_{e,\lambda}$ and each $W^0_{e,\lambda}$ is free of finite rank generated by some monomials of the form $\prod_{1\le i<j\le n}T_{ij}^{s_{ij}}\cdot\prod_{1\le k\le l\le n}Z_{kl}^{t_{kl}}$, $s_{ij},t_{kl}\ge0$, $\sum t_{kl}=e$. The endomorphism $\varphi(\kappa^{\mathrm{un}},e)$ restricts to an $\mathcal{A}(\mathcal{U})$-linear map $\varphi_\lambda: W^0_{e,\lambda}\to W^0_{e,\lambda}$ for each λ and the corresponding matrix, with respect to the basis consisting of monomials, has entries in $\mathcal{O}_K[\mathrm{Log}_1,\dots,\mathrm{Log}_n]$. The first claim is that the determinant of φ_λ in non-zero. For $\kappa\in\mathcal{U}$ write $\varphi_{\lambda,\kappa}$ to denote the specialization of φ_λ at κ. Fix an arbitrary $\kappa=(\kappa_1,\dots,\kappa_n)\in\mathcal{U}$ and consider $\kappa+k=(\kappa_1+k,\dots,\kappa_n+k)$ with k varying in $\mathbb{N}$. Set $Q(k)$ to be the determinant of $\varphi_{\lambda,\kappa+k}$. It is a polynomial in k and is non-zero as observed in [Shi00, (14.3)]. Hence the determinant of φ_λ cannot be zero. Then in order to show the existence of the $\tilde{\varphi}$, it suffices to show that there exists $\eta\in\mathcal{O}_K[\mathrm{Log}_1,\dots,\mathrm{Log}_n]$ such that the minimal polynomial P_λ of φ_λ divides η in $\mathcal{O}_K[\mathrm{Log}_1,\dots,\mathrm{Log}_n]$ for all λ. Let L be the algebraic closure of the field $K(\mathrm{Log}_1,\dots,\mathrm{Log}_n)$. For a generic $\kappa\in\mathcal{U}$, the specialization W^0_κ of W^0 at κ is isomorphic to the irreducible Verma module with highest weight κ. According to [BGG71, Lemma 5], for generic κ, the $\mathfrak{gl}(n)$-module $W^0_{e,\kappa}=W^0_\kappa\otimes\mathrm{Sym}^e\tau^\vee$ admits a Jordan-Hölder series with irreducible Verma modules as graded pieces and the length is finite, independent of κ. Let l be this length. It follows that the subset of L, consisting all the eigenvalues of φ_λ for all λ, is finite, and also for each vector $u\in W^0_{e,\lambda,\kappa}$ with κ generic, the dimension of the space $\mathrm{Span}\{\varphi^m_{\lambda,\kappa}(u):m\in\mathbb{N}\}$ is bounded by l. Therefore as λ varies the degree of the minimal polynomial P_λ is uniformly bounded and all the roots are contained in a finite set. This implies the existence of the desired $\eta\in\mathcal{O}_K[\mathrm{Log}_1,\dots,\mathrm{Log}_n]$. ☐

Proof of Proposition 3.7.1. Let $\tilde{\varphi}, \eta$ be as in the previous lemma for $e = r$. Then $\eta^{-1}\tilde{\varphi}$ induces the morphism

$$\Phi_r : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^r \tau^\vee, w} \longrightarrow \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^r \tau^\vee, w} \otimes_K K(\mathrm{Log}_1, \dots, \mathrm{Log}_n),$$

which is the inverse of $E^r\theta^r D^r$. Set $\mathscr{A}_r = 1 - \theta^r D^r \Phi_r E^r$. Then

$$E^r \mathscr{A}_r = E^r(1 - \theta^r D^r \Phi_r E^r) = E^r - (E^r\theta^r D^r \Phi_r)E^r = E^r - E^r = 0,$$

showing that $\mathscr{A}_r$ sends $N^{\dagger,r}_{\mathcal{U},w,v}$ into $N^{\dagger,r-1}_{\mathcal{U},w,v} \otimes_K K(\mathrm{Log}_1, \dots, \mathrm{Log}_n)$. Meanwhile $\mathscr{A}_r$ is identity on $N^{\dagger,r-1}_{\mathcal{U},w,v}$ because E^r annihilates $N^{\dagger,r-1}_{\mathcal{U},w,v}$. By induction we obtain the desired $\mathscr{A} = \mathscr{A}_1 \circ \mathscr{A}_2 \circ \cdots \circ \mathscr{A}_r$. $\square$

Corollary 3.7.5. *There exists a nonzero $\eta \in K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$ such that each $F \in N^{\dagger,r}_{\mathcal{U},w,v}$ can be written as*

$$\eta F = F_0 + \theta D F_1 + \cdots + \theta^r D^r F_r$$

with $F_i \in N^{\dagger,0}_{\mathcal{U}\otimes \mathrm{Sym}^i \tau^\vee, w, v}$.

3.8. **Unramified Hecke operators.** Let ℓ be a prime integer with $(\ell, Np) = 1$. For $\gamma_\ell \in \mathrm{GSp}(2n, \mathbb{Z}_\ell)\backslash \mathrm{GSp}(2n, \mathbb{Q}_\ell)/\mathrm{GSp}(2n, \mathbb{Z}_\ell)$, the action of the Hecke operator T_{γ_ℓ} on $N^{\dagger,r}_{\kappa,w,v}$ can be defined in the standard way using algebraic correspondence of ℓ-(quasi-)isogenies of type γ_ℓ. Let $Y_{\mathrm{Iw},K}$ be the moduli scheme over K parametrizing principally polarized abelian schemes (A, λ) with a principal level N structure and a self-dual full flag $\mathrm{Fil}_\bullet A[p]$. Define $C_{\gamma_\ell} \subset Y_{\mathrm{Iw},K} \times Y_{\mathrm{Iw},K}$ to be the moduli space, whose R-points $C_{\gamma_\ell}(R)$ for any K-algebra R consists of (quasi-)isogenies

$$\pi : (A_1, \lambda_1, \psi_{N,1}, \mathrm{Fil}_\bullet A_1[p]) \to (A_2, \lambda_2, \psi_{N,2}, \mathrm{Fil}_\bullet A_2[p])$$

of type γ_ℓ with degree being a power of ℓ. Here for $i = 1, 2$, $\lambda_i, \psi_{N,i}$ and $\mathrm{Fil}_\bullet A_i[p]$ need to satisfy $\pi^*\lambda_2 = \nu(\gamma_\ell)\lambda_1$, $\pi \circ \psi_{N,1} = \psi_{N,2}$, $\pi \circ \mathrm{Fil}_\bullet A_1[p] = \mathrm{Fil}_\bullet A_2[p]$. Being of type γ_ℓ means that under certain $\mathbb{Z}_\ell$-basis of the Tate modules $T_\ell(A_i)$, the matrix of the morphism induced by π on Tate modules is γ_ℓ. Denote by p_1 (resp. p_2) the projection of C_{γ_ℓ} to the first (resp. second) factor. Put $\mathcal{C}_{\gamma_\ell}(v) = C_{\gamma_\ell,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},K,\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. Then we have the picture

(3.8.1)
$$\begin{array}{ccccc} & & \mathcal{C}_{\gamma_\ell}(v) & & \\ & {}^{p_1}\swarrow & & \searrow^{p_2} & \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & & & \mathcal{Y}_{\mathrm{Iw}}(v). \end{array}$$

Write $p_i^*\mathcal{T}^\times_{\mathcal{H},w}(v) = \mathcal{C}_{\gamma_\ell}(v) \times_{p_i, \mathcal{Y}_{\mathrm{Iw}}(v)} \mathcal{T}^\times_{\mathcal{H},w}(v)$. Due to the functoriality of the Hodge–Tate map and the canonical subgroups, the (quasi-)isogeny π induces an isomorphism $\pi^* : p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \to p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$ (cf. [AIP15, Lemma 6.1.1]). Applying π^* to the first factor of the contracted product $p_i^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,w}$ we obtain

$$\pi^* : p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w} \xrightarrow{\sim} p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}.$$

The Hecke operator T_{γ_ℓ} is defined as the composition

$$H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{p_{2*}} H^0(\mathcal{C}_{\gamma_\ell}(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{\pi^*} H^0(\mathcal{C}_{\gamma_\ell}(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{\mathrm{Tr}\, p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}).$$

Such defined T_{γ_ℓ} maps bounded functions to bounded functions so it defines an action on $N^{\dagger,r}_{\kappa,w,v}$ by the discussion of [AIP15, §5.5]. Its action also preserves the cuspidal part (see Remark 3.9.5).

3.9. **The $\mathbb{U}_p$-operators.** Let $T^+ = \{\mathrm{diag}(p^{a_1}, \dots, p^{a_n}, p^{a_0-a_1}, \dots, p^{a_0-a_n}) \in \mathbf{T}(\mathbb{Q}) : a_1 \leq \dots \leq a_n, a_0 \geq 2a_n\}$. Set

$$\gamma_{p,i} = \begin{pmatrix} I_i & 0 & 0 & 0 \\ 0 & pI_{n-i} & 0 & 0 \\ 0 & 0 & p^2 I_i & 0 \\ 0 & 0 & 0 & pI_{n-i} \end{pmatrix} \quad 1 \leq i \leq n-1 \quad \text{and} \quad \gamma_{p,n} = \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix}. \tag{3.9.1}$$

We want to attach a Hecke operator to each element of T^+. All such operators will be called $\mathbb{U}_p$-operators. An element $\gamma_p \in T^+$ can be uniquely written as $\gamma_p = p^{s_0} \prod_{j=1}^n \gamma_{p,j}^{s_j}$ with $s_0 \in \mathbb{Z}$, $s_1 \dots, s_n \in \mathbb{N}$. We make the scalar p act on $Y_{\mathrm{Iw},K}$ by sending $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ to $(A, \lambda, \psi_N \circ p, \mathrm{Fil}_\bullet A[p])$. This action is invertible and induces a map on the global sections of the sheaf $\mathcal{V}_{\kappa,w}^{\dagger,r}$, which we take as the Hecke operator corresponding to $p \in T^+$ and denote by $\langle p \rangle$. We define the Hecke operator attached to p^{s_0} as $\langle p \rangle^{s_0}$ for all $s_0 \in \mathbb{Z}$. It remains to define the operators $U_{p,i}$ associated to $\gamma_{p,i}$ for $1 \leq i \leq n$.

3.9.1. *The operator $U_{p,n}$.* Let $C_n \subset Y_{\mathrm{Iw},K} \times Y_{\mathrm{Iw},K}$ be the moduli space parametrizing the quintuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$, with $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ being the moduli problem defining $Y_{\mathrm{Iw},K}$ and $L \subset A[p]$ satisfying $L \oplus \mathrm{Fil}_n A[p] = A[p]$. Denote by $\pi : A \to A/L$ the universal isogeny. There are two projections p_1, p_2 from C_n to $Y_{\mathrm{Iw},K}$. The first one is by forgetting L, and the other sends $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ to $(A/L, \lambda', \pi \circ \psi_N, \mathrm{Fil}_\bullet A/L[p])$, with λ' defined by $\pi^*\lambda' = p\lambda$ and $\mathrm{Fil}_i A/L[p] = \pi \circ \mathrm{Fil}_i A[p]$, $1 \leq i \leq n$. Consider $\mathcal{C}_n(v) = C_{n,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},K}} \mathcal{Y}_{\mathrm{Iw}}(v) \subset \mathcal{Y}_{\mathrm{Iw}}(v) \times \mathcal{Y}_{\mathrm{Iw}}(v)$, which parametrizes $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ with $\mathrm{Hdg}(A[p^\infty]) \leq v$ and $\mathrm{Fil}_n A[p] = H_1$, the level 1 canonical subgroup. According to [Far11, Theorem 8], there is the diagram

$$\begin{array}{ccc} & \mathcal{C}_n(v) & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(p)(\frac{v}{p}). \end{array} \tag{3.9.2}$$

The universal isogeny π induces an isomorphism $\pi^* : p_2^* \mathcal{T}_{\mathcal{H},w}^{\times}(\frac{v}{p}) \to p_1^* \mathcal{T}_{\mathcal{H},w}^{\times}(v)$ (cf. [AIP15, Lemma 6.2.1.2]) that gives rise to $\pi^* : p_2^* \mathcal{V}_{\kappa,w}^{\dagger,r} \xrightarrow{\sim} p_1^* \mathcal{V}_{\kappa,w}^{\dagger,r}$. The operator $U_{p,n}$ is defined as the composition

$$\begin{aligned} H^0(\mathcal{Y}_{\mathrm{Iw}}(p)(\frac{v}{p}), \mathcal{V}_{\kappa,w}^{\dagger,r}) &\xrightarrow{p_2*} H^0(\mathcal{C}_n(v), p_2^* \mathcal{V}_{\kappa,w}^{\dagger,r}) \xrightarrow{\pi^*} H^0(\mathcal{C}_n(v), p_1^* \mathcal{V}_{\kappa,w}^{\dagger,r}) \\ &\xrightarrow{p^{-n(n+1)/2}} \mathrm{Tr}\, p_1 \; H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}_{\kappa,w}^{\dagger,r}). \end{aligned} \tag{3.9.3}$$

See §3.9.5 for the normalizer $p^{-n(n+1)/2}$.

3.9.2. *The operators $U_{p,i}$, $i = 1, \dots, n-1$.* First for $\underline{w} = (w_{jk})_{1 \leq k < j \leq n}$ satisfying

(i) $w_{jk} = w$ or $w - 1$ for some w as before,
(ii) $w_{j+1,k} \geq w_{j,k}$, and $w_{j,k-1} \geq w_{jk}$,

we introduce the $\underline{w}$-analyticity, generalizing the w-analyticity for a scalar w. Recall $N^-(\mathbb{Z}_p) \subset I(\mathbb{Z}_p)$ is the subgroup of lower triangular elements with 1 as diagonal entries. Let $\mathcal{N}_{\underline{w}}^-$ be the rigid analytic group

$$N^-(\mathbb{Z}_p) \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mathcal{B}(0, p^{w_{21}}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}(0, p^{w_{n1}}) & \mathcal{B}(0, p^{w_{n2}}) & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p\mathbb{Z}_p + \mathcal{B}(0, p^{w_{21}}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p\mathbb{Z}_p + \mathcal{B}(0, p^{w_{n1}}) & p\mathbb{Z}_p + \mathcal{B}(0, p^{w_{n2}}) & \cdots & 1 \end{pmatrix}.$$

Then $\mathcal{I}'_{\underline{w}} = \mathcal{N}^-_{\underline{w}} \mathcal{T}^\circ_{w-1} \mathbf{N}_{an}$ is a rigid analytic space with the group $\mathcal{T}^\circ_{w-1} \mathbf{N}_{\mathrm{an}}$ acting by the right multiplication. Due to the requirement (i)(ii) on $\underline{w}$, the space $\mathcal{I}'_{\underline{w}}$ is also stable under the left multiplication by the group $\mathcal{I}_w$. Like in §3.2 we define the $\mathcal{I}_w$-module $W_{\kappa,\underline{w}}$ by

$$W_{\kappa,\underline{w}}(R) = \left\{ \begin{array}{l} f : \mathcal{I}'_{\underline{w}}(R) \to R, \ \ f|_{\mathcal{N}^-_{\underline{w}}} \text{ is analytic and } f(xtn) = \kappa'(t)f(x) \\ \text{for all } x \in \mathcal{I}'_{\underline{w}}(R), t \in \mathcal{T}^\circ_{w-1}(R), n \in \mathbf{N}_{an}(R) \end{array} \right\}$$

for all $R \in \mathfrak{A}_L$. The group $\mathcal{I}_w$ acts on it through the left inverse translation. We write $\underline{w}' \leq \underline{w}$ if $w'_{jk} \leq w_{jk}$ for all $1 \leq k < j \leq n$. For $\underline{w}' \leq \underline{w}$ the module $W_{\kappa,\underline{w}'}$ is contained in $W_{\kappa,\underline{w}}$, and elements in $W_{\kappa,\underline{w}'}$ satisfy stronger analyticity condition. By the same formulas as (3.2.2), (3.2.3) we define $V_{\kappa,\underline{w}}$. The contracted products $\mathcal{T}^\times_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,\underline{w}}$ and $\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V_{\kappa,\underline{w}}$ define sub-Banach sheaves $\omega^\dagger_{\kappa,\underline{w}}$ of $\omega^\dagger_{\kappa,w}$, and $\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}$ of $\mathcal{V}^{\dagger,r}_{\kappa,w}$.

Next we extend the action of $\mathcal{I}_w$ on $W_{\kappa,\underline{w}}$ to $\Delta^-_{I,w} = \mathcal{I}_w T^{\circ -} \mathcal{I}_w$, where $T^{\circ -} = \{\mathrm{diag}(p^{b_1}, \dots, p^{b_n}) \in \mathrm{GL}(n, \mathbb{Q}) : b_1 \geq \cdots \geq b_n\}$. With this extension the $\mathcal{Q}_w$-action on $V^r_{\kappa,\underline{w}}$ extends to $\Delta^-_{Q,w} = \mathcal{Q}_w T^- \mathcal{Q}_w$ where $T^- = \{\mathrm{diag}(p^{b_1}, \dots, p^{b_n}, p^{b_0 - b_1}, \dots, p^{b_0 - b_n}) \in \mathbf{T}(\mathbb{Q}) : b_1 \geq \cdots \geq b_n, b_0 \geq 2b_1\}$. Given $h = h' t_h h''$ with $h', h'' \in \mathcal{I}_w$ and $t_h \in T^{\circ -}$, we make it act on $f \in W_{\kappa,\underline{w}}$ by

$$(f \cdot h)(x) = f(h^{-1} x t_h). \tag{3.9.4}$$

It can be checked that this is a well defined action and has norm less or equal to 1 with respect to the supreme norm on $W_{\kappa,\underline{w}}$. If $t_h = \mathrm{diag}(p^{b_1}, \dots, p^{b_n})$, then h sends $W_{\kappa,\underline{w}}$ into $W_{\kappa,\underline{w}'}$, with $w'_{jk} = \max_{k \leq t < s \leq j} \{w_{st} + b_s - b_t, w_{jk} - 1\} \leq w_{jk}$, increasing the analyticity.

Now fix $1 \leq i \leq n-1$ and consider the moduli scheme C_i over K parametrizing $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$, where $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ is the moduli problem defining Y_{Iw}, and $L \subset A[p^2]$ is a Lagrangian subgroup such that $L[p] \oplus \mathrm{Fil}_i A[p] = A[p]$. Denote by $\pi : A \to A/L$ the universal isogeny. Define the projection $p_1 : C_i \to Y_{\mathrm{Iw},K}$ by forgetting L, and $p_2 : C_i \to Y_{\mathrm{Iw},K}$ by sending $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ to $(A/L, \lambda', \pi \circ \psi_N, \mathrm{Fil}_\bullet A/L[p])$. Here the polarization λ' is defined by $\pi^*\lambda' = p^2\lambda$ and $\mathrm{Fil}_\bullet A/L[p]$ is defined as

$$\begin{aligned} &\mathrm{Fil}_j A/L[p] = \pi\big(\mathrm{Fil}_j A[p]\big), && 1 \leq j \leq i, \\ &\mathrm{Fil}_j A/L[p] = \pi\big(\mathrm{Fil}_j A[p] + p^{-1}(\mathrm{Fil}_j A[p] \cap L)\big), && i < j \leq n, \\ &\mathrm{Fil}_j A/L[p] = \big(\mathrm{Fil}_{2n-j} A/L[p]\big)^\perp, && n+1 \leq j \leq 2n. \end{aligned}$$

For example if $x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}$ is a basis of $A[p^2]$ compatible with $\mathrm{Fil}_\bullet A[p]$ and the Weil pairing, then L can be taken to be $\langle px_{i+1}, \dots, px_n, px_{n+1}, \dots, px_{2n-i}, x_{2n-i+1}, \dots, x_{2n} \rangle$ and correspondingly $\mathrm{Fil}_\bullet A/L[p]$ is

$$\langle p\bar{x}_1 \rangle \subset \cdots \subset \langle p\bar{x}_1, \dots, p\bar{x}_i \rangle \subset \langle p\bar{x}_1, \dots, p\bar{x}_i, \bar{x}_{i+1} \rangle \subset \cdots \subset \langle p\bar{x}_1, \dots, p\bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n \rangle \subset \cdots$$

where $\bar{x}_j$ stands for $x_j \mod L$.

Set $\mathcal{C}_i(v) = C_{i,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw,an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. In order to form a diagram analogous to (3.8.1), (3.9.2) we need

Proposition 3.9.1. *([AIP15, Proposition 6.2.2.1]) If* $\mathrm{Hdg}(A[p^\infty]) < \frac{p-2}{p(2p-2)}$ *and* $\mathrm{Fil}_n A[p]$ *is the canonical subgroup of level* 1*, then* $\mathrm{Hdg}(A[p^\infty]/L) \leq \mathrm{Hdg}(A[p^\infty])$ *and the* $\mathrm{Fil}_n A/L[p]$ *defined above is the canonical subgroup of level* 1 *of* A/L*.*

Then we have the diagram

(3.9.5)
$$\begin{array}{ccc} & \mathcal{C}_i(v) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(v). \end{array}$$

Now the pullback $\pi^* : p_2^*\mathcal{T}^\times_{\mathcal{H},\mathrm{an}}(v) \xrightarrow{\sim} p_1^*\mathcal{T}^\times_{\mathcal{H},\mathrm{an}}(v)$ does not send $p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v)$ into $p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$, but to

$$p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v) \circ \begin{pmatrix} pI_{n-i} & 0 & 0 & 0 \\ 0 & I_i & 0 & 0 \\ 0 & 0 & pI_{n-i} & 0 \\ 0 & 0 & 0 & p^2 I_i \end{pmatrix} \mathcal{Q}_w \subset p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v) \circ \Delta^-_{Q,w}.$$

Given local section (α, u) of the contracted product $p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}}$, there is a $\gamma_\alpha \in \Delta^-_{Q,w}$ such that $(\pi^*\alpha) \circ \gamma_\alpha^{-1}$ lies inside $p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$, and we can define

(3.9.6)
$$\begin{aligned} \widetilde{\pi}^* : p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}} &\longrightarrow p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}'} \\ (\alpha, u) &\longmapsto \big((\pi^*\alpha) \circ \gamma_\alpha^{-1}, \gamma_\alpha \cdot u\big), \end{aligned}$$

with $w'_{jk} = \begin{cases} \max\{w_{jk} - 1, w - 1\}, & \text{if } 1 \le k \le n - i < j \le n, \\ w_{jk}, & \text{otherwise} \end{cases}$. It is easy to see that the right hand side of (3.9.6) does not depend on the choice of γ_α and $\widetilde{\pi}^*$ is well defined.

The operator $U_{p,i}$ is defined as the composition

(3.9.7)
$$\begin{aligned} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}) &\xrightarrow{p_2^*} H^0(\mathcal{C}_i(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}) \xrightarrow{\widetilde{\pi}^*} H^0(\mathcal{C}_i(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}'}) \\ &\xrightarrow{p^{-i(n+1)}\operatorname{Tr} p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,\underline{w}'}). \end{aligned}$$

The normalizer $p^{-i(n+1)}$ is justified in §3.9.5.

3.9.3. *A compact operator* U_p. From (3.9.3), (3.9.7) we see that the composition $U_{p,n} \circ U_{p,n-1} \circ \cdots \circ U_{p,1}$ maps $N^{\dagger,r}_{\kappa,w,v}$ continuously into $N^{\dagger,r}_{\kappa,w-1,pv}$. Let $\mathrm{res} : N^{\dagger,r}_{\kappa,w-1,pv} \to N^{\dagger,r}_{\kappa,w,v}$ be the natural restriction map. Define the operator U_p as

$$U_p = \mathrm{res} \circ U_{p,n} \circ U_{p,n-1} \circ \cdots \circ U_{p,1} : N^{\dagger,r}_{\kappa,w,v} \longrightarrow N^{\dagger,r}_{\kappa,w,v}.$$

In the following we show that the map $\mathrm{res} : N^{\dagger,r}_{\kappa,w-1,pv} \to N^{\dagger,r}_{\kappa,w,v}$ is a compact morphism between two K-Banach modules. To this end it will be convenient to use the interpretation (ii) of $N^{\dagger,r}_{\kappa,w,v}$ in §3.4.3, i.e.

$$N^{\dagger,r}_{\kappa,w,v} = H^0(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J})^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})}.$$

Since the group $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$ is finite, there is a continuous projection from $H^0(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J})$ to its $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$-invariant part. Thus it is enough to show the compactness of the restriction

$$H^0(\mathcal{IW}_{w-1}(pv), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J}) \longrightarrow H^0(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J}).$$

Note that the sheaf $\mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J}$ is coherent. Applying [KL05, Proposition 2.4.1] we reduce to prove that $\mathcal{IW}_w(v)$ is relatively compact inside $\mathcal{IW}_{w-1}(pv)$ (relative to $\mathrm{Spm}(K)$).

According to [KL05, Definition 2.1.1], given a quasi-compact rigid analytic space $\mathcal{Z}$ and $\mathcal{V} \subset \mathcal{Z}$, an admissible open quasi-compact subset, $\mathcal{V}$ is called relatively compact inside $\mathcal{Z}$ (relative to $\mathrm{Spm}(K)$), written as $\mathcal{V} \Subset \mathcal{Z}$, if there exists a formal model $\mathfrak{Z}$ of $\mathcal{Z}$ together with an open sub-formal scheme

$\mathfrak{V} \subset \mathfrak{Z}$ with rigid fibre $\mathfrak{V}_{\mathrm{rig}} = \mathcal{V}$, such that the closure $\overline{\mathfrak{V}_0}$ of the reduction $\mathfrak{V}_0$ inside $\mathfrak{Z}_0$ is proper (over $\mathrm{Spec}(k)$, $k = \mathcal{O}_K/\varpi$).

Lemma 3.9.2. *$\mathcal{X}_1(p^m)(v)$ is relatively compact inside $\mathcal{X}_1(p^m)(pv)$.*

Proof. First note that $\mathcal{X}(pv) \Subset \mathcal{X}$ since X is proper. Then using [KL05, Proposiion 2.3.1] we get $\mathcal{X}(v) \Subset \mathcal{X}(pv)$. Both of the projections $\mathcal{X}_1(p^m)(v) \to \mathcal{X}(v)$ and $\mathcal{X}_1(p^m)(pv) \to \mathcal{X}(pv)$ are finite. The statement follows from [KL05, Lemma 2.1.8]. □

Proposition 3.9.3. *$\mathcal{IW}_w(v)$ is relatively compact inside $\mathcal{IW}_{w-1}(pv)$.*

Proof. By construction we have the formal model $f : \mathfrak{IW}_{w-1}(pv) \to \mathfrak{X}_1(p^m)(pv)$. By the previous lemma we can take an admissible formal blow-up $\mathfrak{X}_1(p^m)(pv)' \to \mathfrak{X}_1(p^m)(pv)$ with an open formal subscheme $\mathfrak{X}_1(p^m)(v)' \subset \mathfrak{X}_1(p^m)(pv)'$, such that $\mathfrak{X}_1(p^m)(v)'_{\mathrm{rig}} = \mathcal{X}_1(p^m)(v)$ and the closure $\overline{\mathfrak{X}_1(p^m)(v)'_0}$ inside $\mathfrak{X}_1(p^m)(pv)'_0$ is proper. Base changing f via the blow-up we get

$$\begin{array}{ccc} \mathfrak{IW}_{w-1}(v)' & \hookrightarrow & \mathfrak{IW}_{w-1}(pv)' \\ \downarrow & \square & \downarrow \\ \mathfrak{X}_1(p^m)(v)' & \hookrightarrow & \mathfrak{X}_1(p^m)(pv)' \end{array} .$$

There is an open covering of $\mathfrak{X}_1(p^m)(pv)'$ by affine open subschemes such that over each member $\mathrm{Spf}(R)$ of it, $\mathfrak{IW}_{w-1}(pv)' \times_{\mathfrak{X}_1(p^m)(pv)'} \mathrm{Spf}(R)$ is isomorphic to

$$\mathrm{Spf}(R) \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p^{w-1}\mathfrak{B}(0,1) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^{w-1}\mathfrak{B}(0,1) & p^{w-1}\mathfrak{B}(0,1) & \cdots & 1 \end{pmatrix} \cong \mathrm{Spf}(R) \times \mathfrak{B}(0,1)^{n(n-1)/2}$$
$$\cong \mathrm{Spf}\left(R\left\langle T_{ij}\right\rangle_{1 \le j < i \le n}\right).$$

Over $\mathfrak{IW}_{w-1}(pv)' \times_{\mathfrak{X}_1(p^m)(pv)'} \mathrm{Spf}(R)$ one can define the ideal sheaf attached to the ideal generated by p and T_{ij}, $1 \le j < i \le n$, which is independent of the choice of the coordinate T_{ij}. Such locally defined ideal sheaves glue together to an ideal sheaf $\mathscr{I}$ over $\mathfrak{IW}_{w-1}(pv)'$. Let $\mathfrak{IW}_{w-1}(pv)''$ be the blow-up of $\mathfrak{IW}_{w-1}(pv)'$ along $\mathscr{I}$, and take $\mathfrak{IW}_w(pv)''$ to be its open sub-formal scheme where the ideal sheaf $\mathscr{I}$ is generated by p. From the local description of $\mathscr{I}$, we know that the closure $\overline{\mathfrak{IW}_w(pv)''_0}$ of $\mathfrak{IW}_w(pv)''_0$ inside $\mathfrak{IW}_{w-1}(pv)''_0$ is proper over the base $\mathfrak{X}_1(p^m)(pv)'_0$. Take $\mathfrak{IW}_w(v)''$ to be the inverse image of $\mathfrak{X}_1(p^m)(v)'$ under the projection $\mathfrak{IW}_w(pv)'' \to \mathfrak{X}_1(p^m)(pv)'$. Then $\mathfrak{IW}_w(v)''$ is an open sub-formal scheme of $\mathfrak{IW}_{w-1}(pv)''$ with rigid fibre equal to $\mathcal{IW}_w(v)$. Now we have the picture

$$\begin{array}{ccccccc} \mathfrak{IW}_w(v)''_0 & \hookrightarrow & \mathfrak{Z}_0 & \hookrightarrow & \overline{\mathfrak{IW}_w(pv)''_0} & \hookrightarrow & \mathfrak{IW}_{w-1}(pv)''_0 \\ \downarrow & & \downarrow g & \square & \downarrow h & \swarrow & \\ \mathfrak{X}_1(p^m)(v)'_0 & \hookrightarrow & \overline{\mathfrak{X}_1(p^m)(v)'_0} & \hookrightarrow & \mathfrak{X}_1(p^m)(pv)'_0 & & \end{array}$$

with the vertical map h being proper. Due to the properness of the scheme $\overline{\mathfrak{X}_1(p^m)(v)'_0}$ and the map g (implied by that of h), the scheme $\mathfrak{Z}_0$ is proper. Then the closure of $\mathfrak{IW}_w(v)''_0$ inside $\mathfrak{IW}_{w-1}(pv)''_0$ must be proper since it is contained in $\mathfrak{Z}$. □

All the arguments apply to the universal weight case by working relatively over $\mathcal{U} \subset \mathcal{W}$ as well as the cuspidal case by replacing $\mathrm{Sym}^r \mathcal{J}$ with $\mathrm{Sym}^r \mathcal{J}(-C)$. We record the following corollary.

Corollary 3.9.4. *The operators* $U_p : N^{\dagger,r}_{\kappa,w,v} \to N^{\dagger,r}_{\kappa,w,v}$ *and* $U_p : N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}} \to N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}}$ *(resp.* $U_p : N^{\dagger,r}_{\mathcal{U},w,v} \to N^{\dagger,r}_{\mathcal{U},w,v}$ *and* $U_p : N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} \to N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$*) are compact operators of* K*-Banach spaces (resp.* $\mathcal{A}(\mathcal{U})$*-Banach modules).*

3.9.4. *Tensoring with* $\tau, \tau^\vee$. We consider the algebraic $\mathrm{GL}(n)$-representations $(\rho_{\mathrm{alg}}, W_{\rho_{\mathrm{alg}}})$ that are obtained by taking tensor products of symmetric powers $\mathrm{Sym}^{e_1} \tau_{\mathrm{alg}}$ and $\mathrm{Sym}^{e_2} \tau^\vee_{\mathrm{alg}}$ with $e_1, e_2 \in \mathbb{N}$. Here we add the subscript to indicate that the action of $\Delta^-_{I,w}$ is the one given by the algebraic action of $\mathrm{GL}(n)$. The notation $\rho, \tau, \tau^\vee$ will be saved for the $\Delta^-_{I,w}$-modules which are obtained from the algebraic ones by a renormalization explained below.

First we define two characters χ_1, χ_2 on the semi-group $\Delta^-_{I,w}$. Given $h = h' t_h h''$ with $h', h'' \in \mathcal{I}_w$ and $t_h = \mathrm{diag}(p^{b_1}, \dots, p^{b_n}) \in T^{\circ -}$, put

$$\chi_1(h) = p^{-2b_n}, \qquad \chi_2(h) = p^{2b_1}.$$

We define the $\Delta^-_{I,w}$-modules $\tau, \tau^\vee$ as

$$\tau := \tau_{\mathrm{alg}} \otimes \chi_1, \qquad \tau^\vee := \tau^\vee_{\mathrm{alg}} \otimes \chi_2.$$

Then by taking tensor products of $\tau, \tau^\vee$, we associate to each ρ_{alg} the renormalized $\Delta^-_{I,w}$-module ρ. The reason we consider this renormalization of ρ_{alg} is that it makes the action of $\Delta^-_{Q,w}$ on V^r_ρ integral.

Then all the $\mathbb{U}_p$-operators can be constructed for $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ in exactly the same way as when ρ is trivial and Corollary 3.9.4 holds for the action of U_p on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ and $N^{\dagger,r}_{\kappa\otimes\rho,w,v,\mathrm{cusp}}$. There is no need to distinguish ρ and ρ_{alg} when constructing of $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w}$. Their difference only concerns the action of $\mathbb{U}_p$-operators.

3.9.5. *The normalizations of the* $\mathbb{U}_p$*-operators.* We show in this section that by our choice of the normalizations of the $\mathbb{U}_p$-operators, all the eigenvalues of the compactor operator U_p acting on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ are p-adically integral for all w-analytic κ. Since $\mathcal{V}^r_{\kappa\otimes\rho,w}$ has a filtration with $\mathcal{V}^0_{\kappa\otimes\rho\otimes\mathrm{Sym}^e \tau^\vee,w}$ as graded pieces, it is enough to consider the case $r = 0$.

For a positive integer $l \in \mathbb{N}$, let Y_l be the Siegel variety modulo p^l and $Y_l[1/\mathrm{Ha}]$ be the ordinary locus. Denote by $S(p^m)_l$ the finite étale cover of $Y_l[1/\mathrm{Ha}]$ parametrizing, the quintuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\le j\le n})$, where (A, λ, ψ_N) is a principally polarized ordinary abelian scheme of relative dimension n with principal level N structure defined over an $\mathcal{O}_K/p^l$-algebra, and $\mathrm{Fil}_\bullet {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ is a complete flag of the free $\mathbb{Z}/p^m\mathbb{Z}$-module ${}^{\mathrm{t}}A[p^m]^{\text{ét}}$ with trivializations of graded pieces $\phi_j : \mathbb{Z}/p^m\mathbb{Z} \simeq \mathrm{Fil}_j / \mathrm{Fil}_{j+1} {}^{\mathrm{t}}A[p^m]^{\text{ét}}$. Put $\mathfrak{S}(p^\infty) = \varinjlim_l \varprojlim_m S(p^m)_l$. The Hodge–Tate map gives rise to the embedding

$$\begin{array}{ccc} \mathfrak{S}(p^\infty) & \hookrightarrow & \mathfrak{IW}^+_w(v) \\ & \searrow & \downarrow \\ & & \mathfrak{X}_{\mathrm{Iw}}(v) \end{array} \tag{3.9.8}$$

which induces an injective map

$$\mathrm{res} : N^{\dagger,r}_{\kappa\otimes\rho,w,v} \to H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho)[1/p], \tag{3.9.9}$$

where $\mathfrak{V}^r_\rho$ is the pullback to $\mathfrak{S}(p^\infty)$ of the locally free sheaf $\mathcal{V}^r_\rho$ of finite rank over X. In the following we define $\mathbb{U}_p$-operators acting on $H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho)$ such that res is $\mathbb{U}_p$-equivariant. Then the integrality of the U_p-eigenvalues on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ follows. We deal with the case of the operator $U_{p,i}$ for $1 \le i \le n-1$. Other cases are basically the same.

First we construct the correspondence analogous to (3.9.5)

$$\begin{array}{ccc} & C_{i,m,l}(0) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ S(p^m)_l & & S(p^{m-1})_l \end{array},$$

where $C_{i,m,l}(0)$ parametrizes the sextuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\le j\le n}, L)$ whose first five components form the quintuple defining $S(p^m)_l$. The flag $\mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ gives a self-dual flag of $\mathrm{Fil}_\bullet\, A[p]$ and $L \subset A[p^2]$ is the one used in defining C_i. The projection p_1 is forgetting L. The universal isogeny $\pi : A \to A' = A/L$ induces a map ${}^{\mathrm{t}}\pi : {}^{\mathrm{t}}A'[p^m]^{\text{ét}} \to {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ and a well-defined map $p\cdot{}^{\mathrm{t}}\pi^{-1} : {}^{\mathrm{t}}A[p^m]^{\text{ét}} \to {}^{\mathrm{t}}A'[p^m]^{\text{ét}}$. Set $\mathrm{Fil}_j\, {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}} = p\cdot{}^{\mathrm{t}}\pi^{-1}(\mathrm{Fil}_j\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}) \cap {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}}$ and $\phi'_j = \begin{cases} p^2\cdot{}^{\mathrm{t}}\pi^{-1}\circ\phi_j, & \text{if } 1\le j\le n-i, \\ p\cdot{}^{\mathrm{t}}\pi^{-1}\circ\phi_j, & \text{if } n-i+1\le j\le n. \end{cases}$ The projection p_2 sends $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\le j\le n}, L)$ to $(A', \lambda', \pi\circ\psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}}, (\phi'_j)_{1\le j\le n})$. Taking the inverse limit with respect to m followed by the direct limit with respect to l, we get $\mathfrak{C}_{i,\infty}(0) = \varinjlim_l \varprojlim_m C_{i,m,l}(0)$ and the correspondence

$$\begin{array}{ccc} & \mathfrak{C}_{i,\infty}(0) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathfrak{S}(p^\infty) & & \mathfrak{S}(p^\infty). \end{array}$$

By our choice of the normalization of the $\Delta^-_{I,w}$-action on V^r_ρ in the previous section, the group $I(\mathbb{Z}_p)T^{\circ-}I(\mathbb{Z}_p)$ acts on it integrally. This guarantees that the map $\widetilde{\pi}^* : p_2^*\mathfrak{V}_\rho \to p_1^*\mathfrak{V}_\rho$ can be defined in a manner similar to (3.9.6). Once we have checked that $\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)}H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho)$, we can define the operator $U_{p,i}$ as

$$H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho) \xrightarrow{p_2*} H^0(\mathfrak{C}_{i,\infty}(0), p_2^*\mathfrak{V}^r_\rho) \xrightarrow{\widetilde{\pi}^*} H^0(\mathfrak{C}_{i,\infty}(0), p_1^*\mathfrak{V}^r_\rho) \xrightarrow{p^{-i(n+1)}\mathrm{Tr}\, p_1} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho).$$

It is not difficult to see that with such defined $\mathbb{U}_p$-operators on $H^0(\mathfrak{S}(p^\infty), \mathfrak{w}_\rho)$, the map res is $\mathbb{U}_p$-equivariant.

In the rest of this section we show the inclusion

$$\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)}H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho).$$

Essentially this containment reflects the fact that the projection p_1 is ramified and $p^{i(n+1)}$ is its pure inseparability degree. Thanks to the projection formula we have

$$p_{1,*}p_1^*\mathfrak{V}^r_\rho = p_{1,*}\mathcal{O}_{\mathfrak{C}_{i,\infty}(0)} \otimes \mathfrak{V}^r_\rho.$$

Therefore it suffices to show

$$\mathrm{Tr}\, p_1(p_{1,*}\mathcal{O}_{\mathfrak{C}_{i,\infty}(0)}) \subset p^{i(n+1)}\mathcal{O}_{\mathfrak{S}(p^\infty)}. \tag{3.9.10}$$

Let $S(p^\infty)_0$ be the reduction of $\mathfrak{S}(p^\infty)$ and take $y_0 \in S(p^\infty)_0$, $y'_0 \in p_2(p_1^{-1}(y_0))$. We show (3.9.10) in the formal neighborhoods $\widehat{\mathfrak{S}(p^\infty)}_{y_0}, \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y'_0)}$. We explicate the projection p_1 using the Serre–Tate coordinates [Hid04, §8.2, 8.3]. The formal neighborhood $\widehat{\mathfrak{S}(p^\infty)}_{y_0}$ is isomorphic to $\mathrm{Hom}_{\mathrm{sym}}(T_pA^{\text{ét}}_{y_0} \times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}, \widehat{\mathbb{G}}_m)$. A point $z \in \widehat{\mathfrak{S}(p^\infty)}_{y_0}$ corresponds to a bilinear map $q : T_pA^{\text{ét}}_{y_0} \times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0} \to \widehat{\mathbb{G}}_m$ that is symmetric if we identify ${}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ with $A^{\text{ét}}_{y_0}$ via the polarization. Given any basis $x_1, \ldots, x_n$ of $T_pA^{\text{ét}}_{y_0}$, let ${}^{\mathrm{t}}x_1, \ldots, {}^{\mathrm{t}}x_n$ be its image under the polarization, which is a basis of

${}^{\mathrm{t}}A^{\text{ét}}_{y_0}$. Write $q(x_i, {}^{\mathrm{t}}x_j) = 1 + T_{jk}$, $1 \le j, k \le n$. We know that $T_{jk} = T_{kj}$. The $\{T_{jk}\}_{1\le j\le k\le n}$ is a Serre–Tate coordinate of $\widehat{\mathfrak{S}(p^\infty)}_{y_0}$. Similarly for $\widehat{\mathfrak{S}(p^\infty)}_{y'_0}$ with a given basis $x'_1, \dots, x'_n$ of $T_pA^{\text{ét}}_{y'_0}$ we get a corresponding Serre–Tate coordinate $\left\{T'_{jk}\right\}_{1\le j\le k\le n}$. The isogeny $\pi : A_{y_0} \to A_{y'_0}$ induces a map on the Tate modules. Now fix basis $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ of $T_pA^{\text{ét}}_{y_0}$ and $T_pA^{\text{ét}}_{y'_0}$, such that with respect to them the matrix for the map $\pi : T_pA^{\text{ét}}_{y_0} \to T_pA^{\text{ét}}_{y'_0}$ is given by $\begin{pmatrix} pI_{n-i} & 0 \\ 0 & p^2I_i \end{pmatrix}$. Then under the basis ${}^{\mathrm{t}}x_1, \dots, {}^{\mathrm{t}}x_n$ and ${}^{\mathrm{t}}x'_1, \dots, {}^{\mathrm{t}}x'_n$ of $T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ and $T_p{}^{\mathrm{t}}A^{\text{ét}}_{y'_0}$, obtained from $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ by the polarization, the matrix for ${}^{\mathrm{t}}\pi : T_p{}^{\mathrm{t}}A^{\text{ét}}_{y'_0} \to T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ is given by $\begin{pmatrix} pI_{n-i} & 0 \\ 0 & I_i \end{pmatrix}$. For each $(z, z') \in \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y'_0)} \subset \widehat{\mathfrak{S}(p^\infty)}_{y_0} \times \widehat{\mathfrak{S}(p^\infty)}_{y'_0}$, let $q : T_pA^{\text{ét}}_{y_0} \times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0} \to \widehat{\mathbb{G}}_m$ (resp. $q' : T_pA^{\text{ét}}_{y'_0} \times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y'_0} \to \widehat{\mathbb{G}}_m$) be the corresponding bilinear map for z (resp. z'). We have $q\big(x_j, {}^{\mathrm{t}}\pi(x'_k)\big) = q'\big(\pi(x_j), x'_k\big)$. Translating to the coordinates T_{jk} and T'_{jk}, we see that T'_{jk} can be taken to be the local coordinates of $\widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y'_0)}$, and the projection $p_1 : \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y'_0)} \to \widehat{\mathfrak{S}(p^\infty)}_{y_0}$ is given by

$$\begin{aligned}
\mathcal{O}_K[[T_{jk}]] &\longrightarrow \mathcal{O}_K[[T'_{jk}]] \\
T_{jk} &\longmapsto T'_{jk} && \text{if } 1 \le j \le k \le n-i, \\
T_{jk} &\longmapsto (T'_{jk}+1)^p - 1 && \text{if } 1 \le j \le n-i < k \le n, \\
T_{jk} &\longmapsto (T'_{jk}+1)^{p^2} - 1 && \text{if } n-i+1 \le j \le k \le n.
\end{aligned}$$

An easy computation shows that the pure inseparability degree of p_1 is $p^{i(n+1)}$ and $\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)}\mathcal{O}_K[[T_{jk}]]$.

Before ending this section we include the following remark concerning the Hecke actions preserving the cuspidality.

Remark 3.9.5. The injection (3.9.9) is equivariant under the action of both unramified Hecke operators and $\mathbb{U}_p$-operators. It is also easy to check that

$$N^{\dagger,r}_{\kappa\otimes\rho,w,v,\mathrm{cusp}} = N^{\dagger,r}_{\kappa\otimes\rho,w,v} \cap H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho(-C))[1/p].$$

Hence it is enough to notice that the space $H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho(-C))$ is preserved under those operators. This follows from the fact that classical cuspidal nearly homomorphic forms are stable under Hecke actions, and that the classical cuspidal nearly homomorphic forms are dense inside $H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho(-C))$.

3.10. **Interchanging the Hecke and differential operators.** Let ρ be as in §3.9.4. In this section we discuss the commutator of the $\mathbb{U}_p$-operators and unramified Hecke operators, with the operators $D_{\kappa\otimes\rho,w}$ and $E_{\kappa\otimes\rho,w}$ acting on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$. Recall that the operators $D_{\kappa\otimes\rho,w}$ and $E_{\kappa\otimes\rho,w}$ are defined as the compositions

$$\begin{aligned}
D_{\kappa\otimes\rho,w} &: \mathcal{V}^r_{\kappa\otimes\rho,w,v} \xrightarrow{\nabla_{\kappa\otimes\rho,w}} \mathcal{V}^{r+1}_{\kappa\otimes\rho\otimes\tau_{\mathrm{alg}},w,v}(-1) \xrightarrow{t^+} \mathcal{V}^{r+1}_{\kappa\otimes\rho\otimes\tau,w,v}, \\
E_{\kappa\otimes\rho,w} &: \mathcal{V}^r_{\kappa\otimes\rho,w,v} \xrightarrow{\epsilon_{\kappa\otimes\rho,w}} \mathcal{V}^{r-1}_{\kappa\otimes\rho\otimes\tau^\vee_{\mathrm{alg}},w,v}(1) \xrightarrow{t^-} \mathcal{V}^{r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w,v}.
\end{aligned}$$

We first show that the $\mathbb{U}_p$-operators and unramified Hecke operators commute with the connection $\nabla_{\kappa\otimes\rho,w}$ and the operator $\epsilon_{\kappa\otimes\rho,w}$, and then see how interchanging the order of the $\mathbb{U}_p$-operators and the maps t^+, t^- leads to a certain power of p.

Lemma 3.10.1. *The $\mathbb{U}_p$-operators and unramified Hecke operators commute with the connection $\nabla_{\kappa\otimes\rho,w}$ and the operator $\epsilon_{\kappa\otimes\rho,w}$.*

Proof. The $\mathbf{Q}$-representation J admits a filtration $0\to \mathrm{triv}\to J\to \tau_{\mathrm{alg}}^\vee(1)\to 0$. The operator $\epsilon_{\kappa\otimes\rho,w}$ by definition is induced from the quotient morphism $J\to\tau_{\mathrm{alg}}^\vee(1)$, and is easily seen to commute with all $\mathbb{U}_p$-operators as well as unramified Hecke operators.

The commutativity of the connection $\nabla_{\kappa\otimes\rho,w}$ with all the Hecke operators is a result of the functoriality of the Gauss–Manin connection, which says that for any map of abelian schemes

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \downarrow & \circlearrowleft & \downarrow \\ S & \xrightarrow{f} & R \end{array},$$

we have

$$\begin{array}{ccccc} f^*\mathcal{H}^1_{dR}(A'/R) & \xrightarrow{f^*\nabla} & f^*\mathcal{H}^1_{dR}(A'/R)\otimes f^*\Omega^1_R & \longrightarrow & f^*\mathcal{H}^1_{dR}(A'/R)\otimes\Omega^1_S \\ \downarrow{\scriptstyle\varphi^*} & & \circlearrowleft & & \downarrow{\scriptstyle\varphi^*\otimes 1} \\ \mathcal{H}^1_{dR}(A/S) & & \xrightarrow{\quad\nabla\quad} & & \mathcal{H}^1_{dR}(A/S)\otimes\Omega^1_S \end{array}.$$

Let π be the universal isogeny $A\to A'=A/L$ over $\mathcal{C}_i(v)$. By the definition of the operator $U_{p,i}$, $1\le i\le n-1$, in order to prove that it commutes with the connection $\nabla_{\kappa\otimes\rho,w}$, we only need to show the following diagram commutes.

$$\begin{array}{ccc} p_2^*\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} & \xrightarrow{\tilde\pi^*} & p_1^*\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \\ \downarrow{\scriptstyle p_2^*\nabla_{\kappa\otimes\rho,w}} & & \downarrow{\scriptstyle p_1^*\nabla_{\kappa\otimes\rho,w}} \\ p_2^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}\otimes\Omega^1_{\mathcal{C}_i(v)} & \xrightarrow{\tilde\pi^*\otimes 1} & p_1^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}\otimes\Omega^1_{\mathcal{C}_i(v)} \end{array}$$

Write a local section of $p_2^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}=p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w}V^r_{\kappa\otimes\rho,w}$ as (α,u), with $u\in V^r_{\kappa\otimes\rho,w}$ and α a w-compatible local basis of $p_2^*\mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{Y}_{\mathrm{Iw}}(v))$. For any $g\in\mathcal{Q}_w$, $(\alpha,u)=(\alpha\circ g,g^{-1}\cdot u)$. Take $\gamma\in\Delta^-_{Q,w}$ such that $\pi^*\alpha\circ\gamma^{-1}\in p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$. If D is a local section of the tangent bundle of $\mathcal{C}_i(v)$ then

$$\begin{aligned}
\big(p_1^*\nabla_{\rho,w}(D)\circ\tilde\pi^*\big)(\alpha,v) &= p_1^*\nabla_{\rho,w}(D)\big((\pi^*\alpha\circ\gamma^{-1},\,\gamma\cdot v)\big)\\
&=\big(\pi^*\alpha\circ\gamma^{-1},\,D(\gamma\cdot v)+X(D,\pi^*\alpha\circ\gamma^{-1})\cdot\gamma\cdot v\big)\\
&=\big(\pi^*\alpha\circ\gamma^{-1},\,D(\gamma\cdot v)+X(D,\alpha\circ\gamma^{-1})\cdot\gamma\cdot v\big)\\
&=\tilde\pi^*\big(\alpha,\,\gamma^{-1}\cdot D(\gamma\cdot v)+\gamma^{-1}\cdot X(D,\alpha\circ\gamma^{-1})\cdot\gamma\cdot v\big)\\
&=\tilde\pi^*\Big(\alpha,\,Dv+\big(\gamma^{-1}D\gamma+\mathrm{Ad}(\gamma^{-1})X(D,\alpha\circ\gamma)\big)\cdot v\Big)\\
&=\tilde\pi^*\big(\alpha,\,Dv+X(D,\alpha)\cdot v\big)\\
&=\big(\tilde\pi^*\circ p_2^*\nabla_{\rho,w}(D)\big)(\alpha,\,v),
\end{aligned}$$

where the third equality follows from the functoriality of the Gauss–Manin connection. The commutativity of $\nabla_{\kappa\otimes\rho,w}$ with other Hecke operators are shown similarly. □

Define two characters $\nu_{p,D},\nu_{p,E}$ from T^+ to $\mathbb{Q}^\times$, sending $t=\mathrm{diag}(p^{a_1},\dots,p^{a_n},p^{a_0-a_1},\dots,p^{a_0-a_n})$ to $\nu_{p,D}(t)=p^{a_0-2a_1}$, $\nu_{p,E}(t)=p^{a_0-2a_n}$, where $a_1\le\dots\le a_n$, $a_0\ge 2a_n$. Both $\nu_{p,D}$ and $\nu_{p,E}$ are trivial on scalar matrices. Evaluated at $\gamma_{p,i}\in T^+$ defined as (3.9.1), we have $\nu_{p,D}(\gamma_{p,i})=p^2$, $\nu_{p,E}(\gamma_{p,i})=1$ for $1\le i\le n-1$, and $\nu_{p,D}(\gamma_{p,n})=\nu_{p,E}(\gamma_{p,n})=p$. Let ℓ be an unramified prime.

Define the character $\nu_\ell : \mathrm{GSp}(2n,\mathbb{Z}_\ell)\backslash \mathrm{GSp}(2n,\mathbb{Q}_\ell)/\mathrm{GSp}(2n,\mathbb{Z}_\ell) \to \mathbb{Q}^\times$, sending γ_ℓ to $|\nu(\gamma_\ell)|_\ell^{-1}$ where ν is the multiplier character.

Lemma 3.10.2.

$$\begin{aligned}
&\text{(i)} \quad \nu_{p,D}(\gamma_p)\cdot t^+U_{\gamma_p} = U_{\gamma_p}t^+, && t^-U_{\gamma_p} = \nu_{p,E}(\gamma_p)\cdot U_{\gamma_p}t^-,\\
&\text{(ii)} \quad \nu_\ell(\gamma_\ell)\cdot t^+T_{\gamma_\ell} = T_{\gamma_\ell}t^+, && t^-T_{\gamma_\ell} = \nu_\ell(\gamma_\ell)\cdot T_{\gamma_\ell}t^-.
\end{aligned}$$

Proof. (ii) is obvious since the corresponding representations differ by a twist of the multiplier character. (i) is basically the same as (ii), but when defining the $\mathbb{U}_p$-operators, we renormalized the algebraic representations τ_{alg}, $\tau^\vee_{\mathrm{alg}}$ to the $\Delta^-_{I,w}$-modules τ, $\tau^\vee$ by twisting the characters χ_1, χ_2 to ensure the integrality. Therefore given $\gamma_p = \mathrm{diag}(p^{a_1},\dots,p^{a_n},p^{a_0-a_1},\dots,p^{a_0-a_n}) \in T^+$, the commutators of U_{γ_p} with t^+, t^- should involve the similitude $\nu_p(\gamma_p)$, as well as the character $\chi_1(\gamma_p^\circ) = p^{-2a_1}$, $\chi_2(\gamma_p^\circ) = p^{2a_n}$ with $\gamma_p^\circ = \mathrm{diag}(p^{a_n},\dots,p^{a_1})$. Explicitly the commutators are $\nu_{p,D}(\gamma_p) = \nu_p(\gamma_p)\cdot p^{-2a_1} = \nu_p(\gamma_p)\cdot\chi_1(\gamma_p^\circ)$, and $\nu_{p,E}(\gamma_p) = \nu_p(\gamma_p)\cdot p^{-2a_n} = \nu_p(\gamma_p)\cdot\chi_2(\gamma_p^\circ)^{-1}$. □

Corollary 3.10.3.

$$\begin{aligned}
&\text{(i)} \quad \nu_{p,D}(\gamma_p)\cdot D_{\kappa\otimes\rho}U_{\gamma_p} = U_{\gamma_p}D_{\kappa\otimes\rho}, && E_{\kappa\otimes\rho}U_{\gamma_p} = \nu_{p,E}(\gamma_p)\cdot U_{\gamma_p}E_{\kappa\otimes\rho},\\
&\text{(ii)} \quad \nu_\ell(\gamma_\ell)\cdot D_{\kappa\otimes\rho}T_{\gamma_\ell} = T_{\gamma_\ell}D_{\kappa\otimes\rho}, && E_{\kappa\otimes\rho}T_{\gamma_\ell} = \nu_\ell(\gamma_\ell)\cdot T_{\gamma_\ell}E_{\kappa\otimes\rho}.
\end{aligned}$$

In particular, for the compact operator U_p we have

$$p^{2n-1}\cdot D_{\kappa\otimes\rho}U_p = U_pD_{\kappa\otimes\rho}, \qquad E_{\kappa\otimes\rho}U_p = p\cdot U_pE_{\kappa\otimes\rho}.$$

3.11. **The slope decomposition.** We consider the slope decomposition of the operator U_p acting on $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} := \bigcup_{r\geq 0} N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$. We have seen that each $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is a projective $\mathcal{A}(\mathcal{U})$-Banach module with the action of U_p being compact. Applying the Coleman–Riesz–Serre theory on the spectrum of compact operators as developed in [Buz07], one can define the Fredohlm determinant $P_r(T) = \det\left(1 - TU_p|_{N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}}\right)$, which belongs to $\mathcal{A}(\mathcal{U})\{\{T\}\}$, the $\mathcal{A}(\mathcal{U})$-algebra of power series with convergence radius being infinity. Because of the integrality of the operator U_p, all the coefficients of $P_r(T)$ are power bounded, i.e. $P_r(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$.

Proposition 3.11.1. *The sequence*

$$0 \longrightarrow N^{\dagger,r-1}_{\kappa,w,v,\mathrm{cusp}} \longrightarrow N^{\dagger,r}_{\kappa,w,v\,\mathrm{cusp}} \xrightarrow{\frac{1}{r!}E^r_{\kappa,w}} N^{\dagger,0}_{\kappa\otimes\mathrm{Sym}^r\tau^\vee,w,v\,\mathrm{cusp}} \longrightarrow 0 \tag{3.11.1}$$

is exact.

Proof. Let $\eta : \mathfrak{X}_1(p^m)(v) \to \mathfrak{X}^\star(v)$ be as in §3.5. Combining the vanishing result (3.5.3) there and (3.6.1), we get the exact sequence of small formal Banach sheaves over $\mathfrak{X}^\star(v)$

$$0 \longrightarrow \eta_*\tilde{\mathfrak{V}}^{\dagger,r-1}_{\kappa,w}(-C) \longrightarrow \eta_*\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa,w}(-C) \xrightarrow{\frac{1}{r!}E^r_{\kappa,w}} \eta_*\tilde{\mathfrak{V}}^{\dagger,0}_{\kappa\otimes\mathrm{Sym}^r\tau^\vee,w}(-C) \longrightarrow 0.$$

Due to the smallness we know that the augmented Cěch complexes of the above sheaves are exact after inverting p [AIP15, Theorem A.1.2.2]. Thus we deduce the exactness of the sequence

$$\begin{aligned}
0 \longrightarrow H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger,r-1}_{\kappa,w}(-C)\big) \longrightarrow &H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w}(-C)\big)\\
&\xrightarrow{\frac{1}{r!}E^r_{\kappa,w}} H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger,0}_{\kappa\otimes\mathrm{Sym}^r\tau^\vee,w}(-C)\big) \longrightarrow 0.
\end{aligned}$$

The proposition follows by taking the invariants of $I(\mathbb{Z}/p^m\mathbb{Z})$. □

Combining (3.11.1) and the equality

$$E^r_{\kappa,w}U_p = p^r U_p E^r_{\kappa,w},$$

we see that there exist $C_r(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$ such that

$$P_r(T) = P_{r-1}(T)C_r(p^r T).$$

Therefore we can define $P_\infty(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$ as the limit

$$P_\infty(T) := \lim_{r\to\infty} P_r(T).$$

Given $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ dividing $P_\infty(T)$ one checks by definition [Col97, p.434-435] that for sufficiently large r, the resultant $\mathrm{Res}\big(Q(T), P_\infty(T)/P_r(T)\big)$ is a unit in $\mathcal{A}(\mathcal{U})$, so $Q(T)$ divides $P_r(T)$.

Now take $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ whose constant term is 1 and the leading coefficient is a unit of $\mathcal{A}(U)$, such that $P_\infty(T) = Q(T)S(T)$ with $S(T)$ relatively prime to $Q(T)$. We call such a $Q(T)$ admissible for $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}}$. If $r \geq 0$ we can apply [Buz07, Thm.3.3] to get the slope decomposition

$$N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} = N^r_{Q,\mathcal{U},\mathrm{cusp}} \oplus F^r_{Q,\mathcal{U}}, \tag{3.11.2}$$

satisfying

(i) the direct summand $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ is a projective $\mathcal{A}(\mathcal{U})$-Banach module of finite rank, and also we have $\det\left(1 - TU_p|_{N^r_{Q,\mathcal{U},\mathrm{cusp}}}\right) = Q(T)$,

(ii) the operator $Q^*(U_p)$ is invertible on $F^r_{Q,\mathcal{U}}$, where $Q^*(T) = T^{\deg Q}Q(1/T)$.

Since $Q(T)$ is of finite degree and is picked such that $\mathrm{Res}\big(Q(T), P_\infty(T)/P_r(T)\big)$ is a unit in $\mathcal{A}(\mathcal{U})$ for $r \gg 0$, the module $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ stops increasing after r is sufficiently large. We define $N_{Q,\mathcal{U},\mathrm{cusp}}$ as $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ for $r \gg 0$. The subscripts w, v are omitted since all eigenvalues of U_p acting on $N_{Q,\mathcal{U},\mathrm{cusp}}$ are nonzero, and it follows from the property of increasing analyticity and overconvergence of the operator U_p, that the module does not depend on w, v. Elements in the finite rank projective $\mathcal{A}(\mathcal{U})$-Banach module $N_{Q,\mathcal{U},\mathrm{cusp}}$ are Q-finite slope families of cuspidal nearly overconvergnent forms, and we have the Q-finite slope projection

$$e_{Q,\mathcal{U}} : N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} \longrightarrow N_{Q,\mathcal{U},\mathrm{cusp}}.$$

3.12. **p-adic splitting of $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over ordinary locus.** Let Y, X, $\mathfrak{X}$, $\mathfrak{X}(v)$, $\mathfrak{X}_{\mathrm{Iw}}(v)$, $\mathcal{X} = X_{\mathrm{rig}}$, $\mathcal{X}(v)$, $\mathcal{X}_{\mathrm{Iw}}(v)$ be defined as in §3.3. Over X (resp. Y) there is the semi-abelian scheme $\mathbf{p} : \mathcal{G} \to X$ (resp. the universal abelian scheme $\mathbf{p} : \mathcal{A} \to Y$). Denote by $\mathbf{p} : G_0 \to X_0$ (resp. $\mathbf{p} : A_0 \to Y_0$) the reduction modulo ϖ. Set $X_{0,\mathrm{ord}}$, $Y_{0,\mathrm{ord}}$ to be the ordinary locus of X_0, Y_0. Fix a lift $\sigma : \mathcal{O}_K \to \mathcal{O}_K$ of the Frobenius of the residue field $k = \mathcal{O}_K/\varpi$. Let $F : X_{0,\mathrm{ord}} \to X_{0,\mathrm{ord}}$ be the absolute Frobenius and consider the commutative diagram

$$\begin{array}{ccccc} X_{0,\mathrm{ord}} & \hookrightarrow & \mathfrak{X}(0) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \\ \downarrow F & & \downarrow u & & \downarrow \sigma \\ X_{0,\mathrm{ord}} & \hookrightarrow & \mathfrak{X}(0) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \end{array},$$

where u is the lift of the absolute Frobenius defined by sending an ordinary semi-abelian scheme $\mathcal{G}$ to its quotient by $\mathcal{G}[p]^o$, the connected part of $\mathcal{G}[p]$, and composing with the base change by σ. The isogeny $\mathcal{A} \to \mathcal{A}/\mathcal{A}[p]^o$ induces, by pullback, a morphism

$$\Phi : u^*\mathcal{H}^1_{dR}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{dR}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}}$$

of formal coherent sheaves over $\mathfrak{X}(0)$. By [Kat73a, Theorem 4.1], the locally free formal sheaf $\mathcal{H}^1_{dR}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}}$ of rank $2n$ has a unique Φ-stable locally free formal sub-sheaf $\mathfrak{U}_{\mathcal{H}}$ of rank n, over which Φ restricts to an isomorphism. This $\mathfrak{U}_{\mathcal{H}}$ gives rise to a splitting, called the unit-root splitting, of the Hodge filtration:

$$\mathcal{H}^1_{dR}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}} = \omega(\mathcal{G}/\mathfrak{X}(0)) \oplus \mathfrak{U}_{\mathcal{H}}.$$

Moreover $\mathfrak{U}_{\mathcal{H}}$ is stable under the Gauss–Manin connection. The unit-root splitting pulls back to $\mathfrak{X}_{\mathrm{Iw}}(p)(0)$, and induces a projection $\mathfrak{J} \to \mathcal{O}_{\mathfrak{X}_{\mathrm{Iw}}(p)(0)}$. Taking the generic fibre we get the projection

$$H^0(\mathcal{X}_{\mathrm{Iw}}(0), \mathcal{V}^{\dagger,r}_{\kappa,w}) = H^0(\mathcal{X}_{\mathrm{Iw}}(0), \omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}) \longrightarrow H^0(\mathcal{X}_{\mathrm{Iw}}(0), \omega^{\dagger}_{\kappa,w}). \tag{3.12.1}$$

The Igusa tower $\mathfrak{S}(p^\infty)$ defined in §3.9.5 is étale over $\mathfrak{X}_{\mathrm{Iw}}(0)$ with the group $\mathbf{T}^\circ(\mathbb{Z}_p)$ acting on it. The space of p-adic forms of weight κ consists of functions on $\mathfrak{S}(p^\infty)$ that are κ'-invariant under the action of $\mathbf{T}^\circ(\mathbb{Z}_p)$, i.e.

$$M^{p\text{-adic}}_\kappa = H^0(\mathfrak{S}(p^\infty), \mathcal{O}_{\mathfrak{S}(p^\infty)})[\kappa'].$$

Composing (3.9.9) with $r = 0$ and (3.12.1) we obtain the map

$$\xi_p : N^{\dagger,r}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_\kappa[1/p],$$

sending nearly overconvergent forms to p-adic forms.

Let $\kappa \in \mathcal{W}(K)$ be an arithmetic weight with algebraic part κ_{alg} and finite order κ_{f}. Set

$$\Gamma_1(N, p^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) : c \equiv 0 \mod p^m, \quad a \mod p^m \in \mathbf{N}(\mathbb{Z}/p^m\mathbb{Z}) \right\}.$$

Denote by $N^r_\kappa(\Gamma_1(N, p^m), K)$ the space of weight κ_{alg}, degree r classical nearly holomorphic Siegel modular forms of level $\Gamma_1(N, p^m)$ with nebentype κ_{f} at p.

Proposition 3.12.1. *The following restriction of ξ_p to classical nearly holomorphic Siegel modular forms*

$$\xi_{p,\mathrm{cl}} : N^r_\kappa(\Gamma_1(N, p^m), K) \hookrightarrow N^{\dagger,r}_{\kappa,w,v} \xrightarrow{\xi_p} M^{p\text{-}adic}_\kappa[1/p]$$

is injective.

Proof. Take $f \in \operatorname{Ker} \xi_{p,\mathrm{cl}}$. Under the map $\phi : N^r_\kappa(\Gamma_1(N, p^m), K) \otimes_K \mathbb{C} \to N^r_\kappa(\mathfrak{h}_n, \Gamma_1(N, p^m))$ defined as (2.5.3), the image $\phi(f)$ of f is a polynomial in $(\operatorname{Im} z)^{-1}$ with coefficients being holomorphic maps from the upper half space $\mathfrak{h}_n$ to $W_{\kappa_{\mathrm{alg}}}(\mathbb{C})$. By definition ϕ is equivalent to the projection from $\mathcal{V}^r_\kappa$ to $\mathcal{V}^0_\kappa$ through the C^∞ splitting, given by the Hodge decomposition of $\mathcal{H}^1_{dR}(A_{\mathfrak{h}_n}/\mathfrak{h}_n) \otimes C^\infty(\mathfrak{h}_n, \mathbb{C})$. Let $S \subset \mathfrak{h}_n$ be the subset consisting of ordinary CM points. It is analytically dense inside $\mathfrak{h}_n$. At each point of S, the unit-root splitting agrees with the C^∞ splitting [Kat78, Lemma 5.1.27]. Therefore $f \in \operatorname{Ker} \xi_{p,\mathrm{cl}}$ implies that $\phi(f) = 0$ and f=0. □

In general it is conjectured that for all w-analytic weight κ, the map ξ_p is injective. The injectivity is proved in the $n = 1$ case.

Proposition 3.12.2. *([Urb14, Proposition 3.2.4]) When $n = 1$, $\mathbf{G} = \mathrm{GL}(2)_{/\mathbb{Q}}$, the map*

$$\xi_p : N^{\dagger,r}_{\kappa,w,v} \to M^{p\text{-}adic}_\kappa[1/p]$$

is injective.

Below we replicate the proof given in [Urb14] with more details.

Proof. Suppose that there exists a nonzero $f \in \operatorname{Ker} \xi_p$. In the GL(2) case, we can identify $\mathcal{Y}_{\mathrm{Iw}}(v)$ with the open subset $\mathcal{Y}(v)$ of $\mathcal{Y}$. Let $\{\mathcal{U}_i\}$ be an admissible cover of $\mathcal{Y}(v)$, such that each $\mathcal{U}_i$ is an affinoid subdomain, and there is a basis (α_i, β_i) of $\mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{U}_i)$ giving rise to a section of $\mathcal{T}^\times_{\mathcal{H},w}(v) \to \mathcal{Y}(v)$ over $\mathcal{U}_i$. Denote by $\mathcal{U}_{\mathcal{H}}$ the rigid fibre of the formal invertible sheaf $\mathfrak{U}_{\mathcal{H}}$. After a refinement of $\{\mathcal{U}_i\}$ if necessary, we can assume that over $\mathcal{U}_{i,\mathrm{ord}} = \mathcal{U}_i \cap \mathcal{Y}(0)$ there is a section

β_i' of $\mathcal{U}_{\mathcal{H}}$ such that (α_i, β_i') gives a section of $\mathcal{T}^{\times}_{\mathcal{H},w}(v) \to \mathcal{Y}(v)$ over $\mathcal{U}_{i,\mathrm{ord}}$. Then there exists $\lambda_i \in \mathcal{A}(\mathcal{U}_{i,\mathrm{ord}})$ such that $(\alpha_i, \beta_i') = (\alpha_i, \beta_i)\begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix}$. Evaluating f at $(\mathcal{A}/\mathcal{U}_i, (\alpha_i, \beta_i))$, we get $P_{f,i}(Y) \in \mathcal{A}(\mathcal{U}_i)[Y]_{\leq r}$. Then $\xi_p(f) = 0$ implies that $P_{f,i}(\lambda_i) = 0$, i.e. λ_i is algebraic over the function field of $\mathcal{U}_i$ for all i. Applying [BDR80, Theorem 1] we know that there is $0 < v' < v$ such that $\lambda_i \in \mathcal{A}(\mathcal{U}_i \cap \mathcal{Y}(v'))$ for all i. It follows that there is an invertible subsheaf $\mathcal{U}'_{\mathcal{H}} \subset \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{Y}(v'))$ extending $\mathcal{U}_{\mathcal{H}} \subset \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{Y}(0))$. By the rigidity of analytic functions, one deduces that $\mathcal{U}'_{\mathcal{H}}$ is stable under the Gauss–Manin connection ∇. Now consider the convergent F-isocrystal $R^1\mathbf{p}_{\mathrm{rig},*}(\mathcal{A}/\mathcal{Y})$ over $Y_0/(\mathcal{O}_K, \sigma)$, and we use $\mathcal{E}$ to denote its restriction to $Y_{0,\mathrm{ord}}/(\mathcal{O}_K, \sigma)$. By definition $\mathcal{E}$ is an overconvergent F-isocrystal over $(Y_{0,\mathrm{ord}}, Y_0)/(\mathcal{O}_K, \sigma)$. The inclusions $Y_0 \hookrightarrow \mathfrak{Y}(v')$, $Y_0 \hookrightarrow \mathfrak{Y}(v'/p)$ are both closed embeddings and fit into the following commutative diagram

$$\begin{array}{ccccccc} Y_{0,\mathrm{ord}} & \hookrightarrow & Y_0 & \hookrightarrow & \mathfrak{Y}(v'/p) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K)\ , \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \sigma} \\ Y_{0,\mathrm{ord}} & \hookrightarrow & Y_0 & \hookrightarrow & \mathfrak{Y}(v') & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \end{array}$$

where u is the map defined by sending an abelian scheme A to A/H_1, its quotient by the level 1 canonical subgroup, and composing with the base change by σ. The isogeny $A \to A/H_1$ induces, by pullback, a morphism

$$\Phi : u^*\mathcal{H}^1_{dR}(\mathcal{A}/\mathfrak{Y}(pv')) \longrightarrow \mathcal{H}^1_{dR}(\mathcal{A}/\mathfrak{Y}(v')).$$

The triple $(\mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{Y}(v')), \nabla, \Phi)$ is a $\mathfrak{Y}(v')$ realization of the overconvergent F-isocrystal $\mathcal{E}$ (cf. [Ber96, §2.3.2]). The unit-root splitting $\mathfrak{U}_{\mathcal{H}}$ corresponds to a convergent sub-F-isocrystal $\mathcal{E}'$ of $\mathcal{E}$ over $Y_0/(\mathcal{O}_K, \sigma)$. The extension $\mathcal{U}'_{\mathcal{H}}$ of $\mathcal{U}_{\mathcal{H}}$ over $\mathcal{Y}(v')$, stable under ∇, makes $\mathcal{E}'$ an overconvergent isocrystal over $(Y_{0,\mathrm{ord}}, Y_0)/\mathcal{O}_K$. By the discussion at the end of [Ber96, §2.3.9], $\mathcal{E}'$ is actually a unit-root overconvergent F-isocrystal over $(Y_{0,\mathrm{ord}}, Y_0)/(\mathcal{O}_K, \sigma)$. Then [Cre87, Theorem 4.12] (cf. also Remark 4.15 there) says that the representation $\rho_{\mathcal{E}'} : \pi_1(Y_{\mathrm{ord},0}) \to \mathbb{Z}_p^{\times}$ associated to $\mathcal{E}'$ has finite local monodromy. However according to a theorem of Igusa [Kat73b, Theorem 4.3], the image of $\rho_{\mathcal{E}'}$ of the inertia group at each supersingular point of Y_0 surjects onto $\mathbb{Z}_p^{\times}$. □

3.13. **Polynomial q-expansions and p-adic q-expansions.** The embedding (3.9.8) induces, by restriction, the injective map

(3.13.1)
$$N^{\dagger,r}_{\kappa,w,v} \longrightarrow H^0(\mathfrak{S}(p^{\infty}), \mathrm{Sym}^r \mathfrak{J})[1/p].$$

For each geometrically connected component $\mathfrak{S}(p^{\infty})^{\circ}$, with the Mumford object constructed in §2.6, one can define a map

$$\iota : \mathrm{Spf}(\mathcal{O}_K[1/t][[N^{-1}S_{L,\geq 0}]]) \longrightarrow \mathfrak{S}(p^{\infty})^{\circ}.$$

The canonical basis $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$ induces an isomorphism $\iota^* \mathrm{Sym}^r \mathfrak{J} \simeq \mathcal{O}_K[1/t][[N^{-1}S_{L,\geq 0}]][\underline{Y}]_{\leq r}$, which, together with (3.13.1), defines a p-adic polynomial q-expansion map

$$\epsilon_{\iota,q,\mathrm{poly}} : N^{\dagger,r}_{\kappa,w,v} \longrightarrow \mathcal{O}_K[[N^{-1}S_{L,\geq 0}]][\underline{Y}]_{\leq r}[1/p].$$

Remark 3.13.1. Note that the image of $\epsilon_{\iota,q,\mathrm{poly}}$ are polynomials in $\underline{Y}$ with scalar coefficients, while the polynomial q-expansion $f(q, \underline{Y})$, defined as (2.6.1) for a classical nearly holomorphic form f of an arithmetic weight κ, is a polynomial in $\underline{Y}$ with coefficients inside the representation W_{κ}. To obtain the polynomial q-expansion here from the polymonial q-expansion in (2.6.1), one simply applies the canonical map $\mathfrak{e}_{\mathrm{can}} : W_{\kappa_{\mathrm{alg}}} \to \mathbb{A}^1$, defined as the evaluation at the identity matrix in $\mathrm{GL}(n)$.

If c is the number of geometrically connected components of $\mathfrak{Y}_1(p^\infty)(0)$, we can choose $\iota_1, \dots, \iota_c$ such that ι_j maps $\mathrm{Mum}_N(q)$ to the j-th component . We define the polynomial q-expansion map $\epsilon_{q,\mathrm{poly}}$ as $\bigoplus_{j=1}^c \epsilon_{\iota_j,q,\mathrm{poly}}$. Then it follows from the irreduciblity of the Igusa tower $\mathfrak{S}(p^\infty)$ [Hid04, Corollary 8.17], that the map $\epsilon_{q,\mathrm{poly}}$ is injective. Similarly we can define the polynomial q-expansion map for families of nearly overconvergent forms which is again injective.

Proposition 3.13.2. *The polynomial q-expansion maps*

$$\epsilon_{q,\mathrm{poly}} : N^{\dagger,\infty}_{\kappa,w,v} \longrightarrow \left(\mathcal{O}_K[[N^{-1}S_{L,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c},$$
$$\epsilon_{q,\mathrm{poly}} : N^{\dagger,\infty}_{\mathcal{U},w,v} \longrightarrow \left(\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c}$$

are injective.

In §3.12 we defined a map $\xi_p : N^{\dagger,r}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_\kappa[1/p]$ using the unit root splitting. Composing ξ_p with the q-expansion map for p-adic forms, we get the map

$$\epsilon_q : N^{\dagger,\infty}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_\kappa[1/p] \longrightarrow \left(\mathcal{O}_K[[N^{-1}S_{L,\geq 0}]][1/p]\right)^{\oplus c},$$

and call it the p-adic q-expansion of nearly overconvergent forms. Recall that in the construction of $\mathrm{Mum}_N(q)$ we defined a basis $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$. The locally free sheaf spanned by the δ_{can} is exactly the unit-root part. Therefore $\varepsilon_{q,p\text{-adic}}$ is nothing but $\varepsilon_{q,\mathrm{poly}}|_{\underline{Y}=0}$. In the case when the map ξ_p is injective, the p-adic q-expansion $\varepsilon_{q,p\text{-adic}}$ will also be injective. For families we define the p-adic q-expansion simply as $\varepsilon_{q,\mathrm{poly}}|_{\underline{Y}=0}$.

Proposition 3.13.3. *Suppose that the subdomain $\mathcal{U} \subset \mathcal{W}$ is a closed ball centered at an arithmetic point and $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ is admissible for $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}}$. Then after being restricted to $N_{Q,\mathcal{U},\mathrm{cusp}}$, the p-adic q-expansion map*

$$\varepsilon_{q,p\text{-adic}} : N_{Q,\mathcal{U},\mathrm{cusp}} \longrightarrow \left(\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L,\geq 0}]][1/p]\right)^{\oplus c}$$

is injective.

Proof. Take $F \in N_{Q,\mathcal{U},\mathrm{cusp}}$ with $\varepsilon_{q,p\text{-adic}}(F) = 0$. Then Proposition 3.12.1 implies that for each arithmetic weight $\kappa \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ such that the specialization F_κ is a classical nearly holomorphic form, we have $F_\kappa = 0$. We reduce to show that the subset of $\mathcal{U}(\overline{\mathbb{Q}}_p)$ consisting of points κ with F_κ being classical is Zariski dense inside $\mathcal{U}$. By the construction of $N_{Q,\mathcal{U},\mathrm{cusp}}$, we know that $F \in N^r_{Q,\mathcal{U},\mathrm{cusp}}$ for some $r \in \mathbb{N}$. Then F can be written as (Corollary 3.7.5)

$$\eta F = F_0 + \theta D F_1 + \cdots + \theta^r D^r F_r$$

with $F_i \in N^{\dagger,0}_{\mathcal{U}\otimes \mathrm{Sym}^i \tau^\vee,w,v}$ and $\eta \in K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$ nonzero. By Corollary 3.10.3 there is a bound, depending on Q and r, on the slopes of $F_0, F_1, \dots, F_n$. Therefore if an arithmetic weight $\kappa \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ is outside the zeroes of η with κ_{alg} dominant and sufficiently regular with respect to that bound on slopes, then the classicity of $F_{0,\kappa}, \dots, F_{n,\kappa}$ can be deduced from [AIP15, Proposition 7.3.1] and [BPS16], from which the classicity of F_κ follows. Since $\mathcal{U}$ is a closed ball centered at an arithmetic point, such arithmetic points outside the zeroes of η with algebraic parts dominant and sufficiently regular are Zariski dense in $\mathcal{U}$. □

3.14. **Families by q-expansions.** Keep the assumption on $\mathcal{U}, Q$ as in Prop. 3.13.3. Let $\Sigma \subset \mathcal{U}(\overline{\mathbb{Q}}_p)$ be a Zariski dense subset consisting of arithmetic points. Define $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$ (resp. $N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}}$) to be the sub-$\mathcal{A}(\mathcal{U})$-module of $\left(\mathcal{A}^\circ(\mathcal{U})[[N^{-1}S_{L,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c}$ (resp. $\left(\mathcal{A}^\circ(\mathcal{U})[[N^{-1}S_{L,\geq 0}]][1/p]\right)^{\oplus c}$) consisting of those elements whose specialization at almost all $\kappa \in \Sigma$ is contained in $\varepsilon_{q,\mathrm{poly}}(N_{Q_\kappa,\kappa,\mathrm{cusp}})$ (resp. $\varepsilon_{q,p\text{-adic}}(N_{Q_\kappa,\kappa,\mathrm{cusp}})$).

Proposition 3.14.1. *With $\mathcal{U}, Q$ as in Proposition 3.13.3, the polynomial q-expansion map induces an isomorphism from $N_{Q,\mathcal{U},\mathrm{cusp}}$ to $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$.*

Proof. We follow the argument of [Wil88, Theorem 1.2.2], [Hid93, Theorem 7.3.1]. Abbreviate $\mathcal{A}(\mathcal{U})$, $N_{Q,\mathcal{U}\,\mathrm{cusp}}$, $N_{Q_\kappa,\kappa,\mathrm{cusp}}$, $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$, as $\mathcal{A}$, N, N_κ, $N^{\Sigma,\mathrm{poly}}$. Let I be the set consisting of monomials $q^{\beta_i}\prod Y_{jk}^{a_{jk}}$, where $a_{jk}\in\mathbb{N}$, $1\le j\le k\le n$, and $\beta_i\in N^{-1}S_{L,\ge0}$ with the subscript $1\le i\le c$ meaning the i-th connected component. By taking coefficients there is a natural embedding of $\big(\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L,\ge0}]][\underline{Y}][1/p]\big)^{\oplus c}$ into the direct product $\mathcal{A}^I$. Denote by $K(\mathcal{A})$ the fraction field of $\mathcal{A}$. The $\mathcal{A}$-module N is finite projective. Let $d=\mathrm{rank}_{\mathcal{A}}(N)=\dim_{K(\mathcal{A})}(N\otimes K(\mathcal{A}))<\infty$, and pick $F_1,\dots,F_d\in N$ such that they span $N\otimes K(\mathcal{A})$ over $K(\mathcal{A})$. Write their images inside $\mathcal{A}^I$ under the polynomial q-expansion map as $\big(a(F_j,i)\big)_{i\in I}$, $1\le j\le d$. Thanks to the injectivity of the map $\varepsilon_{q,\mathrm{poly}}$, we can choose $i_1,\dots,i_d$ such that $D=\det\big(a(F_j,i_t)\big)_{1\le j,t\le d}\ne0$. We claim that $DN^{\Sigma,\mathrm{poly}}\subset\varepsilon_{q,\mathrm{poly}}(N)$. Otherwise there exists $G=\big(a(G,i)\big)_{i\in I}\in DN^{\Sigma,\mathrm{poly}}\backslash\varepsilon_{q,\mathrm{poly}}(N)$. Subtracting from G a linear combination of the $\varepsilon_{q,\mathrm{poly}}(F_j)$'s, we get a nonzero $G'\in N^{\Sigma,\mathrm{poly}}$ with $a(G,i_t)=0$ for all $1\le t\le d$. Since Σ is Zariski dense there exists some $\kappa\in\Sigma$, such that specializing at κ, the vectors $\varepsilon_{q,\mathrm{poly}}(F_1)_\kappa,\dots,\varepsilon_{q,\mathrm{poly}}(F_d)_\kappa$ and G'_κ are $\overline{\mathbb{Q}}_p$-linearly independent and $G'_\kappa=\varepsilon_{q,\mathrm{poly}}(f)$ for some $f\in N_\kappa$. The injectivity of $\varepsilon_{q,\mathrm{poly}}$ shows that $F_{1,\kappa},\dots,F_{d,\kappa},f$ are linearly independent inside N_κ which is impossible. Therefore $N^{\Sigma,\mathrm{poly}}=\varepsilon_{q,\mathrm{poly}}(N)\otimes K(\mathcal{A})\cap\mathcal{A}^I$. We also deduce that $N^{\Sigma,\mathrm{poly}}$ is a finitely generated $\mathcal{A}$-module because $\mathcal{A}$ is noetherian. In fact $\mathcal{A}$ is a noetherian UFD and a Jacobson ring [BGR84, §5.2.6 Theorem 1, 3]. Now take an arbitrary $G''\in\varepsilon_{q,\mathrm{poly}}(N)\otimes K(\mathcal{A})\cap\mathcal{A}^I$, we want to prove that G'' actually lies inside $\varepsilon_{q,\mathrm{poly}}(N)$. Since $\mathcal{A}$ is a UFD we can take some $\eta\in\mathcal{A}$ such that $\eta G''\in\varepsilon_{q,\mathrm{poly}}(N)$ and for any η' strictly divides η, we have $\eta'G''\notin\varepsilon_{q,\mathrm{poly}}(N)$. Take $F\in N$ such that $\eta G''=\varepsilon_{q,\mathrm{poly}}(F)$. If $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}$ containing η, then the polynomial q-expansion $\varepsilon_{q,\mathrm{poly}}(F_{\kappa_\mathfrak{m}})=\eta(\kappa_\mathfrak{m})G''_{\kappa_\mathfrak{m}}=0$, which implies that $F_{\kappa_\mathfrak{m}}=0$ and $F\in\mathfrak{m}N$ by Proposition 3.5.1. This shows that $F\in\bigcap_{\eta\in\mathfrak{m}}\mathfrak{m}N$. The $\mathcal{A}$-module N is finite projective so there exists $a_1,\dots,a_l\in\mathcal{A}$ such that each localization N_{a_i} is free of finite rank over A_{a_i} which is still a noetherian UFD [Mat80, Lemma (19.B)] and a Jacobson ring [Sta15, Tag 00G6]. Let $\eta_1,\dots,\eta_b$ be all the prime factors of η. Each $\eta_jA_{a_i}$ is a prime ideal that is the intersection of all maximal ideals in A_{a_i} containing η_j. It follows that $\sqrt{\eta}A_{a_i}=\bigcap_j\eta_jA_{a_i}=\bigcap_{\eta\in\mathfrak{m},\mathfrak{m}\in\mathrm{Max}(A_{a_i})}\mathfrak{m}A_{a_i}$ and $\sqrt{\eta}N_{a_i}=\bigcap_{\eta\in\mathfrak{m},\mathfrak{m}\in\mathrm{Max}(A_{a_i})}\mathfrak{m}N_{a_i}$. Then from $F\in\bigcap_{\eta\in\mathfrak{m}}\mathfrak{m}N$, we deduce that $F\in\sqrt{\eta}N_{a_i}$ for all i, and hence $F\in\sqrt{\eta}N$. By our choice of η this implies that η is a unit in $\mathcal{A}$. $\square$

If we apply the same argument to $N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}}$, due to the lack of injectivity of the map $\varepsilon_{q,p\text{-adic}}$ at all points in $\mathcal{U}$, we only get a weaker result.

Proposition 3.14.2. *With $\mathcal{U}, Q$ as in Proposition 3.13.3, there exists a nonzero $\eta\in\mathcal{A}(\mathcal{U})$ such that $\eta N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}}$ belongs to $\varepsilon_{q,p\text{-adic}}(N_{Q,\mathcal{U},\mathrm{cusp}})$.*

References

[AI17] F. Andreatta and A. Iovita. Triple product p-adic L-functions associated to finite slope p-adic families of modular forms. 2017, https://arxiv.org/abs/1708.02785. with an appendix by Eric Urban. 4

[AIP15] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. p-adic families of Siegel modular cuspforms. *Ann. of Math. (2)*, 181(2):623–697, 2015. 2, 3, 21, 22, 24, 26, 27, 32, 33, 34, 41, 45

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. Generalized Heegner cycles and p-adic Rankin L-series. *Duke Math. J.*, 162(6):1033–1148, 2013. With an appendix by Brian Conrad. 2

[BDR80] S. Bosch, B. Dwork, and P. Robba. Un théorème de prolongement pour des fonctions analytiques. *Math. Ann.*, 252(2):165–173, 1979/80. 44

[Ber96] Pierre Berthelot. *Cohomologie rigide et cohomologie rigide à supports propres*. Université de Rennes 1. Institut de Recherche Mathématique de Rennes [IRMAR], 1996. 44

[BGG71] I. N. Bernshtein, I. M. Gel'fand, and S. I. Gel'fand. Structure of representations that are generated by vectors of highest weight. *Funckcional. Anal. i Priložen.*, 5(1):1–9, 1971. 31

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry. 46

[Böc85] S Böcherer. Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II. *Mathematische Zeitschrift*, 189(1):81–110, 1985. 2, 18

[BPS16] Stéphane Bijakowski, Vincent Pilloni, and Benoît Stroh. Classicité de formes modulaires surconvergentes. *Ann. of Math. (2)*, 183(3):975–1014, 2016. 45

[BS00] S. Böcherer and C.-G. Schmidt. p-adic measures attached to Siegel modular forms. *Ann. Inst. Fourier (Grenoble)*, 50(5):1375–1443, 2000. 2, 16

[Buz07] Kevin Buzzard. Eigenvarieties. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 59–120. Cambridge Univ. Press, Cambridge, 2007. 41, 42

[Col97] Robert F Coleman. p-adic Banach spaces and families of modular forms. *Inventiones mathematicae*, 127(3):417–479, 1997. 42

[CP04] Michel Courtieu and Alexei Panchishkin. *Non-Archimedean L-functions and arithmetical Siegel modular forms*, volume 1471 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2004. 16

[Cre87] Richard Crew. F-isocrystals and p-adic representations. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 111–138. Amer. Math. Soc., Providence, RI, 1987. 44

[EFMV] Ellen Eischen, Jessica Fintzen, Elena Mantovan, and Ila Varma. Differential operators and families of automorphic forms on unitary groups of arbitrary signature. *To appear in Documenta Mathematica*. 4

[EHLS16] Ellen Eischen, Michael Harris, Jianshu Li, and Christopher Skinner. p-adic L-functions for unitary groups, part II: zeta-integral calculations, 2016, http://arxiv.org/abs/1602.01776. 2, 16

[Eis12] Ellen E. Eischen. p-adic differential operators on automorphic forms on unitary groups. *Ann. Inst. Fourier (Grenoble)*, 62(1):177–243, 2012. 4, 8

[EW16] Ellen Eischen and Xin Wan. p-adic Eisenstein series and L-functions of certain cusp forms on definite unitary groups. *J. Inst. Math. Jussieu*, 15(3):471–510, 2016. 2, 16

[Far11] Laurent Fargues. La filtration canonique des points de torsion des groupes p-divisibles. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(6):905–961, 2011. With collaboration of Yichao Tian. 21, 33

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. 6, 9, 10, 11, 15, 28

[FvdP04] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004. 28

[Har86] Michael Harris. Arithmetic vector bundles and automorphic forms on Shimura varieties. II. *Compositio Math.*, 60(3):323–378, 1986. 2, 18

[Har97] Michael Harris. L-functions and periods of polarized regular motives. *J. Reine Angew. Math.*, 483:75–161, 1997. 2, 16

[Har07] Michael Harris. Cohomological automorphic forms on unitary groups. II. Period relations and values of L-functions. In *Harmonic analysis, group representations, automorphic forms and invariant theory*, volume 12 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 89–149. World Sci. Publ., Hackensack, NJ, 2007. 2

[Har08] Michael Harris. A simple proof of rationality of Siegel-Weil Eisenstein series. In *Eisenstein series and applications*, volume 258 of *Progr. Math.*, pages 149–185. Birkhäuser Boston, Boston, MA, 2008. 16

[Hid88] Haruzo Hida. A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II. *Ann. Inst. Fourier (Grenoble)*, 38(3):1–83, 1988. 2

[Hid93] H. Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, 1993. 46

[Hid04] Haruzo Hida. *p-adic automorphic forms on Shimura varieties*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004. 38, 45

[HX14] Robert Harron and Liang Xiao. Gauss-Manin connections for p-adic families of nearly overconvergent modular forms. *Ann. Inst. Fourier (Grenoble)*, 64(6):2449–2464, 2014. 4

[Ibu99] Tomoyoshi Ibukiyama. On differential operators on automorphic forms and invariant pluri-harmonic polynomials. *Comment. Math. Univ. St. Paul.*, 48(1):103–118, 1999. 2, 18

[Ich15] T. Ichikawa. Integrality of nearly (holomorphic) Siegel modular forms. 2015, https://arxiv.org/abs/1508.03138. 4

[JV79] Hans Plesner Jakobsen and Michèle Vergne. Restrictions and expansions of holomorphic representations. *J. Funct. Anal.*, 34(1):29–53, 1979. 9

[Kat73a] Nicholas Katz. Travaux de Dwork. pages 167–200. Lecture Notes in Math., Vol. 317, 1973. 42

[Kat73b] Nicholas M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, 1973. 11, 44

[Kat78] Nicholas M. Katz. p-adic L-functions for CM fields. *Invent. Math.*, 49(3):199–297, 1978. 2, 43

[KL05] Mark Kisin and King Fai Lai. Overconvergent Hilbert modular forms. *Amer. J. Math.*, 127(4):735–783, 2005. 35, 36

[KO68] Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, 8:199–213, 1968. 6

[Lan12] Kai-Wen Lan. Toroidal compactifications of PEL-type Kuga families. *Algebra Number Theory*, 6(5):885–966, 2012. 5, 6, 10

[Lan17] Kai-Wen Lan. *Compactifications of PEL-type Shimura varieties and Kuga families with ordinary loci.* World Scientific Publishing Co., Singapore, 2017. 27, 28

[Liu15] Zheng Liu. p-adic L-functions for ordinary families of symplectic groups. *Preprint*, 2015, http://http://www.math.ias.edu/ zliu/SLF2

[Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980. 46

[Mum70] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970. 28

[Nap92] Mark Alan Nappari. Holomorphic forms canonically attached to nearly holomorphic automorphic forms. *Thesis Brandeis University*, 1992. 2

[PSS15] Ameya Pitale, Abhishek Saha, and Ralf Schmidt. Lowest weight modules of $\mathrm{Sp}_4(\mathbb{R})$ and nearly holomorphic Siegel modular forms. 2015, http://arxiv.org/abs/1501.00524. 2

[ShE97] *Euler products and Eisenstein series*, volume 93 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. 2

[Shi76] Goro Shimura. The special values of the zeta functions associated with cusp forms. *Comm. Pure Appl. Math.*, 29(6):783–804, 1976. 1

[Shi90] Goro Shimura. Invariant differential operators on Hermitian symmetric spaces. *Ann. of Math.*, 132(2):237–272, 1990. 14

[Shi00] Goro Shimura. *Arithmeticity in the theory of automorphic forms*, volume 82 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000. 1, 2, 12, 14, 16, 17, 28, 29, 31

[Sta15] The Stacks Project Authors. Stacks Project. `http://stacks.math.columbia.edu`, 2015. 46

[Til11] Jacques Tilouine. Companion forms and classicality in the $\mathbf{GL}_2(\mathbb{Q})$-case. In *Number theory*, volume 15 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 119–141. Ramanujan Math. Soc., Mysore, 2011. 8

[Urb13] Eric Urban. On the rank of Selmer groups for elliptic curves over $\mathbb{Q}$. In *Automorphic representations and L-functions*, volume 22 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 651–680. Tata Inst. Fund. Res., Mumbai, 2013. 11

[Urb14] Eric Urban. Nearly overconvergent modular forms. In *Iwasawa theory 2012*, volume 7 of *Contrib. Math. Comput. Sci.*, pages 401–441. Springer, Heidelberg, 2014. 1, 2, 4, 10, 11, 15, 25, 43

[Wil88] Andrew J. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988. 46

School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540
E-mail address: `zliu@math.ias.edu`