NON-CUSPIDAL HIDA THEORY FOR SIEGEL MODULAR FORMS AND TRIVIAL ZEROS OF P-ADIC L-FUNCTIONS

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ABSTRACT. We study the derivative of the standard p-adic L-function associated with a P-ordinary Siegel modular form (for P a parabolic subgroup of $\mathrm{GL}(n)$) when there presents a semi-stable trivial zero. This implies part of Greenberg's conjecture on the order and leading coefficient of p-adic L-functions at such trivial zero. We use the method of Greenberg–Stevens. For the construction of the *improved* p-adic L-function we develop Hida theory for non-cuspidal Siegel modular forms.

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INTRODUCTION

In the seminal paper [MTT86] the three authors consider an elliptic curve E and a prime p such that E has split multiplicative reduction at p (for example $E = X_0(11)$ and $p = 11$). In this case the p-adic L-function $\mathcal{L}_p(s, E)$ presents a trivial zero at $s = 1$ because of the modified Euler factor at p. If the complex L-value $L(1, E)$ is not vanishing, they conjecture that the first derivative of the p-adic L-function at $s = 1$ interpolates the algebraic part of the complex L-value, up to an error factor of the form $\log_p(q_E)/\mathrm{ord}_p(q_E)$, which they call the ℓ-invariant. Here q_E is the Tate period of E.

This conjecture has been proved in [GS93] using Hida theory and a two-variable p-adic L-function togheter with a one-variable improved p-adic L-function. At the same time, Greenberg generalized the conjecture of Mazur–Tate–Taitelbaum to the class of p-adic Galois representations V that satisfy the so-called Pantchichkine condition. Assuming $L(0, V) \neq 0$, his conjecture, roughly speaking, predicts that the multiplicity of the trivial zero of $\mathcal{L}_p(s, V)$ at $s = 0$ equals the order of vanishing of $\mathcal{L}_p(s, V)$ at $s = 0$, and gives an exact formula for the leading coefficient of the p-adic L-function. This precise formula involves a factor $\ell(V)$, called the ℓ-invariant of V, which is defined in purely Galois theoretic terms and coincides with $\log_p(q_E)/\mathrm{ord}_p(q_E)$ when V is the Tate module of an elliptic curve E. This conjecture has been recently generalized to all semi-stable Galois representations [Ben11]. For the precise statement, see Conjecture 3.1.1.

Let n be an integer and let P be the parabolic of $\mathrm{GL}(n)$ associated with the partition $n = n_1 + \ldots + n_d$, *i.e.*

$$P = \left\{ \begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_d \end{pmatrix} \in \mathrm{GL}(n) \;\middle|\; a_i \in \mathrm{GL}(n_i),\, 1 \leq i \leq d \right\}.$$

The main objective of the paper is to study Conjecture 3.1.1 when V is the standard Galois representation associated to an irreducible cuspidal automorphic representation π of $\mathrm{Sp}(2n, \mathbb{A})$ which is P-ordinary, *i.e.* the archimedean component π_∞ is isomorphic to a holomorphic discrete series $\mathcal{D}_{\underline{t}}$ of weight $\underline{t} = (\underbrace{t_1^P, \ldots, t_1^P}_{n_1}, \underbrace{t_2^P, \ldots, t_2^P}_{n_2}, \ldots, \underbrace{t_d^P, \ldots, t_d^P}_{n_d})$, and the action of certain $\mathbb{U}_p^P$-operators (which are Hecke operators at p whose normalization depends on $\underline{t}$) on π admits a non-zero eigenvector with eigenvalues being p-adic units.

Denote by $L(s, \pi \times \xi)$ the standard L-function for π twisted by a finite order Dirichlet character ξ. It is defined as an infinite Euler product. The local L-factors for a place v where both π and ξ are unramified is given as

$$L_v(s, \pi_v \times \xi_v) = (1 - \xi(q_v)q_v^{-s})^{-1} \prod_{i=1}^{n} (1 - \xi(q_v)\alpha_{v,i}q_v^{-s})^{-1}(1 - \xi(q_v)\alpha_{v,i}^{-1}q_v^{-s})^{-1},$$

where $\alpha_{v,i}^{\pm 1}$, $1 \leq i \leq n$, are the Satake parameters of π_v and q_v is the cardinality of the residue field. The Deligne critical points for $L(s, \pi \times \xi)$ are the integers s_0 such that

$$1 \leq s_0 \leq t_n - n,\ (-1)^{s_0+n} = \xi(-1), \text{ or } n + 1 - t_n \leq s_0 \leq 0,\ (-1)^{s_0+n+1} = \xi(-1).$$

The algebracity of these critical L-values divided by certain Petersson norm period has been shown in [Har81, Shi00, BS00]. In [Liu16b], the first author constructed an $n+1$-variable p-adic L-function interpolating the critical values to the right of the center of the partial standard L-function with π varying in a Hida family (ordinary for $P = B$).

In this paper we generalize the results of [Liu16b] and construct a $d+1$-variable p-adic L-function for P-ordinary Hida families (where P is general), interpolating critical values to the left of the center of the partial standard L-function.

Let $T_P = P/SP$ be the maximal quotient torus of P. We say that $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ is arithmetic if it is a product of an algebraic character corresponding to integers $(t_1^P, \dots, t_d^P)$ and a finite order character $(\epsilon_1^P, \dots, \epsilon_d^P)$; we say that it is admissible if moreover $t_1^P \geq t_2^P \dots \geq t_d^P \geq n+1$. We fix a sufficiently large p-adic field F as coefficient field and denote by $\mathcal{O}_F$ its valuation ring.

Suppose $p \geq 3$. Hida theory for P-ordinary cuspidal Siegel modular forms has been developed in [Pil12] generalizing the case $P = B$ in [Hid02]. Let $\mathscr{C}_P$ be a geometrically irreducible component of the spectrum of $\mathbb{T}^{0,N}_{P\text{-ord}}$, the Hecke algebra acting on P-ordinary Hida families of cuspidal Siegel modular forms of tame principal level N, and let $F_{\mathscr{C}_P}$ be its function field. We denote by $\mathbb{I}_{\mathscr{C}_P}$ the integral closure of $\Lambda_P := \mathcal{O}_F[\![T_P(\mathbb{Z}_p)^\circ]\!]$ in $F_{\mathscr{C}_P}$, where $T_P(\mathbb{Z}_p)^\circ$ is the maximal p-profinite subgroup of $T_P(\mathbb{Z}_p)$. We prove the following theorem:

Theorem (Theorem 2.6.2)**.** *Let $\mathscr{C}_P$ be as above. For a Dirichlet character ϕ with conductor dividing N and $\phi^2 \neq 1$, a pair $(\beta_1, \beta_2) \in N^{-1}\,\mathrm{Sym}(n, \mathbb{Z})^{*\oplus 2}_{>0}$, and $j \in \mathbb{Z}/(p-1)$ such that $\phi\omega^j(-1) = 1$, there is a p-adic L-function $\mathcal{L}_{\mathscr{C}_P, \phi\omega^j, \beta_1, \beta_2} \in \mathbb{I}_{\mathscr{C}_P}[[S]] \otimes_{\mathbb{I}_{\mathscr{C}_P}} F_{\mathscr{C}_P}$ which satisfies the following interpolation property.*

Let $x : \mathbb{I}_{\mathscr{C}_P} \to F'$ be an F'-point of $\mathscr{C}_P$ (with F' being a finite extension of F). Suppose that the weight map $\Lambda_P \to \mathbb{T}^{0,N}_{P\text{-ord}}$ is étale at x and maps x to an admissible point $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}\,(T_P(\mathbb{Z}_p), F'^\times)$. For an integer $n+1 \leq k \leq t_d^P$ and a finite order character $\chi^\circ : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}^\times$ trivial on $(\mathbb{Z}/p)^\times$, we have

$$\mathcal{L}_{\mathscr{C}_P, \phi\omega^j, \beta_1, \beta_2}(\chi^\circ(1+p)(1+p)^k - 1, x) = C_{k, \underline{t}^P} \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\mathfrak{c}(\varphi, \beta_1)\mathfrak{c}(e_P \mathcal{W}(\varphi), \beta_2)}{\langle \varphi, \overline{\varphi} \rangle}$$
$$\times E_p(n+1-k, \pi_x \times \phi\chi^\circ\omega^{j-k}) \cdot L^{Np\infty}(n+1-k, \pi_x \times \phi\chi^\circ\omega^{j-k}),$$

Here the factor $E_p(n+1-k, \pi_x \times \phi\chi^\circ\omega^{j-k})$ is the modified Euler factor at p as predicted by Coates–Perrin-Riou [Coa91].

We refer to Theorem 2.6.2 (which is formuated as p-adic measures) for the undefined notation and §2.3 for the definition of the modified Euler factor at p. The construction of this p-adic L-function is similar to the one in [Liu16b] and uses the doubling method [Gar84, PSR87].

Remark. For the whole paper, we assume that $\phi^2 \neq 1$. This hypothesis is absolutely not necessary, but when $\phi^2 = 1$ the p-adic L-function could have a possible pole in the cyclotomic variable (outside the range of interpolation), which comes from the pole of the Kubota–Leopoldt p-adic function appearing in the Fourier coefficients of the Siegel Eisenstein series on $\mathrm{Sp}(4n)$. When $n = 1$ this pole cancels out if and only if $\mathscr{C}_P$ has no CM [Hid90, Proposition 5.2]. In general we expect a cyclotomic pole if and only if the standard representation associated with $\mathscr{C}_P$ is reducible and the trivial representation appears as a sub-quotient of it.

When $n_d = 1$ and ϵ_d^P is trivial, the factor $1 - \phi_p(p)^{-1}\alpha_{n,x}^{-1}p^{s-1}$ appears in $E_p(s, \pi_x \times \phi)$, where $\alpha_{n,x}$ is an algebraic number related with the $\mathbb{U}_p^P$-eigenvalues (see §2.3 for the precise definition). Supposing that x is classical, then $\alpha_{n,x}$ corresponds to the Frobenius eigenvalue of p-adic valuation $-(t_d^P - n)$ in the Weil representation associated to $\pi_{x,p}$. If $\phi(-1) = (-1)^{n+1}$, $\phi_p(p) = 1$, and $\alpha_{n,x_0} = p^{-1}$ for a classical point $x_0 \in \mathscr{C}_P(F)$, then the factor $1 - \phi_p(p)^{-1}\alpha_{n,x_0}^{-1}p^{k-n}$ vanishes if $k = n+1$, and a trivial zero occurs at the point $((1+p)^{n+1}-1, x_0)$ for $\mathcal{L}_{\mathscr{C}_P, \phi\omega^{n+1}, \beta_1, \beta_2}$. Denote by $\rho_{x_0} : G_{\mathbb{Q}} \to \mathrm{GL}(2n+1, \overline{\mathbb{Q}}_p)$ the Galois representation attached to x_0 [Art13, CH13]. We shall call $((1+p)^{n+1} - 1, x_0)$ a semi-stable trivial zero for $\mathcal{L}_{\mathscr{C}_P, \phi\omega^{n+1}, \beta_1, \beta_2}$ if furthermore $\mathrm{Fil}^0\,\rho_{x_0}|_{G_{\mathbb{Q}_p}} / \mathrm{Fil}^2\,\rho_{x_0}|_{G_{\mathbb{Q}_p}}$

is a two dimensional indecomposable $G_{\mathbb{Q}_p}$-representation. When $n_d = 1$ and π_{x_0} is P-ordinary with $\alpha_{n,x_0} = p^{-1}$, the condition on $\rho_{x_0}|_{G_{\mathbb{Q}_p}}$ is expected to be always satisfied (see Remark 2.3.3). It is for this special type of trivial zeros that we can use the Greenberg–Stevens method to study the derivative of the p-adic L-function.

The step of expressing the ℓ-invariant in terms of the derivative of $\mathbb{U}_p^P$-eigenvalues in the Greenberg–Stevens method for the so-called trivial zero of type M (as named in [Gre94]) has already been done [Ros15]. The other step in the method, which relates the derivative with respect to the cyclotomic variable of the p-adic L-function to the derivative with respect to the weight variable of the $\mathbb{U}_p^P$-eigenvalues, applies in the following situation. Suppose that there is a $d+1$-variable p-adic L-function $\mathcal{L}(S, T_1, \dots, T_d)$ with S as the cyclotomic variable, and it has a trivial zero at $(s_0, t_1, \dots, t_d)$. If there exists a d-tuple integer $(a_1, \dots, a_d) \neq 0$, and integers $a_0 \neq a_0'$, such that $\mathcal{L}(S, T_1, \dots, T_d)$ vanishes along the line $(s_0, t_1, \dots, t_d) + S(a_0, a_1, \dots, a_d)$ and can be improved (in the sense of saving the factor that causes the trivial zero in the interpolation result) along the line $(s_0, t_1, \dots, t_d) + S(a_0', a_1, \dots, a_d)$, then the strategy applies.

In our above mentioned case of the semi-stable zero, the assumption on $\rho_x|_{G_{\mathbb{Q}_p}}$ implies that the trivial zero is of type M. The p-adic L-function $\mathcal{L}_{\mathscr{C}_P, \phi\omega^{n+1}, \beta_1, \beta_2}$ vanishes along the hyperplane $S = (1+p)^{n+1} - 1$ (because of the missing factor $1 - \phi_p(p)p^{-s}$ for π with π_p unramified). Meanwhile, when $k = t_d^P$ the factor $1 - \phi_p(p)^{-1}\alpha_n^{-1}p^{n-k}$ is a p-adic analytic function as $\alpha_n p^{t_d^P - n}$ can be expressed in terms of $\mathbb{U}_p^P$ eigenvalues. Hence there is the possibility to improve the p-adic L-function along the hyperplane $S = (1+p)^{t_d^P} - 1$. The lines $((1+p)^{n+1} - 1, x_0) + S(0, 0, \dots, 0, 1)$ and $((1+p)^{n+1} - 1, x_0) + S(1, 0, \dots, 0, 1)$ satisfy the conditions in the previous paragraph.

Now in order to carry out the Greenberg–Stevens method, we need to construct the improved p-adic L-function. Indeed, by a different choice of the local sections at p for the Siegel Eisenstein series on $\mathrm{Sp}(4n)$ (compare the tables in §2.4.8), we obtain a new Eisenstein series such that applying to it the pullback formula from the doubling method produces the complex L-function without the factor $1 - \phi_p(p)^{-1}\alpha_n^{-1}p^{s-1}$.

However, a new difficulty arises. One useful fact about the sections selected for constructing the p-adic L-function in Theorem 2.6.2 is that the restrictions to $\mathrm{Sp}(2n, \mathbb{A}) \times \mathrm{Sp}(2n, \mathbb{A})$ of the corresponding Siegel Eisenstein series are cuspidal, so Hida theory for cuspidal Siegel modular forms can be applied to finish the construction. However, the new Eisenstein series for the improved p-adic L-function do not restrict to (p-adic) cuspidal forms on $\mathrm{Sp}(2n, \mathbb{A}) \times \mathrm{Sp}(2n, \mathbb{A})$. Therefore, Hida theory for non-cuspidal Siegel modular forms needs to be developed in order to construct the improved p-adic L-function. Such a theory has been developed for Siegel modular forms with $P = \mathrm{GL}(n)$ [Pil12], and for $\mathrm{U}(2,2)$ [SU14] which is later generalized to $\mathrm{U}(n,1)$ [Hsi14a]. In the second section we develop Hida theory for p-adic Siegel modular forms vanishing along the strata of the toroidal compactification associated with cusp labels of rank strictly bigger than r, for an integer $r \leq n_d$.

Our approach is different from that in [SU14, Hsi14a] where they introduce the subsheaf $\omega_{\underline{t}}^\flat$ inside $\omega_{\underline{t}}$ and prove the base change property for its global sections. Instead, ours is based on a careful analysis of the quotient $\mathcal{V}_{m,l}^{SP,r} \Big/ \mathcal{V}_{m,l}^{SP,r-1}$, where $\mathcal{V}_{m,l}^{SP,r}$ (resp. $\mathcal{V}_{m,l}^{SP,r-1}$) denotes the space of functions on the l-th layer of the Igusa tower modulo p^m which vanish along the strata of the toroidal compactification associated with cusp labels of rank strictly bigger than r (resp. $r-1$). This allows us to define a useful subspace $\mathcal{V}_{m,l}^{SP,r,\flat} \subset \mathcal{V}_{m,l}^{SP,r}$, and to establish the exact sequence

$$0 \longrightarrow \mathcal{V}^{SP,r-1} \longrightarrow \mathcal{V}^{SP,r,\flat} \longrightarrow \bigoplus_{\substack{V \in \mathfrak{C}_{\mathbb{V}}/\Gamma \\ \mathrm{rk}\, V = r}} \mathbb{Z}_p[[T_P(\mathbb{Z}_p)]] \otimes_{\mathbb{Z}_p[[T_{P_{n-r}}(\mathbb{Z}_p)]]} \mathcal{V}_V^{SP_{n-r},0} \longrightarrow 0, \tag{0.0.1}$$

from which one can establish Hida theory for $\mathcal{V}^{SP,r}$ by using cuspidal Hida theory and induction on r.

The idea of using exact sequences involving non-cupidal Siegel modular forms and Siegel modular forms of lower genus also appears in [Wei83, BR89]. As they work in characteristic 0, for an irreducible algebraic representation W of $\mathrm{GL}(n)$, and a congruence subgroup $L \subset \mathrm{GL}(n)$ consisting of elements of the form $\begin{pmatrix} I_{n_r} & * \\ 0 & * \end{pmatrix}$, one has

$$W(R)^L = W(\mathbb{Q})^L \otimes R, \quad \text{for a } \mathbb{Q}\text{-algebra } R. \tag{0.0.2}$$

However, (0.0.2) fails if $\mathbb{Q}$ is replaced by $\mathbb{Z}_p$. The failure of equation (0.0.2) causes the difficulty for directly generalizing the Hida theory for cuspidal Siegel modular forms to non-cuspidal Siegel modular forms. The sheaf $\omega^\flat_{\underline{t}}$ in [SU14, Hsi14a] is about remedying the failure of (0.0.2) when $\mathbb{Q}$ is replaced by $\mathbb{Z}_p$. This issue is bypassed in our approach, as we study the space $\mathcal{V}^{SP,r}$ via the terms on the two ends of the exact sequence (0.0.1).

Our results are summarized as follows:

Theorem (Theorem 1.3.1). *For the given parabolic subgroup $P \subset \mathrm{GL}(n)$ and an integer $1 \le r \le n_d$, the following holds:*

(i) An ordinary projector $e_P = e_P^2$ can be defined on $\mathcal{V}^{SP,r}$, and the Pontryagin dual of its ordinary part

$$\mathcal{V}^{r,*}_{P\text{-ord}} = \mathrm{Hom}_{\mathbb{Z}_p}\left(e_P\mathcal{V}^{SP,r}, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

(which is naturally an $\mathcal{O}_F[[T_P(\mathbb{Z}_p)]]$-module) is finite free over $\Lambda_P = \mathcal{O}_F[[T_P(\mathbb{Z}_p)^\circ]]$.

(ii) Define

$$\mathcal{M}^r_{P\text{-ord}} = \mathrm{Hom}_{\Lambda_n}\left(\mathcal{V}^{r,*}_{P\text{-ord}}, \Lambda_P\right).$$

Given a dominant arithmetic weight $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ with dominant algebraic part $\underline{t}^P \in X(T_P)^+$ and finite order part $\underline{\epsilon}^P \in \mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}^\times)$, let $\mathcal{P}_{\underline{\tau}^P}$ be the corresponding prime ideal of $\mathcal{O}_F[[T_P(\mathbb{Z}_p)]]$. Then

$$\mathcal{M}^r_{P\text{-ord}} \otimes_{\mathcal{O}_F[[T_P(\mathbb{Z}_p)]]} \mathcal{O}_F[[T_P(\mathbb{Z}_p)]]/\mathcal{P}_{\underline{\tau}^P} \xrightarrow{\sim} \varprojlim_m \varinjlim_l e_P V^{SP,r}_{m,l}[\underline{\tau}^P],$$

and (see (0.0.5)(1.2.2) and (0.0.7) for the definition of the congruence subgroup $\Gamma_{SP} \subset \mathrm{Sp}(2n,\mathbb{Z})$ and weight $\iota(\underline{t}^P) \in X(T)$ associated to $\underline{t}^P \in X(T_P)$)

$$\varinjlim_l e_P M^r_{\iota(\underline{t}^P)}\left(\Gamma \cap \Gamma_{SP}(p^l), \underline{\epsilon}^P; F\right) \hookrightarrow \left(\mathcal{M}^r_{P\text{-ord}} \otimes_{\mathcal{O}_F[[T_P(\mathbb{Z}_p)]]} \mathcal{O}_F[[T_P(\mathbb{Z}_p)]]/\mathcal{P}_{\underline{\tau}^P}\right)[1/p].$$

Here the maps are equivariant under the action of the unramfied Hecke algebra away from Np and the $\mathbb{U}_p^P$-operators.

(iii) When $\underline{\epsilon}^P$ is trivial and $t_1^P \gg t_2^P \gg \cdots \gg t_d^P \gg 0$, the above embedding is an isomorphism.

(iv) There is the following so-called fundamental exact sequence (in the study of Klingen Eisenstein congruence),

$$0 \longrightarrow \mathcal{M}^{r-1}_{P\text{-ord}} \longrightarrow \mathcal{M}^r_{P\text{-ord}} \longrightarrow \bigoplus_{\substack{V \in \mathfrak{C}_{\mathbb{V}}/\Gamma \\ \mathrm{rk}\, V = r}} \mathcal{M}^0_{V,P_{n-r}\text{-ord}} \otimes_{\mathcal{O}_F[[T_{P_{n-r}}(\mathbb{Z}_p)]]} \mathcal{O}_F[[T_P(\mathbb{Z}_p)]] \longrightarrow 0,$$

and $\mathcal{M}^0_{V,P_{n-r}\text{-ord}}$ is the $\mathcal{O}_F[[T_{P_{n-r}}(\mathbb{Z}_p)]]$-module of families of p-adic ordinary Siegel modular forms of degree $n-r$ over $Y_{V,\mathrm{ord}}$ for the parabolic $P_{n-r} \subset \mathrm{GL}(n-r)$ defined by the partition $n-r = n_1 + \cdots + n_{d-1} + (n_d - r)$.

This construction can be generalized to other PEL type Shimura varieties, both with (using [Hid02]) and without (using [EM17, BR17]) ordinary locus.

With the new choice of sections at p for the Siegel Eisenstein series and the Hida theory for non-cuspidal Siegel modular forms we can then construct the improved p-adic L-function:

Theorem (Theorem 2.6.2). *With the same notation as above, assume that the parity of $\mathscr{C}_P$ is compatible with ϕ (i.e. $\phi(-1) = \tau_d^P(-1)$ for a $\underline{\tau}^P$ in the image of the projection of $\mathscr{C}_P$ to the weight space). There is a p-adic L-function $\mathcal{L}^{P\text{-imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(x) \in F_{\mathscr{C}_P}$ which satisfies the following interpolation property. If x is étale and its projection $\underline{\tau}^P$ in the weight space is admissible, then*

$$\mathcal{L}^{P\text{-imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(x) = C_{\underline{t}^P} \cdot \sum_{\varphi\in\mathfrak{s}_x} \frac{\mathfrak{c}(\varphi,\beta_1)\mathfrak{c}(e_P\mathcal{W}(\varphi),\beta_2)}{\langle\varphi,\overline{\varphi}\rangle} \\ \times E_p^{P\text{-imp}}(n+1-t_d^P,\pi_x\times\phi\epsilon_d^P)\cdot L^{Np\infty}(n+1-t_d^P,\pi_x\times\phi\epsilon_d^P),$$

for $E_p^{P\text{-imp}}$ defined as in §2.3.

The Greenberg–Stevens method [GS93] allows us to prove the following theorem on semi-stable trivial zeroes:

Theorem (Theorem 3.3.5). *Let x_0 be an F-point of $\mathscr{C}_P$ where the weight projection map $\Lambda_P \to \mathbb{T}^{1,N}_{P\text{-ord}}$ is étale and maps x_0 to $\underline{\tau}_0^P$. Suppose that the p-adic L-function $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2} \in \mathbb{I}_{\mathscr{C}_P}[[S]]\otimes_{\mathbb{I}_{\mathscr{C}_P}} F_{\mathscr{C}_P}$ has a semi-stable trivial zero at $((1+p)^{n+1}-1,x_0)$ and the local-global compatibility is satisfied by the p-adic Galois representation ρ_{x_0}. Then we have*

$$\left.\frac{\mathrm{d}\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}(S,x_0)}{\mathrm{d}S}\right|_{S=(1+p)^{n+1}-1} = -\,\ell(\rho_{x_0})\cdot C_{\underline{t}_0^P}\cdot \sum_{\varphi\in\mathfrak{s}_{x_0}} \frac{\mathfrak{c}(\varphi,\beta_1)\mathfrak{c}(e_P\mathcal{W}(\varphi),\beta_2)}{\langle\varphi,\overline{\varphi}\rangle} \\ \times E_p^{P\text{-imp}}(0,\pi_{x_0}\times\phi)\cdot L^{Np\infty}(0,\pi_{x_0}\times\phi),$$

where $\ell(\rho_{x_0})$ is the ℓ-invariant as defined by Greenberg.

This result almost implies the conjecture of Greenberg, up to the non-vanishing of the ℓ-invariant and of the imprimitive L-function. The non-vanishing of the ℓ-invariant is a very hard problem and it is known only in the case of [MTT86], thanks to a deep result in transcendental number theory stating that q_E is trascendental [BSDGP96]. Note that for $n = 2$ we know the non-vanishing of $\ell(\rho_{x_0})$ whenever $\pi_{x_0} = \mathrm{Sym}^3(\pi_{f_E})$, where f_E is the weight two modular form associated with an elliptic curve with semi-stable reduction at p. The imprimitive L-function could vanish because of the vanishing of some of the Euler factors at a prime ℓ dividing N. One may deal with such vanishing by selecting better sections at $\ell|N$.

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Notation. For the whole length of the paper we fix an odd prime p as well as isomorphism between $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$. Also, we fix a positive integer $N \geq 3$ prime to p and an integer $n \geq 1$ together with a partition $n = n_1 + \cdots + n_d$ with $n_1, \ldots, n_d \geq 1$.

We denote by $\mathbb{V}$ a free $\mathbb{Z}$-module of rank $2n$ with a standard basis $e_1, \ldots, e_n, f_1, \cdots, f_n$ equipped with a symplectic pairing given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ with respect to the standard basis. Then $e_1, \cdots, e_n$

span a maximal isotropic subspace V_n inside $\mathbb{V}$. Set $V_n^* = \mathbb{V}/V_n^\perp$. One can canonically identify V_n^* with the maximal isotropic subspace of $\mathbb{V}$ spanned by $f_1, \cdots, f_n$ and there is the polarization $\mathbb{V} = V_n \oplus V_n^*$.

Let $G = \mathrm{Sp}(2n)$ be the algebraic group acting on $\mathbb{V}$ preserving the symplectic pairing. In matrix form it is

$$\left\{ g \in \mathrm{GL}(2n) : {}^{\mathrm{t}}g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Let Q_G be the standard Siegel parabolic subgroup of G preserving V_n, whose unipotent subgroup we denote by U_{Q_G}. We identify the Levi subgroup of Q_G with $\mathrm{GL}(n)$ via the map

$$\begin{aligned} Q_G &\longrightarrow \mathrm{GL}(n) \\ \begin{pmatrix} a & b \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix} &\longmapsto a. \end{aligned} \tag{0.0.3}$$

Denote by B the standard Borel subgroup of $\mathrm{GL}(n)$ consisting of upper triangular matrices, and by U_B, T its unipotent radical and maximal torus respectively. We fix the isomorphism of $\mathbb{G}_m^n$ with T which sends $(a_1, \ldots, a_n)$ to $\mathrm{diag}(a_1, \ldots, a_n)$. The inverse image under (0.0.3) of B constitutes the standard Borel subgroup B_G of G with unipotent radical N_G and maximal torus T_G. The tori T and T_G are identified via the map (0.0.3).

We put ourselves in the setting of [Pil12], i.e. considering Siegel modular forms ordinary with respect to a general parabolic subgroup of $\mathrm{GL}(n)$ containing B associated to our fixed partition $n = n_1 + \cdots + n_d$ (the ordinarity considered in [Hid02] is the ordinarity with respect to B). Set $N_i = \sum_{j=1}^{i} n_i$, $1 \leq i \leq d$. Define

$$P = \left\{ \begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_d \end{pmatrix} \in \mathrm{GL}(n) \;\middle|\; a_i \in \mathrm{GL}(n_i),\, 1 \leq i \leq d \right\}, \tag{0.0.4}$$

$$SP = \left\{ \begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_d \end{pmatrix} \in \mathrm{SL}(n) \;\middle|\; a_i \in \mathrm{SL}(n_i),\, 1 \leq i \leq d \right\}, \tag{0.0.5}$$

and U_P to be the unipotent radical of P. When the partition is taken as $n = 1 + 1 + \cdots + 1$, the group P, (resp. both SP and U_P) is just B (resp. U_B). Let $T_P = P/SP$ and we fix the following isomorphism

$$\begin{aligned} T_P = P/SP &\xrightarrow{\sim} \mathbb{G}_m^d \\ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix} u &\longmapsto (\det(a_1), \ldots, \det(a_d)). \end{aligned} \tag{0.0.6}$$

Note here that T_P is not the maximal torus in P. Denote by $X(T_P)$ the group of characters of T_P (which are also naturally viewed as characters of P). We identify it with $\mathbb{Z}^d$ by associating to $\underline{t}^P := (t_1^P, \ldots, t_d^P)$ the character sending $\mathrm{diag}(a_1, \ldots, a_d)$ to $\prod_{i=1}^{d} \det(a_i)^{t_i^P}$. When working with B, we shall drop the superscript from the notation for the characters when there is unlikely confusion. The map (0.0.6) restricts to a map $T \to T_P$, which induces an embedding

$$\begin{aligned} \imath : X(T_P) &\longrightarrow X(T) \\ \underline{t}^P := (t_1^P, \ldots, t_d^P) &\longmapsto (\underbrace{t_1^P, \ldots, t_1^P}_{n_1}, \underbrace{t_2^P, \ldots, t_2^P}_{n_2}, \ldots, \underbrace{t_d^P, \ldots, t_d^P}_{n_d}). \end{aligned} \tag{0.0.7}$$

Denote by $X(T)^+$ the subset of $X(T)$ of dominant weights with respect to B and set $X(T_P)^+ = X(T_P) \cap X(T)^+$. Then $\underline{t}^P \in X(T_P)$ belongs to $X(T_P)^+$ if and only if $t_1^P \geq t_2^P \geq \cdots \geq t_d^P$. Fix a finite extension F of $\mathbb{Q}_p$ (assumed to be sufficiently large in the context) and denote by $\mathcal{O}_F$ its ring of integers. The weight space in Hida theory for P-ordinary Siegel modular forms over F is $\operatorname{Spec}(\mathcal{O}_F[[T_P(\mathbb{Z}_p)]])$. For an arithmetic point of the weight space $\underline{\tau}^P \in \operatorname{Spec}(\mathcal{O}_F[[T_P(\mathbb{Z}_p)]])(\overline{\mathbb{Q}}_p)$, i.e. a character in $\operatorname{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right)$ that is the product of an algebraic and a finite order character, we write its algebraic part (resp. finite order part) as $\underline{\tau}_{\mathrm{alg}}^P = \underline{t}^P = (t_1^P, \ldots, t_d^P)$ (resp. $\underline{\tau}_{\mathrm{f}}^P = \underline{\epsilon}^P = (\epsilon_1^P, \ldots, \epsilon_d^P)$).

We fix the standard additive character $\mathbf{e}_{\mathbb{A}} = \bigotimes_v \mathbf{e}_v : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ with local component $\mathbf{e}_v$ defined as $\mathbf{e}_v(x) = \begin{cases} e^{-2\pi i \{x\}_v}, & v \neq \infty \\ e^{2\pi i x}, & v = \infty \end{cases}$ where $\{x\}_v$ is the fractional part of x.

1. Non-cuspidal Hida theory

In this section we develop Hida theory for non-cuspidal Siegel modular forms, or more precisely, for the P-ordinary Siegel modular forms vanishing along the strata with cusp labels of rank $> r$ for some $0 \leq r \leq n_d$. Later, the family of Siegel Eisenstein series on $\mathrm{Sp}(4n)$ we shall use to construct the improved p-adic L-function, unlike the ones for the usual p-adic L-function, do not restrict to cuspidal forms on $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$ (see the discussion at the end of §2.4.7). The Hida theory developed here will be applied to them. Also, we expect such a theory to be of independent interest and to find applications elsewhere, for instance in the study of Eisenstein congruences.

The main difficulty in directly generalizing Hida theory for cuspidal forms on PEL Shimura varieties to non-cuspidal forms is that for an algebraic representation W of $\mathrm{GL}(n)_{/\mathbb{Z}}$ of finite rank, an algebra R and a subgroup $L \subset \mathrm{GL}(n,\mathbb{Z})$ of the form

$$L = \left\{ \begin{pmatrix} I_{n-r} & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}(n,\mathbb{Z}) \right\}, \quad 1 \leq r \leq n,$$

the module $W(R/p^m)^L$ is not necessarily equal to $W(R)^L \otimes R/p^m$.

In [SU14, §6][Hsi14b, §4], a subsheaf of $\omega_{\underline{t}}^\flat \subset \omega_{\underline{t}}$ is introduced to remedy this failure of base change property. The sheaf $\omega_{\underline{t}}^\flat$ is not free and differs from $\omega_{\underline{t}}$ along the boundary of the toroidal compactification. The base change property for global sections of $\omega_{\underline{t}}^\flat$ is shown *loc. cit.* With this base change property, by mimicking Hida's method [Hid02], Hida theory for certain non-cuspidal forms on $\mathrm{U}(2,2)$ and $\mathrm{U}(n,1)$ is established *loc. cit.*

Here we take a different approach. Let $\mathcal{V}^{SP,r}$ be the space of p-adic forms for the parabolic P vanishing along strata indexed by cusp labels of rank $> r$ with p-power torsion coefficients. Instead of studying the space $\mathcal{V}^{SP,r}$ via the classical Siegel modular forms embedded in it through (1.2.1) (for which a base change property for the space of certain non-cuspidal classical Siegel modular forms is required), we make a careful analysis of the Igusa tower over the boundary and define a nice subspace $\mathcal{V}^{SP,r,\flat}$ inside $\mathcal{V}^{SP,r}$. The exact sequences in Proposition 1.7.1 plus Proposition 1.9.3 allow us to deduce desired properties for the space $\mathcal{V}^{SP,r,\flat}$ from those for the space of cuspidal p-adic Siegel modular forms. Meanwhile, Proposition 1.9.4 shows that the desired properties for $\mathcal{V}^{SP,r,\flat}$ imply the existence of a nice ordinary projection on $\mathcal{V}^{SP,r}$. Then we obtain Hida theory for non-cuspidal Siegel modular forms as summarized in Theorem 1.3.1.

Exact sequences for automorphic bundles and p-adic analytic deformation of automorphic bundles, similar to those in Proposition 1.7.1, are used in [Wei83] and [BR15], where things are simpler than our case here because everything is in characteristic zero and the issue of base change does not appear.

1.1. **Compactifications of Siegel varieties.** We start by briefly recalling some facts on the toroidal and minimal compactifications of Siegel varieties of principal level. We mainly follow the notation in [Pil12]. Fix an integer $N \geq 3$ coprime to p. Let Y be the degree n Siegel variety of principal level N defined over $\mathbb{Z}[1/N, \zeta_N]$. All the objects we consider in the following are endowed with principal level N structure and we shall omit N to lighten the notation.

Recall that $\mathbb{V} = \mathbb{Z}^{2n}$ with standard basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ and the symplectic pairing given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. For $1 \leq r \leq n$, let V_r be the submodule of $\mathbb{V}$ spanned by $e_1, \cdots, e_r$ (we sometimes call it the standard submodule of rank r in $\mathbb{V}$), and we put $V_0 = \{0\}$. The group $\mathrm{Sp}(\mathbb{V}) \cong \mathrm{Sp}(2n, \mathbb{Z})$ acts on $\mathbb{V}$ preserving the symplectic pairing. Denote by Γ the kernel of the projection $\mathrm{Sp}(2n, \mathbb{Z}) \to \mathrm{Sp}(2n, \mathbb{Z}/N\mathbb{Z})$.

Denote by $\mathfrak{C}_{\mathbb{V}}$ the set of cotorsion free isotropic $\mathbb{Z}$-submodules of $\mathbb{V}$. The group $\mathrm{Sp}(2n, \mathbb{Z})$ acts naturally on $\mathfrak{C}_{\mathbb{V}}$. The quotient $\mathfrak{C}_{\mathbb{V}}/\Gamma$ is called the set of cusp labels of level Γ (or of principal level N). For a free $\mathbb{Z}$-module X of finite rank, we write $C(X)$ to denote the cone of positive semi-definite symmetric bilinear forms on $X \otimes \mathbb{R}$ with rational radicals. A surjective morphism $X \to X'$ of free $\mathbb{Z}$-modules induces an inclusion $C(X') \hookrightarrow C(X)$. Define $\mathcal{C}_{\mathbb{V}}$ as the quotient of the disjoint union $\coprod_{V \in \mathfrak{C}_{\mathbb{V}}} C(\mathbb{V}/V^{\perp})$ by the equivalence relations induced by the inclusions $C(\mathbb{V}/V^{\perp}) \hookrightarrow C(\mathbb{V}/V'^{\perp})$ for $V \subset V'$, $V, V' \in \mathfrak{C}_{\mathbb{V}}$. The group $\mathrm{Sp}(2n, \mathbb{Z})$ acts on $\mathcal{C}_{\mathbb{V}}$.

A $\mathrm{GL}(n, \mathbb{Z})$-admissible smooth rational polyhedral cone decomposition Σ of $C(\mathbb{Z}^n)$ ([FC90, Definition 2.2]) gives rise to a rational polyhedral cone decomposition $\Sigma_{\mathcal{C}_{\mathbb{V}}}$ of $\mathcal{C}_{\mathbb{V}}$. Corresponding to it is a toroidal compactification X^{Σ} of Y endowed with an action of $\mathrm{Sp}(2n, \mathbb{Z})$ [FC90, §IV.6].

The toroidal compactification X^{Σ} comes with a stratification indexed by $\Sigma_{\mathcal{C}_{\mathbb{V}}}/\Gamma$, and we denote by Z_{σ} the stratum in X^{Σ} associated with $\sigma \in \Sigma_{\mathcal{C}_{\mathbb{V}}}$. There is a canonical map $\Sigma_{\mathcal{C}_{\mathbb{V}}} \to \mathfrak{C}_{\mathbb{V}}$ sending σ to the unique $V_{\sigma} \in \mathfrak{C}_{\mathbb{V}}$ satisfying $\sigma \subset C(\mathbb{V}/V_{\sigma}^{\perp})^{\circ}$. The locally closed subscheme $Z_V \subset X^{\Sigma}$ is defined as the union $\coprod_{\sigma \in \Sigma_{\mathcal{C}_{\mathbb{V}}}/\Gamma, V_{\sigma} = V} Z_{\sigma}$. For $0 \leq r \leq n$, define $\mathcal{I}^{r}_{X^{\Sigma}}$ to the sheaf of ideals associated to the closed subscheme $\coprod_{V \in \mathfrak{C}_{\mathbb{V}}/\Gamma, \mathrm{rk} V > r} Z_V$.

Over X^{Σ}, there is the canonical semi-abelian scheme $\mathcal{G}_{/X^{\Sigma}}$ whose restriction to Y is the universal principally polarized abelian scheme of genus n with principal level N structure. The coherent sheaf ω over X^{Σ} is defined as the sheaf of invariant differentials of $\mathcal{G}_{/X^{\Sigma}}$, which is locally free of rank n.

From the toroidal compactification, the minimal compactification is constructed as

$$X^{\star} = \mathrm{Proj}\left(\bigoplus_{k \geq 0} \mathrm{H}^0(X^{\Sigma}, \det{}^k \omega) \right).$$

The projection $\pi : X^{\Sigma} \to X^{\star}$ is proper with connected fibres. The minimal compactification $X^{\star}$ is stratified by $\mathfrak{C}_{\mathbb{V}}/\Gamma$. The stratum Y_V corresponding to $V \in \mathfrak{C}_{\mathbb{V}}/\Gamma$ is defined as the image of Z_V. As a scheme it is isomorphic to the Siegel variety of degree $(n - \mathrm{rk}\, V)$ and principal level N.

1.2. **The Igusa tower over the ordinary locus and p-adic forms.** The invertible sheaf $\det^k \omega$ descends to an invertible sheaf on $X^{\star}$, which we still denote by $\det^k \omega$. For sufficiently large k, it is very ample over $X^{\star}$. Choose k such that the k-th power of the Hasse invariant, which is an element in $\mathrm{H}^0\left(X^{\star}_{/\mathbb{F}_p[\zeta_N]}, \det^{p-1} \omega\right)$, lifts to a global section $E \in \mathrm{H}^0\left(X^{\star}, \det^{t(p-1)} \omega\right)$. We write its pull-back in $\mathrm{H}^0\left(X^{\Sigma}, \det^{t(p-1)} \omega\right)$ also as E.

Let $X^{\star,\mathrm{ord}} = X^{\star}[1/E]$ and $X^{\Sigma,\mathrm{ord}} = X^{\Sigma}[1/E]$, and define Y_V^{ord}, Z_σ^{ord}, Z_V^{ord} similarly. The reductions modulo powers of p of these schemes are independent of the choice of E and are called the ordinary loci. Note that $X^{\star,\mathrm{ord}}$ is affine, while $X^{\Sigma,\mathrm{ord}}$ is not (except when $n = 1$).

Fix a finite extension F of $\mathbb{Q}_p$ containing all N-th roots of unity. We regard all above schemes as defined over $\mathcal{O}_F$. For $m \geq 1$, we use a subscript $_m$ to indicate the reduction modulo p^m. Over $X_m^{\Sigma,\mathrm{ord}}$, we consider the full Igusa tower

$$\mathscr{T}_{m,l}^{\Sigma} = \underline{\mathrm{Isom}}_{X_m^{\Sigma,\mathrm{ord}}}\left(\mu_{p^l}^n, \mathcal{G}_{/X^{\Sigma,\mathrm{ord}}}[p^l]^\circ\right),$$

for $l \geq 1$. It is an étale cover of $X^{\Sigma,\mathrm{ord}}$ with Galois group isomorphic to $\mathrm{GL}(n, \mathbb{Z}/p^l\mathbb{Z})$. The group

$$\Gamma_0(p^l) = \left\{ g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \in G(\mathbb{Z}) : c_g \equiv 0 \mod p^l \right\}$$

naturally acts on $\mathscr{T}_{m,l}^{\Sigma}$ with g acting on the principal level N structure and a_g acting by the Galois action of $\mathrm{GL}(n, \mathbb{Z}/p^l\mathbb{Z})$ on $\mathscr{T}_{m,l}^{\Sigma}$ over $X^{\Sigma,\mathrm{ord}}$.

Define

$$\mathscr{T}_{SP,m,l}^{\Sigma} = \mathscr{T}_{m,l}^{\Sigma}\Big/SP(\mathbb{Z}_p/p^l\mathbb{Z}_p)$$

(see (0.0.4)(0.0.5) for the definition of the algebraic subgroups P, SP of $\mathrm{GL}(n)$). It parametrizes (in addition to the structure parametrized by $X_m^{\Sigma,\mathrm{ord}}$) the level structure $(E_i, \varepsilon_i)_{1\leq i\leq d,\, p^l}$, where $\{0\} = E_0 \subset E_1 \subset \cdots \subset E_d = \mathcal{G}[p^l]^\circ$ is a d-step increasing filtration and ε_i is an isomorphism $\bigwedge^{n_i} \mu_{p^l}^{n_i} \cong \bigwedge^{n_i} E_i/E_{i-1}$. There is a natural $T_P(\mathbb{Z}_p)$-action on $\mathscr{T}_{SP,m,l}^{\Sigma}$.

Write $f_{m,l} : \mathscr{T}_{SP,m,l}^{\Sigma} \to X_m^{\Sigma,\mathrm{ord}}$ for the natural projection. Define

$$\begin{aligned} V_{m,l}^{SP,r} &= \mathrm{H}^0\left(\mathscr{T}_{SP,m,l}^{\Sigma},\, f_{m,l}^* \mathcal{I}_{X_m^\Sigma}^r\right), \\ \mathcal{V}^{SP,r} &= \varinjlim_m \varinjlim_l V_{m,l}^{SP,r}. \end{aligned}$$

The space $\mathcal{V}^{SP,r}$ is called the space of p-adic Siegel modular forms for the parabolic P vanishing along the strata indexed by cusp labels of rank $> r$ with p-power torsion coefficients. When $P = B$ we shall drop the P from the notation, and when $r = n$ we shall drop r from the notation.

The $T_P(\mathbb{Z}_p)$-action on $\mathscr{T}_{SP,m,l}^{\Sigma}$ equips $V_{m,l}^{SP,r}$ and $\mathcal{V}^{SP,r}$ with an $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-module structure. The space $\mathcal{V}^0$ is the space of cuspidal p-adic Siegel modular forms with p-power torsion coefficients, which is considered in Hida theory (for the Borel B) for cuspidal Siegel modular forms, while $\mathcal{V}^{SP,0}$ (for general P) is the one in [Pil12].

Besides the torsion $\mathbb{Z}_p$-module $\mathcal{V}^{SP,r}$ (which is in fact p-divisible by Remark 1.5.1), we will also consider the $\mathbb{Z}_p$-module $\varprojlim_m \varinjlim_l V_{m,l}^r$, *i.e.* taking the inverse instead of direct limit with respect to m (which is torsion free over $\mathbb{Z}_p$ by Remark 1.5.1). It is the torsion $\mathbb{Z}_p$-module $\mathcal{V}^{SP,r}$ that will be used to construct the $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-module of Hida families. Meanwhile, the space $\varprojlim_m \varinjlim_l V_{m,l}^r$ is more easily seen related to the classical Siegel modular forms.

More precisely, for $\underline{t}^P \in X(T_P)^+$, $\underline{\epsilon}^P \in \mathrm{Hom}\left(T_P(\mathbb{Z}/p^l), \mathbb{C}^\times\right)$, and $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right)$ which is the product of $\underline{t}^P$ and $\underline{\epsilon}^P$, there is a canonical Hecke-equivariant embedding [Pil12, §4.2.1]

$$\varinjlim_l \mathrm{M}_{\imath(\underline{t}^P)}^r\left(\Gamma \cap \Gamma_{SP}(p^l), \underline{\epsilon}^P; F\right) \hookrightarrow \left(\varprojlim_m \varinjlim_l V_{m,l}^{SP,r}[\underline{\tau}^P]\right)[1/p]. \tag{1.2.1}$$

Here $\mathrm{M}^r_{\underline{t}}\left(\Gamma \cap \Gamma_{SP}(p^l), \underline{\epsilon}^P; F\right)$ denotes the space of classical holomorphic Siegel modular forms of weight $\underline{t} = \imath(\underline{t}^P)$ and level $\Gamma \cap \Gamma_{SP}(p^l)$ with nebentypus $\underline{\epsilon}^P$ vanishing along strata with cusp labels of rank > 1, and the congruence subgroup $\Gamma_{SP}(p^l)$ is defined as

$$\Gamma_{SP}(p^l) = \left\{ g \in \mathrm{Sp}(2n, \mathbb{Z}) : g \bmod p^l \text{ belongs to } SP(\mathbb{Z}/p^l) \right\}. \tag{1.2.2}$$

The vanishing condition here for the classical Siegel modular forms is equivalent to requiring that at all cusps all the Fourier coefficients with indices of corank $> r$ vanish. The space $\varprojlim_m \varinjlim_l V^{SP,r}_{m,l}[\underline{\tau}^P]$ is the $\underline{\tau}^P$-eigenspace for the action of $T_P(\mathbb{Z}_p)$ on $\varprojlim_m \varinjlim_l V^{SP,r}_{m,l}$. The construction of the embedding (1.2.1) mainly relies on the Hodge–Tate map

$$\mathrm{Hom}_R(\mathcal{G}[p^\infty]^\circ, \mu_{p^\infty}) \otimes_{\mathbb{Z}_p} R \xrightarrow{\sim} \omega_{\mathcal{G}/R}$$

for an ordinary semi-abelian scheme $\mathcal{G}$ over a $\mathbb{Z}_p$-algebra R.

1.3. **The main theorem.** Our goal is to establish the following theorem.

Theorem 1.3.1. *For given $P \subset \mathrm{GL}(n)$ as in (0.0.4) and an integer $1 \leq r \leq n_d$, we have the following.*

(i) An ordinary projector $e_P = e_P^2$ can be defined on $\mathcal{V}^{SP,r}$, and the Pontryagin dual of its ordinary part

$$\mathcal{V}^{r,*}_{P\text{-ord}} = \mathrm{Hom}_{\mathbb{Z}_p}\left(e_P \mathcal{V}^{SP,r}, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

(which is naturally an $\mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket$-module) is finite free over $\Lambda_P = \mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)^\circ]]$, where $T_P(\mathbb{Z}_p)^\circ$ is the maximal p-profinite subgroup of $T_P(\mathbb{Z}_p)$.

(ii) Define

$$\mathcal{M}^r_{P\text{-ord}} = \mathrm{Hom}_{\Lambda_P}\left(\mathcal{V}^{r,*}_{P\text{-ord}}, \Lambda_P\right).$$

Given an arithmetic weight $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ with dominant algebraic part $\underline{t}^P \in X(T_P)^+$ and finite order part $\underline{\epsilon}^P \in \mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}^\times)$, let $\mathcal{P}_{\underline{\tau}^P}$ be the corresponding prime ideal of $\mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket$. Then

$$\mathcal{M}^r_{P\text{-ord}} \otimes_{\mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket} \mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket/\mathcal{P}_{\underline{\tau}^P} \xrightarrow{\sim} \varprojlim_m \varinjlim_l e_P V^{SP,r}_{m,l}[\underline{\tau}^P],$$

which combining with (1.2.1) gives

$$\varinjlim_l e_P \mathrm{M}^r_{\imath(\underline{t}^P)}\left(\Gamma \cap \Gamma_{SP}(p^l), \underline{\epsilon}^P; F\right) \hookrightarrow \left(\mathcal{M}^r_{P\text{-ord}} \otimes_{\mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket} \mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket/\mathcal{P}_{\underline{\tau}^P}\right)[1/p]. \tag{1.3.1}$$

Here the maps are equivariant under the action of the unramfied Hecke algebra away from Np and the $\mathbb{U}^P_p$-operators.

(iii) When $\underline{\epsilon}^P$ is trivial and $t^P_1 \gg t^P_2 \gg \cdots \gg t^P_d \gg 0$, the embedding (1.3.1) is an isomorphism.

(iv) There is the following so-called fundamental exact sequence (in the study of Klingen Eisenstein congruence),

$$0 \longrightarrow \mathcal{M}^{r-1}_{P\text{-ord}} \longrightarrow \mathcal{M}^r_{P\text{-ord}} \longrightarrow \bigoplus_{\substack{V \in \mathfrak{C}_{\mathbb{V}}/\Gamma \\ \mathrm{rk}\, V = r}} \mathcal{M}^0_{V, P_{n-r}\text{-}\mathrm{ord}} \otimes_{\mathcal{O}_F\llbracket T_{P_{n-r}}(\mathbb{Z}_p)\rrbracket} \mathcal{O}_F\llbracket T_P(\mathbb{Z}_p)\rrbracket \longrightarrow 0,$$

where

$$P_{n-r} = \left\{ \begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_d \end{pmatrix} \in \mathrm{GL}(n-r) \;\middle|\; a_i \in \mathrm{GL}(n_i),\, 1 \leq i \leq d-1,\, a_d \in \mathrm{GL}(n_d - r) \right\},$$

and $\mathcal{M}^0_{V,P_{n-r}\text{-ord}}$ *is the* $\mathcal{O}_F[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]$*-module of families of* p*-adic cuspidal ordinary Siegel modular forms of degree* $n-r$ *over* $Y_{V,\mathrm{ord}}$ *for the parabolic* P_{n-r}*.*

Remark 1.3.2. For two different P, P', there are no inclusion relations between $\mathcal{M}^r_{P\text{-ord}}$ and $\mathcal{M}^r_{P'\text{-ord}}$, thus in order to study certain Siegel modular forms by using Hida theory, one needs first to specify a parabolic P containing the standard Borel subgroup of $\mathrm{GL}(n)$ such that the Siegel modular forms under inspection are $SP(\mathbb{Z}_p)$-invariant and P-ordinary. There is not such Hida theory (as to the authors' knowledge) that studies P-ordinary Siegel modular forms for all P simultaneously. On the contrary, when studying families of finite slope families, there is no need to specify a parabolic as the theory established for the Borel (e.g. [AIP15]) treats all Siegel modular forms of finite slope.

The remaining part of this section is devoted to proving this theorem. The proof relies on a careful study of the quotient $\mathcal{V}^{SP,r}/\mathcal{V}^{SP,r-1}$ and the boundary of the Igusa tower, which leads to the definition of the subspace $\mathcal{V}^{SP,r,\flat}_{m,l} \subset \mathcal{V}^{SP,r}_{m,l}$. This subspace is characterized by the vanishing along certain connected components of the Igusa tower over $\coprod_{\substack{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma \\ \mathrm{rk}\, V=r}} Z_{V,\mathrm{ord}}$, and plays an important role in our proof of the above theorem.

1.4. **The Mumford construction.** We quickly recall the Mumford construction which will be used in the description of the fibre of the push-forward of the ideal sheaf $\mathcal{I}^r_{X^\Sigma}$, as well as in the definition of q-expansions.

Given a free $\mathbb{Z}$-module X_r of rank r with basis $x_1, \cdots, x_r$, set X_r^* to be its dual free $\mathbb{Z}$-module with dual basis $x_1^*, \cdots, x_r^*$. Let $\mathscr{T}_{n-r,m,l}$ the Igusa tower, $m, l \geq 1$, over the degree $(n-r)$ Siegel variety of principal level N and $(\mathcal{A}_{/\mathscr{T}_{n-r,m,l}}, \psi_{N,\mathscr{T}_{n-r,m,l}}, \phi_{p,\mathscr{T}_{n-r,m,l}})$ be the universal object over it.

The extensions of $\mathcal{A}_{/\mathscr{T}_{n-r,m,l}}$ by the torus $X_r^*\otimes\mathbb{G}_m$ are parametrized by $\mathrm{Hom}_{\mathscr{T}_{n-r,m,l}}(X_r, \mathcal{A}_{/\mathscr{T}_{n-r,m,l}})$. Let $\mathcal{B}_{X_r^*,m,l}$ be an abelian scheme which is isogenous to

$$\mathrm{Hom}_{\mathscr{T}_{n-r,m,l}}(N^{-1}X_r, \mathcal{A}_{/\mathscr{T}_{n-r,m,l}})$$

via an isogeny of degree a power of p, related to the p-level structure of the Igusa tower. Given $\mu \in N^{-1}X_r$, there is tautologically a map $c(\mu) : \mathcal{B}_{X_r^*,m,l} \to \mathcal{A}_{/\mathscr{T}_{n-r,m,l}}$ through evaluation at μ. Denote by $S^2(X_r)$ the symmetric quotient of $X_r \otimes_{\mathbb{Z}} X_r$. Let $\mathcal{P} \to \mathcal{A}_{/\mathscr{T}_{n-r,m,l}} \times_{\mathscr{T}_{n-r,m,l}} \mathcal{A}_{/\mathscr{T}_{n-r,m,l}}$ be the Poincaré bundle and $\mathcal{P}^\times \to \mathcal{A}_{/\mathscr{T}_{n-r,m,l}} \times_{\mathscr{T}_{n-r,m,l}} \mathcal{A}_{/\mathscr{T}_{n-r,m,l}}$ be its associated $\mathbb{G}_m$-torsor.

Pick a basis $[\mu_i \otimes \nu_i]$, $1 \leq i \leq r(r-1)/2$, of $N^{-1}S^2(X_r)$ with μ_i, ν_i belonging to $N^{-1}X_r$. Associated to each $[\mu_i \otimes \nu_i]$ there is a map

$$c(\mu_i) \times c(\nu_i) : \mathcal{B}_{X_r^*,m,l} \longrightarrow \mathcal{A}_{/\mathscr{T}_{n-r,m,l}} \times_{\mathscr{T}_{n-r,m,l}} \mathcal{A}_{/\mathscr{T}_{n-r,m,l}},$$

along which one can pull back the Poincaré bundle and its associated $\mathbb{G}_m$-torsor. Define

$$\mathcal{M}_{X_r^*,m,l} = \prod_i (c(\mu_i) \times c(\nu_i))^*(\mathcal{P}^\times)^{\otimes N},$$

which is a torsor over $\mathcal{B}_{X_r^*,m,l}$ for the torus $\mathrm{Hom}_{\mathbb{Z}}(N^{-1}S^2(X_r), \mathbb{G}_m)$. For $\lambda = \sum_{i=1}^{r(r-1)/2} a_i[\mu_i \otimes \nu_i] \in S^2(X_r)$, define the invertible sheaf $\mathcal{L}(\lambda)$ over $\mathcal{B}_{X_r^*,m,l}$ as

$$\mathcal{L}(\lambda) = \bigotimes_i c(\mu_i) \times c(\nu_i))^*\mathcal{P}^{\otimes a_i N}.$$

We have

$$\left(\mathcal{M}_{X_r^*,m,l} \to \mathscr{T}_{n-r,m,l}\right)_* \mathcal{O}_{\mathcal{M}_{X_r^*,m,l}} = \bigoplus_{\lambda\in N^{-1}S^2(X_r)} \mathrm{H}^0\left(\mathcal{B}_{X_r^*,m,l}, \mathcal{L}(\lambda)\right).$$

Now suppose $\sigma \subset C(X_r)^\circ$ is a cone generated by a set of elements that extends to a basis of the space of symmetric bilinear forms on X_r. Let $\mathcal{M}_{X_r^*,m,l} \hookrightarrow \mathcal{M}_{X_r^*,m,l,\sigma}$ be the affine torus embedding over $\mathcal{B}_{X_r^*,m,l}$ corresponding to σ. Denote by $S^2(X_r)$ the symmetric quotient of $X_r \otimes X_r$, and by $\sigma^\vee$ the dual cone of σ consisting of elements in $S^2(X_r) \otimes \mathbb{R}$ whose pairing with any element in σ is non-negative. Let $\sigma^{\vee,\circ}$ be the interior of $\sigma^\vee$. Let $\mathcal{I}^\sigma_{\mathcal{M}_{X_r^*,m,l,\sigma}}$ be the ideal sheaf inside $\mathcal{O}_{\mathcal{M}_{X_r^*,m,l,\sigma}}$ attached to the boundary of the affine torus embedding. Then

$$\left(\mathcal{M}_{X_r^*,m,l,\sigma} \to \mathscr{T}_{n-r,m,l}\right)_* \mathcal{O}_{\mathcal{M}_{X_r^*,m,l,\sigma}} = \bigoplus_{\lambda \in N^{-1}S^2(X_r) \cap \sigma^\vee} \mathrm{H}^0\left(\mathcal{B}_{X_r^*,m,l},\, \mathcal{L}(\lambda)\right), \tag{1.4.1}$$

$$\left(\mathcal{M}_{X_r^*,m,l,\sigma} \to \mathscr{T}_{n-r,m,l}\right)_* \mathcal{I}^\sigma_{\mathcal{M}_{X_r^*,m,l,\sigma}} = \bigoplus_{\lambda \in N^{-1}S^2(X_r) \cap \sigma^{\vee,\circ}} \mathrm{H}^0\left(\mathcal{B}_{X_r^*,m,l},\, \mathcal{L}(\lambda)\right). \tag{1.4.2}$$

Let $\widehat{\mathcal{M}}_{X_r^*,m,l,\sigma}$ be the formal completion along the boundary of the torus embedding. The natural map $X_r \to \mathrm{Hom}(X_r, S^2(X_r))$ defines a period subgroup $N^{-1}X_r \subset X_r^* \otimes \mathbb{G}_{m/\mathbb{Z}[N^{-1}S^2(X_r)]}$ with a polarization given by the duality between X_r and X_r^*. The Mumford construction gives a principally polarized semi-abelian scheme $\mathcal{G}_{/\widehat{\mathcal{M}}_{X_r^*,m,l,\sigma}}$, together with a canonical principal level N structure $\psi_{N,\mathrm{can}} : (\mathbb{Z}/N\mathbb{Z})^{2n} \to \mathcal{G}_{/\widehat{\mathcal{M}}_{X_r^*,m,l,\sigma}}[N]$ and a canonical trivialization $\phi_{p,\mathrm{can}} : \mu_{p^l}^n \xrightarrow{\sim} \mathcal{G}_{/\widehat{\mathcal{M}}_{X_r^*,m,l,\sigma}}[p^l]^\circ$, which comes from the level structure parametrized by $\mathscr{T}_{n-r,m,l}$, the extension data parametrized by $\mathcal{B}_{X_r^*,m,l}$ plus the fixed basis of X_r.

1.5. **The fibre of the push-forward to the minimal compactification.** Let $\mathscr{T}^\star_{SP,m,l}$ be the Stein factorization of $\mathscr{T}^\Sigma_{SP,m,l} \to X^{\star,\mathrm{ord}}$,

$$\begin{tikzcd}
\mathscr{T}^\Sigma_{SP,m,l} \arrow[r, "f_{m,l}"] \arrow[d, "\pi_{\mathscr{T}}"] & X_m^{\Sigma,\mathrm{ord}} \arrow[d, "\pi"] \\
\mathscr{T}^\star_{SP,m,l} \arrow[r] & X_m^{\star,\mathrm{ord}}.
\end{tikzcd} \tag{1.5.1}$$

The scheme $\mathscr{T}^\Sigma_{SP,m,l}$ (resp. $\mathscr{T}^\star_{SP,m,l}$) can also be viewed as the partial toroidal (resp. minimal) compactification of the Igusa tower $\mathscr{T}_{SP,m,l}$ over Y_m^{ord}, which is a special case of the construction in [Lanar]. They admit a similar description as X^Σ, $X^\star$.

Let $\mathfrak{C}_{\mathbb{V},p^l} \subset \mathfrak{C}_\mathbb{V}$ be the orbit of $\{V_0, V_1, \cdots, V_r\}$ under the action of the group

$$\Gamma_0(p^l)\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{Z}) : c \equiv 0 \mod p^l\right\},$$

and define $\mathcal{C}_{\mathbb{V},p^l}$ from $\mathfrak{C}_{\mathbb{V},p^l}$ in the same way as $\mathcal{C}_\mathbb{V}$ from $\mathfrak{C}_\mathbb{V}$. The partial compactification $\mathscr{T}^\star_{SP,m,l}$ (resp. $\mathscr{T}^\Sigma_{SP,m,l}$) is stratified by $\mathfrak{C}_{\mathbb{V},p^l}/\Gamma \cap \Gamma_{SP}(p^l)$ (resp. the rational polyhedral cone decomposition $\Sigma_{\mathcal{C}_{\mathbb{V},p^l}}$ induced from Σ). The natural maps

$$\begin{aligned} \mathfrak{p}_{\mathfrak{C},l} &: \mathfrak{C}_{\mathbb{V},p^l}/\Gamma \cap \Gamma_{SP}(p^l) \longrightarrow \mathfrak{C}_\mathbb{V}/\Gamma, \\ \mathfrak{p}_{\mathcal{C},l} &: \Sigma_{\mathcal{C}_{\mathbb{V},p^l}}/\Gamma \cap \Gamma_{SP}(p^l) \longrightarrow \Sigma_{\mathcal{C}_\mathbb{V}}/\Gamma \end{aligned} \tag{1.5.2}$$

are surjective.

In order to distinguish from the notation for the stratum indices for $X^\star$, X^Σ, we use an extra $\widetilde{\ }$ to denote the stratum indices for $\mathscr{T}^\star_{SP,m,l}$, $\mathscr{T}^\Sigma_{SP,m,l}$, *i.e.* we write $\widetilde{V}$, $\widetilde{\sigma}$ for elements in $\mathcal{C}_{\mathbb{V},p^l}/\Gamma \cap \Gamma_{SP}(p^l)$, $\Sigma_{\mathcal{C}_{\mathbb{V},p^l}}/\Gamma \cap \Gamma_{SP}(p^l)$. The stratum in $\mathscr{T}^\star_{SP,m,l}$ (resp. $\mathscr{T}^\Sigma_{SP,m,l}$) associated to $\widetilde{V}$ (resp. $\widetilde{\sigma}$, $\widetilde{V}$) will be denoted as $\mathscr{T}_{\widetilde{V},m,l}$ (resp. $Z_{\widetilde{\sigma},m,l}$, $Z_{\widetilde{V},m,l}$).

The stratum $\mathscr{T}_{\widetilde{V},m,l}$ is isomorphic to the quotient of the full level l Igusa tower over $Y_{\widetilde{V},m}$ by $\mathrm{Im}\left(\Gamma_{\widetilde{V}}\cap\Gamma_{SP}(p^l)\to\mathrm{Sp}(\widetilde{V}^{\perp}/\widetilde{V},\mathbb{Z}/p^l)\right)$, where $\Gamma_{\widetilde{V}}\subset\Gamma$ is the subgroup mapping $\widetilde{V}$ to itself. For $\widetilde{V}=V_{\widetilde{\sigma}}$, the diagram below describes the completion of $\mathscr{T}^{\Sigma}_{SP,m,l}$ along the stratum $Z_{\widetilde{\sigma},m,l}$,

$$\begin{tikzcd}
\mathcal{M}_{\widetilde{V},m,l} \arrow[r, hook] \arrow[d] & \mathcal{M}_{\widetilde{V},m,l,\widetilde{\sigma}} \arrow[r, hook] \arrow[dl] & \mathcal{M}_{\widetilde{V},m,l,\Sigma_{\widetilde{V}}} \arrow[dll] \\
\mathcal{B}_{\widetilde{V},m,l} \arrow[d] & & \\
\mathscr{T}_{\widetilde{V},m,l}. & &
\end{tikzcd}$$

Here $\mathcal{B}_{\widetilde{V},m,l}$, $\mathcal{M}_{\widetilde{V},m,l}$, $\mathcal{M}_{\widetilde{V},m,l,\widetilde{\sigma}}$ are the objects constructed in §1.4 with $X^*_r=\widetilde{V}$, and $\mathcal{M}_{\widetilde{V},m,l,\Sigma_{\widetilde{V}}}$ is the torus embedding associated to the rational polyhedral cone decomposition $\Sigma_{\widetilde{V}}$ of $C(\mathbb{V}/\widetilde{V}^{\perp})$ given by $\Sigma_{\mathcal{C}_{\mathbb{V}},p^l}$. Let $\widehat{\mathcal{M}}_{\widetilde{V},m,l,\widetilde{\sigma}}$ be the completion of $\mathcal{M}_{\widetilde{V},m,l,\Sigma_{\widetilde{V}}}$ along the closure of the stratum attached to σ. Then the completion of $\mathscr{T}^{\Sigma}_{SP,m,l}$ along $Z_{\widetilde{\sigma},m,l}$ is isomorphic to $\widehat{\mathcal{M}}_{\widetilde{V},m,l,\widetilde{\sigma}}/\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{V}^{\perp})}(p^l)$, where $\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{V}^{\perp})}(p^l)$ equals $\mathrm{Im}\left(\Gamma_{\widetilde{V}}\cap\Gamma_{SP}(p^l)\to\mathrm{GL}(\mathbb{V}/\widetilde{V}^{\perp})\right)$.

Denote by $\mathcal{I}^r_{\mathscr{T}^{\Sigma}_{SP,m,l}}$ (resp. $\mathcal{I}^r_{\mathscr{T}^{\star}_{SP,m,l}}$) the ideal sheaf attached to the union of all strata inside $\mathscr{T}^{\Sigma}_{SP,m,l}$ (resp. $\mathscr{T}^{\star}_{SP,m,l}$) with cusp labels of rank $>r$. The ideal sheaf $\mathcal{I}^r_{\mathscr{T}^{\Sigma}_{SP,m,l}}$ equals $f^*_{m,l}\mathcal{I}^r_{X^{\Sigma,\mathrm{ord}}_m}$ as $f_{m,l}$ is étale.

Since $\pi_{\mathscr{T},*}\mathcal{O}_{\mathscr{T}^{\Sigma}_{SP,m,l}}=\mathcal{O}_{\mathscr{T}^{\star}_{SP,m,l}}$, applying $\pi_{\mathscr{T},*}$ to the short exact sequence

$$0\longrightarrow\mathcal{I}^r_{\mathscr{T}^{\Sigma}_{SP,m,l}}\longrightarrow\mathcal{O}_{\mathscr{T}^{\Sigma}_{SP,m,l}}\longrightarrow\iota_{r,*}\mathcal{O}_{\coprod_{\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l),\,\mathrm{rk}\widetilde{V}>r}Z_{\widetilde{V},m,l}}\longrightarrow 0,$$

we get

$$0\longrightarrow\pi_{\mathscr{T},*}\mathcal{I}^r_{\mathscr{T}^{\Sigma}_{SP,m,l}}\longrightarrow\mathcal{O}_{\mathscr{T}^{\star}_{SP,m,l}}\longrightarrow(\pi\circ\iota_r)_*\mathcal{O}_{\coprod_{\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l),\,\mathrm{rk}\widetilde{V}\geq r}Z_{\widetilde{V},m,l}},\tag{1.5.3}$$

where $\iota_r:\coprod_{\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l),\,\mathrm{rk}\widetilde{V}>r}Z_{\widetilde{V},m,l}\to\mathscr{T}^{\Sigma}_{SP,m,l}$ is the canonical closed embedding. Since by definition the stratum $\mathscr{T}_{\widetilde{V},m,l}$ is the image of $Z_{\widetilde{V},m,l}$, we see from (1.5.3) that

$$\mathcal{I}^r_{\mathscr{T}^{\star}_{SP,m,l}}=\pi_{\mathscr{T},*}\mathcal{I}^r_{\mathscr{T}^{\Sigma}_{SP,m,l}}.$$

The above description of the completion of $\mathscr{T}^{\Sigma}_{SP,m,l}$ along $Z_{\widetilde{\sigma},m,l}$, combined with (1.4.1) and (1.4.2), gives the following description of the fibre of the structure and ideal sheaves at a closed point $x\in\mathscr{T}_{\widetilde{V},m,l}\subset\mathscr{T}^{\star}_{SP,m,l}$,

(1.5.4)

$$\begin{aligned}\left(\mathcal{O}_{\mathscr{T}^{\star}_{SP,m,l}}\right)^{\wedge}_x=\left(\pi_{\mathscr{T},*}\mathcal{O}_{\mathscr{T}^{\Sigma}_{SP,m,l}}\right)^{\wedge}_x&=\left(\bigcap_{\sigma\in C(\mathbb{V}/\widetilde{V}^{\perp})^{\circ}}\prod_{\lambda\in N^{-1}S^2(\mathbb{V}/\widetilde{V}^{\perp})\cap\sigma^{\vee}}\mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{V},m,l,x},\,\mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{V}^{\perp})}(p^l)}\\&=\left(\prod_{\lambda\in N^{-1}S^2(\mathbb{V}/\widetilde{V}^{\perp})_{\geq 0}}\mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{V},m,l,x},\,\mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{V}^{\perp})}(p^l)},\end{aligned}$$

and

$$\begin{aligned}
(1.5.5)\qquad \left(\mathcal{I}^r_{\mathscr{T}^\star_{SP,m,l}}\right)^\wedge_x &= \left(\pi_{\mathscr{T},*}\mathcal{I}^r_{\mathscr{T}^\Sigma_{SP,m,l}}\right)^\wedge_x \\
&= \left(\pi_{\mathscr{T},*}\mathcal{O}_{\mathscr{T}^\Sigma_{SP,m,l}}\right)^\wedge_x \cap \left(\bigcap_{\substack{\sigma\in C(\mathbb{V}/\widetilde{V}^\perp)^\circ \\ \mathrm{rk}\,\widetilde{V}_{\widetilde{\sigma}}>r}} \prod_{\lambda\in N^{-1}S^2(\mathbb{V}/\widetilde{V}^\perp)\cap\sigma^{\vee,\circ}} \mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{V},m,l,x},\, \mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{V}^\perp)}(p^l)} \\
&= \left(\prod_{\substack{\lambda\in N^{-1}S^2(\mathbb{V}/\widetilde{V}^\perp)_{\geq 0} \\ \mathrm{rk}\,\lambda\geq \mathrm{rk}\,\widetilde{V}-r}} \mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{V},m,l,x},\, \mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{V}^\perp)}(p^l)},
\end{aligned}$$

where $\widehat{\mathcal{B}}_{\widetilde{V},m,l,x}$ denotes the completion of $\mathcal{B}_{\widetilde{V},m,l}$ along its fibre over x.

Remark 1.5.1. The invertible sheaf $\mathcal{L}(\lambda)$ is the pull-back of an ample line bundle on a quotient of the abelian scheme $\mathcal{B}_{\widetilde{V},m,l}$. Thus in particular, taking the global sections commutes with base change (cf. [FC90, p. 155]). Therefore (1.5.5) implies that for the ideal sheaf $\mathcal{I}^r_{\mathscr{T}^\star_{SP,m,l}}$, the push-forward $\pi_{\mathscr{T},*}$ commutes with the base change. Since $\mathscr{T}^\star_{SP,m,l}$ is affine, we see that the base change property holds, *i.e.*

$$(1.5.6)\qquad V^{SP,r}_{m,l} = V^{SP,r}_{m+1,l}\otimes\mathbb{Z}/p^m.$$

1.6. **The quotient** $V^{SP,r}_{m,l}\Big/V^{SP,r-1}_{m,l}$**.** Since $X^{\star,\mathrm{ord}}$ is affine, we have

$$V^{SP,r}_{m,l}\Big/V^{SP,r-1}_{m,l} = \mathrm{H}^0\left(X^{\star,\mathrm{ord}}_m,\ \pi_*f_{m,l,*}\mathcal{I}^r_{\mathscr{T}^\Sigma_{SP,m,l}}\Big/\pi_*f_{m,l,*}\mathcal{I}^{r-1}_{\mathscr{T}^\Sigma_{SP,m,l}}\right).$$

We need to analyze the quotient $\pi_*f_{m,l,*}\mathcal{I}^r_{\mathscr{T}^\Sigma_{SP,m,l}}\Big/\pi_*f_{m,l,*}\mathcal{I}^{r-1}_{\mathscr{T}^\Sigma_{SP,m,l}}$. It is easily seen that this quotient sheaf is supported on $\coprod_{W\in\mathfrak{C}_{\mathbb{V}}/\Gamma,\mathrm{rk}W\geq r} Y^{\mathrm{ord}}_{W,m}\subset X^{\star,\mathrm{ord}}_m$. For $x\in\mathscr{T}_{\widetilde{W},m,l}$ with $\widetilde{W}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l)$

of rank $\geq r$, using (1.5.5) we get

(1.6.1)

$$\left(\pi_{\mathscr{T},*}\mathcal{I}^{r}_{\mathscr{T}^{\Sigma}_{SP,m,l}}\Big/\pi_{\mathscr{T},*}\mathcal{I}^{r-1}_{\mathscr{T}^{\Sigma}_{SP,m,l}}\right)^{\wedge}_{x}=\left(\prod_{\substack{\lambda\in N^{-1}S^2(\mathbb{V}/\widetilde{W}^{\perp})_{\geq 0}\\ \mathrm{rk}\lambda=\mathrm{rk}\widetilde{W}-r}}\mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{W},m,l,x},\,\mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{W}^{\perp})}(p^l)}$$

$$=\left(\prod_{\substack{\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l)\\ \widetilde{V}\subset\widetilde{W},\,\mathrm{rk}\widetilde{V}=r}}\ \prod_{\substack{\lambda\in N^{-1}S^2(\mathbb{V}/\widetilde{W}^{\perp})_{\geq 0}\\ \ker\lambda=\widetilde{V}}}\mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{W},m,l,x},\,\mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{W}^{\perp})}(p^l)}$$

$$=\prod_{\substack{\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l)\\ \widetilde{V}\subset\widetilde{W},\,\mathrm{rk}\widetilde{V}=r}}\left(\prod_{\lambda\in N^{-1}S^2(\widetilde{V}^{\perp}/\widetilde{W}^{\perp})_{>0}}\mathrm{H}^0(\widehat{\mathcal{B}}_{\widetilde{W},\widetilde{V},m,l,x},\,\mathcal{L}(\lambda))\right)^{\Gamma_{\mathrm{GL}(\widetilde{V}^{\perp}/\widetilde{W}^{\perp})}(p^l)}.$$

Here $\mathcal{B}_{\widetilde{W},\widetilde{V},m,l}$ is the abelian scheme over $\mathscr{T}_{\widetilde{W},m,l}$ obtained as the quotient of $\mathcal{B}_{\widetilde{W},m,l}$ by $\widetilde{V}$. It is p-power isogenous to

$$\mathrm{Hom}_{\mathscr{T}_{\widetilde{W},m,l}}(N^{-1}(\widetilde{V}^{\perp}/\widetilde{W}^{\perp}),\,\mathcal{A}_{\mathscr{T}_{\widetilde{W},m,l}}).$$

The invertible sheaf $\mathcal{L}(\lambda)$ over $\mathcal{B}_{\widetilde{W},\widetilde{V},m,l}$ with $\lambda\in N^{-1}S^2(\widetilde{V}^{\perp}/\widetilde{W})_{>0}$ is defined in the way as described in §1.4. The group $\Gamma_{\mathrm{GL}(\widetilde{V}^{\perp}/\widetilde{W}^{\perp})}(p^l)$ is the image of the stablilizer of $\widetilde{V}^{\perp}/\widetilde{W}^{\perp}$ inside $\Gamma_{\mathrm{GL}(\mathbb{V}/\widetilde{W}^{\perp})}(p^l)$.

For each $\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l)$, there is a closed embedding

$$\iota^{\star}_{\widetilde{V}}:\mathscr{T}^{\star}_{\widetilde{V},m,l}\longhookrightarrow\mathscr{T}^{\star}_{SP,m,l},$$

where $\mathscr{T}^{\star}_{\widetilde{V},m,l}$ is the partial minimal compactification of the stratum $\mathscr{T}_{\widetilde{V},m,l}$. The image is the Zariski closure of the stratum $\mathscr{T}_{\widetilde{V},m,l}$ inside $\mathscr{T}^{\star}_{SP,m,l}$, which equals the union of all strata with cusp labels containing $\widetilde{V}$. Like before one can define the sheaf of ideals $\mathcal{I}^{s}_{\mathscr{T}^{\star}_{\widetilde{V},m,l}}$ for $0\leq r\leq\mathrm{rk}\,\widetilde{V}$.

We define the group $P^{\circ}_{n,r}(\mathbb{Z}/p^l)$ as the image of the map

$$\Gamma_{V_r}\cap\Gamma_0(p^l)\longrightarrow\mathrm{GL}(n,\mathbb{Z}/p^l)$$

$$\begin{array}{cccccc} r & n-r & r & n-r & & \\ \multicolumn{4}{c}{\left(\begin{array}{cccc}\alpha & u & * & * \\ 0 & a & * & b \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & c & v & d\end{array}\right)} & \begin{array}{c} r \\ n-r \\ r \\ n-r\end{array} & \end{array}\longmapsto\begin{pmatrix}\alpha & u\\ 0 & a\end{pmatrix}\quad\mathrm{mod}\ p^l,$$

which is easily seen equal to $\begin{pmatrix}\mathrm{SL}(r,\mathbb{Z}/p^l) & * \\ 0 & \mathrm{GL}(n-r,\mathbb{Z}/p^l)\end{pmatrix}$.

Proposition 1.6.1. *There are the following short exact sequence,*

$$0 \longrightarrow \pi_{\mathscr{T},*}\mathcal{I}^{r-1}_{\mathscr{T}^{\Sigma}_{SP,m,l}} \longrightarrow \pi_{\mathscr{T},*}\mathcal{I}^{r}_{\mathscr{T}^{\Sigma}_{SP,m,l}} \longrightarrow \bigoplus_{\substack{\widetilde{V}\in\mathfrak{C}_{\mathbb{V},\,p^l}/\Gamma\cap\Gamma_{SP}(p^l)\\ \mathrm{rk}\,\widetilde{V}=r}} \iota^{\star}_{\widetilde{V},*}\mathcal{I}^{0}_{\mathscr{T}^{\star}_{\widetilde{V},m,l}} \longrightarrow 0, \tag{1.6.2}$$

$$0 \longrightarrow \pi_{*}f_{m,l,*}\mathcal{I}^{r-1}_{\mathscr{T}^{\Sigma}_{SP,m,l}} \longrightarrow \pi_{*}f_{m,l,*}\mathcal{I}^{r}_{\mathscr{T}^{\Sigma}_{SP,m,l}} \longrightarrow \bigoplus_{\substack{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma\\ \mathrm{rk}\,V=r}} \left(\bigoplus_{\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)} \iota^{\star}_{\widetilde{V},*}\mathcal{I}^{0}_{\mathscr{T}^{\star}_{\widetilde{V},m,l}}\right) \longrightarrow 0, \tag{1.6.3}$$

where $\mathfrak{p}_{\mathfrak{C},l}$ is the projection defined in (1.5.2) *and*

$$\mathfrak{p}^{-1}_{\mathfrak{C},l}(V) \simeq \Gamma_{\widetilde{V}} \cap \Gamma_0(p^l) \Big\backslash \Gamma \cap \Gamma_0(p^l) \Big/ \Gamma \cap \Gamma_{SP}(p^l) \simeq P^{\circ}_{n,r}(\mathbb{Z}/p^l) \Big\backslash \mathrm{GL}(n,\mathbb{Z}/p^l) \Big/ SP(\mathbb{Z}/p^l). \tag{1.6.4}$$

Proof. The short exact sequence (1.6.2) follows directly from our above description in (1.5.5) and (1.6.1) of the fibres of the relevant sheaves on the partial minimal compactification. The term at the right end is a direct sum because the intersection between $\mathscr{T}^{\star}_{\widetilde{V},m,l}$ and $\mathscr{T}^{\star}_{\widetilde{V}',m,l}$, $\widetilde{V} \neq \widetilde{V}'$, lies inside the closed subscheme defining the ideal sheaf $\mathcal{I}^{0}_{\mathscr{T}^{\star}_{\widetilde{V}m,l}}$. The exact sequence (1.6.3) is obtained from (1.6.2) by rewriting the term at the right end. □

By taking global sections, (1.6.3) gives

$$0 \longrightarrow V^{SP,r-1}_{m,l} \longrightarrow V^{SP,r}_{m,l} \longrightarrow \bigoplus_{\substack{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma\\ \mathrm{rk}\,V=r}} \left(\bigoplus_{\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)} H^0\left(\mathscr{T}^{\star}_{\widetilde{V}m,l}, \mathcal{I}^{0}_{\mathscr{T}^{\star}_{\widetilde{V}m,l}}\right)\right) \longrightarrow 0. \tag{1.6.5}$$

We see that the quotient $V^{SP,r}_{m,l}\Big/V^{SP,r-1}_{m,l}$ is a direct sum of cuspidal p-adic Siegel modular forms with p^m-torsion coefficients of level l and degree $n-r$ with respect to certain parabolics.

The action of the group $T_P(\mathbb{Z}_p)$ permutes the summands of $V^{SP,r}_{m,l}\Big/V^{SP,r-1}_{m,l}$. There are too many summands in the quotient in order for it to form a nice $\mathbb{Z}/p^m[\![T_P(\mathbb{Z}_p)]\!]$-module after taking direct limit with respect to l, or in other words the structure of the $T_P(\mathbb{Z}_p)$-action on (1.6.4) is in some sense too complicated as l grows. The idea is that we pick out a single $T_P[\![\mathbb{Z}_p]\!]$-orbit from (1.6.4) which patch nicely with l growing.

1.7. The space $\mathcal{V}^{SP,r,\flat}$.

For $V \in \mathfrak{C}_{\mathbb{V}}/\Gamma$ of rank r, consider

$$\mathscr{T}_{Z^{\mathrm{ord}}_V,SP,m,l} = Z^{\mathrm{ord}}_{V,m} \times_{X^{\Sigma}_m} \mathscr{T}^{\Sigma}_{SP,m,l} = \coprod_{\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)} Z_{\widetilde{V},m,l},$$

the restriction of the SP-Igusa tower to the stratum $Z^{\mathrm{ord}}_{V,m}$. It is not connected if the set $\mathfrak{p}^{-1}_{\mathfrak{C},l}(V) \simeq P^{\circ}_{n,r}(\mathbb{Z}/p^l)\backslash\mathrm{GL}(n,\mathbb{Z}/p^l)/\,SP(\mathbb{Z}/p^l)$ has more than one element. For $r \leq n_d$, we will define a subscheme $\mathscr{T}^{\flat}_{Z^{\mathrm{ord}}_V,SP,m,l} \subset \mathscr{T}_{Z^{\mathrm{ord}}_V,SP,m,l}$ consisting of certain connected components which form a single orbit for the $T_P(\mathbb{Z}_p)$-action. The space $V^{SP,r,\flat}_{m,l}$ will be defined as the subspace of $V^{SP,r}_{m,l}$ consisting of sections vanishing outside $\mathscr{T}^{\flat}_{Z^{\mathrm{ord}}_V,SP,m,l}$.

Recall that the semi-abelian scheme $\mathcal{G}_{/\widehat{\mathcal{M}}_{X^*_r,m,l,\sigma}}$ in the Mumford construction carries canonical level structures

$$\psi_{N,\mathrm{can}} : (\mathbb{Z}/N\mathbb{Z})^{2n} \longrightarrow \mathcal{G}_{/\widehat{\mathcal{M}}_{X^*_r,m,l,\sigma}}[N], \qquad \phi_{p,\mathrm{can}} : \mu^n_{p^l} \xrightarrow{\sim} \mathcal{G}_{/\widehat{\mathcal{M}}_{X^*_r,m,l,\sigma}}[p^l]^{\circ}.$$

We decide that if $\widetilde{V} = V_r$, the standard submodule of $\mathbb{V}$ of rank r, and $\widetilde{\sigma} \in C(\mathbb{V}/\widetilde{V}^{\perp})$, then the restriction of the semi-abelian scheme $(\mathcal{G}_{/\mathscr{T}^{\Sigma}_{SP,m,l}}, \psi_N, (E_i, \varepsilon_i)_{1\leq i\leq d,\, p^l})$ to the formal completion along $Z^{\mathrm{ord}}_{\widetilde{\sigma},m,l}$ is isomorphic to the one induced from $(\mathcal{G}_{/\widehat{\mathcal{M}}_{X^*_r,m,l,\widetilde{\sigma}}},\ \psi_{N,\mathrm{can}},\ \phi_{p,\mathrm{can}})$ (in other words, the level structures parametrized by cusps at infinity are the canonical ones).

Then for $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_0(p^l)$, $\widetilde{V} = \gamma \cdot V_r$ and $\widetilde{\sigma} = \gamma \cdot \sigma \in C(\mathbb{V}/\widetilde{V}^{\perp})$ for some $\sigma \in \Sigma$, the restriction of $\left(\mathcal{G}_{/\mathscr{T}^{\Sigma}_{SP,m,l}}, \psi_N, (E_i, \varepsilon_i)_{1\leq i\leq d,\, p^l}\right)$ to the formal completion along $Z^{\mathrm{ord}}_{\widetilde{\sigma},m,l}$ is isomorphic to the one induced from

$$\left(\mathcal{G}_{/\widehat{\mathcal{M}}_{X^*_r,m,l,\widetilde{\sigma}}},\ \psi_{N,\mathrm{can}} \circ \gamma,\ \phi_{p,\mathrm{can}} \circ a_\gamma\right).$$

If we fix $V \in \mathfrak{C}_{\mathbb{V},p^l}$, the connected components of $\mathscr{T}_{Z^{\mathrm{ord}}_V,SP,m,l}$ can be thought of in terms of the relation between the two-step filtration of $\mathcal{G}_{/Z^{\mathrm{ord}}_{V,m}}[p^\infty]^\circ$ induced from

$$0 \longrightarrow V \otimes \mathbb{G}_m \longrightarrow \mathcal{G}_{/Z_V,m}[p^\infty]^\circ \longrightarrow \mathcal{A}_{/Y_V} \times_{Y_L} Z_V \longrightarrow 0, \tag{1.7.1}$$

and the d-step filtration

$$\{0\} = E_{\widetilde{V},0} \subset E_{\widetilde{V},1} \subset \cdots \subset E_{\widetilde{V},d} = \mathcal{G}_{/Z^{\mathrm{ord}}_V,m}[p^l]^\circ$$

induced from the universal object $\left(\mathcal{G}_{/\mathscr{T}^{\Sigma}_{SP,m,l}}, \psi_N, (E_i, \varepsilon_i)_{1\leq i\leq d,\, p^l}\right)$ restricted to $Z_{\widetilde{V},m,l} \subset \mathscr{T}^{\Sigma}_{SP,m,l}$.

From now on assume $r \leq n_d$. Define

$$\begin{aligned} \mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^\flat &= \left\{\widetilde{V} \in \mathfrak{p}^{-1}_{\mathfrak{C},l}(V) :\ E_{\widetilde{V},d-1} \cap V \otimes \mu_{p^l} = 0\right\}, \\ \mathscr{T}^\flat_{Z^{\mathrm{ord}}_V,SP,m,l} &= \coprod_{\widetilde{V} \in \mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^\flat} Z_{\widetilde{V},m,l} \subset \mathscr{T}_{Z^{\mathrm{ord}}_V,SP,m,l}, \end{aligned}$$

i.e. the union of the connected components of $\mathscr{T}_{Z^{\mathrm{ord}}_V,SP,m,l}$ for which the first $d-1$ steps of the parametized filtrations of $\mathcal{G}_{/Z^{\mathrm{ord}}_{V,m}}[p^l]^\circ$ intersect trivially with the p^l-torsion of the torus part in (1.7.1).

Under the natural map

$$\begin{aligned} \mathfrak{p}^{-1}_{\mathfrak{C},l}(V) &\longrightarrow P^\circ_{n,r}(\mathbb{Z}/p^l) \Big\backslash \mathrm{GL}(n, \mathbb{Z}/p^l) \Big/ SP(\mathbb{Z}/p^l) \\ \widetilde{V} = \gamma \cdot V_r \quad (\gamma \in \Gamma_0(p^l)) &\longmapsto a_\gamma, \end{aligned}$$

the set $\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^\flat$ corresponds to

$$P^\circ_{n,r}(\mathbb{Z}/p^l) \Big\backslash P_{n,r}(\mathbb{Z}/p^l) \begin{pmatrix} 0 & I_r \\ I_{n-r} & 0 \end{pmatrix} SP(\mathbb{Z}/p^l) \Big/ SP(\mathbb{Z}/p^l) \subset P^\circ_{n,r}(\mathbb{Z}/p^l) \Big\backslash \mathrm{GL}(n, \mathbb{Z}/p^l) \Big/ SP(\mathbb{Z}/p^l),$$

with $P_{n,r} = \begin{pmatrix} \mathrm{GL}(r, \mathbb{Z}/p^l) & 0 \\ 0 & \mathrm{GL}(n-r, \mathbb{Z}/p^l) \end{pmatrix}$. The action of $T_P(\mathbb{Z}_p)$ on $\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^\flat$ is transitive, and we have

$$\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^\flat = \begin{cases} \left\{\begin{pmatrix} 0 & I_r \\ I_{n-r} & 0 \end{pmatrix}\right\}, & \text{if } r < n_d, \\ \begin{pmatrix} 0 & I_r \\ I_{n-r} & 0 \end{pmatrix} \begin{pmatrix} I_{N_{d-1}} & 0 \\ 0 & \mathrm{GL}(n_d, \mathbb{Z}/p^l)/\,\mathrm{SL}(n_d, \mathbb{Z}/p^l) \end{pmatrix} \simeq (\mathbb{Z}/p^l)^\times, & \text{if } r = n_d. \end{cases} \tag{1.7.2}$$

We now define $\mathcal{I}^{r,\flat}_{\mathscr{T}^{\Sigma}_{SP,m,l}} \subset \mathcal{I}^{r}_{\mathscr{T}^{\Sigma}_{SP,m,l}}$ to be the sheaf of ideals associated to the closed subscheme given as the complement of $\coprod\limits_{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma,\,\mathrm{rk}V<r}\mathscr{T}_{Z_V^{\mathrm{ord}},SP,m,l}\cup\coprod\limits_{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma,\,\mathrm{rk}V=r}\mathscr{T}^{\flat}_{Z_V^{\mathrm{ord}},SP,m,l}$ inside $\mathscr{T}^{\Sigma}_{SP,m,l}$, and define

$$\begin{aligned} V^{SP,r,\flat}_{m,l} &= \mathrm{H}^0\left(\mathscr{T}^{\Sigma}_{SP,m,l},\, \mathcal{I}^{r,\flat}_{\mathscr{T}^{\Sigma}_{SP,m,l}}\right) \subset V^{SP,r}_{m,l},\\ \mathcal{V}^{SP,r,\flat} &= \varinjlim_{m}\varinjlim_{l} V^{SP,r,\flat}_{m,l} \subset \mathcal{V}^{SP,r}. \end{aligned}$$

If follows from the definition and (1.6.5) that

$$V^{SP,r,\flat}_{m,l}\Big/V^{SP,r-1}_{m,l} = \bigoplus_{\substack{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma\\ \mathrm{rk}\,V=r}}\left(\bigoplus_{\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^{\flat}}\mathrm{H}^0\left(\mathscr{T}^{\star}_{\widetilde{V}m,l},\,\mathcal{I}^0_{\mathscr{T}^{\star}_{\widetilde{V}m,l}}\right)\right). \tag{1.7.3}$$

The natural $T_P(\mathbb{Z}_p)$-action on the left hand side induces a $T_P(\mathbb{Z}_p)$-action on

$$\bigoplus_{\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^{\flat}}\mathrm{H}^0\left(\mathscr{T}^{\star}_{\widetilde{V}m,l},\,\mathcal{I}^0_{\mathscr{T}^{\star}_{\widetilde{V}m,l}}\right). \tag{1.7.4}$$

Let

$$P_{n-r} = \left\{\begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_d\end{pmatrix}\in\mathrm{GL}(n-r)\;\middle|\; a_i\in\mathrm{GL}(n_i),\ 1\le i\le d-1,\ a_d\in\mathrm{GL}(n_d-r)\right\}, \tag{1.7.5}$$

$$SP_{n-r} = \left\{\begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_d\end{pmatrix}\in\mathrm{SL}(n-r)\;\middle|\; a_i\in\mathrm{SL}(n_i),\ 1\le i\le d-1,\ a_d\in\mathrm{SL}(n_d-r)\right\}, \tag{1.7.6}$$

$$T_{P_{n-r}} = P_{n-r}/SP_{n-r} = \begin{cases}\mathbb{G}_m^d & \text{if } r<n_d,\\ \mathbb{G}_m^{d-1} & \text{if } r=n_d.\end{cases} \tag{1.7.7}$$

We know that for each $\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^{\flat}$, we have

$$\mathrm{Im}\left(\Gamma_{\widetilde{V}}\cap\Gamma_{SP}(p^l)\to\mathrm{Sp}(\widetilde{V}^{\perp}/\widetilde{V},\mathbb{Z}/p^l)\right)\simeq\Gamma(N)\cap SP_{n-r}(\mathbb{Z}). \tag{1.7.8}$$

The embedding $P_{n-r}\hookrightarrow P$ induces a morphism $T_{P_{n-r}}\to T_P$, and the induced action of $T_{P_{n-r}}(\mathbb{Z}_p)$ on (1.7.4) preserves each direct summand, so equips each $\mathrm{H}^0\left(\mathscr{T}^{\star}_{\widetilde{V}m,l},\,\mathcal{I}^0_{\mathscr{T}^{\star}_{\widetilde{V}m,l}}\right)$, $\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^{\flat}$, with an $\mathcal{O}_F[\![T_{P_{n-r}}(\mathbb{Z}_P)]\!]$-module structure.

From (1.7.8), we also know that for each $\widetilde{V}\in\mathfrak{p}^{-1}_{\mathfrak{C},l}(V)^{\flat}$, the scheme $\mathscr{T}^{\star}_{\widetilde{V},m,l}$ is isomorphic to the minimal compactification of the quotient by $SP_{n-r}(\mathbb{Z}/p^l)$ of the full level p^l Igusa tower over $Y^{\mathrm{ord}}_{V,m}$. Denote by $V^{SP_{n-r},0}_{V,m,l}$ the space of cuspidal sections over that Igusa tower over $Y^{\mathrm{ord}}_{V,m}$, which carries a natural $T_{P_{n-r}}(\mathbb{Z}_p)$-action.

Then

$$\mathrm{H}^0\left(\mathscr{T}^{\star}_{\widetilde{V}m,l},\,\mathcal{I}^0_{\mathscr{T}^{\star}_{\widetilde{V}m,l}}\right)\simeq V^{SP_{n-r},0}_{V,m,l}$$

as $\mathcal{O}_F[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]$-modules. Furthermore, we have

$$\bigoplus_{\widetilde{V}\in\mathfrak{p}_{\mathfrak{C},l}^{-1}(V)^\flat} \mathrm{H}^0\left(\mathscr{T}^\star_{\widetilde{V}m,l}, \mathcal{I}^0_{\mathscr{T}^\star_{\widetilde{V}m,l}}\right) \simeq \mathbb{Z}_p[\![T_P(\mathbb{Z}_p)]\!]\otimes_{\mathbb{Z}_p[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]} V^{SP_{n-r},0}_{V,m,l},$$

because by (1.7.2) and (1.7.7) the action of $T_P(\mathbb{Z}/p^l)/T_{P_{n-r}}(\mathbb{Z}/p^l)$ on $\mathfrak{p}_{\mathfrak{C},l}^{-1}(V)^\flat$ is simply transitive.

Summarizing the above discussion, we get

Proposition 1.7.1. *There are the following short exact sequences of $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-modules,*

$$0\longrightarrow V^{SP,r-1}_{m,l}\longrightarrow V^{SP,r,\flat}_{m,l}\longrightarrow \bigoplus_{\substack{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma\\ \mathrm{rk}\,V=r}} \mathbb{Z}_p[\![T_P(\mathbb{Z}_p)]\!]\otimes_{\mathbb{Z}_p[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]} V^{SP_{n-r},0}_{V,m,l}\longrightarrow 0, \tag{1.7.9}$$

$$0\longrightarrow \mathcal{V}^{SP,r-1}\longrightarrow \mathcal{V}^{SP,r,\flat}\longrightarrow \bigoplus_{\substack{V\in\mathfrak{C}_{\mathbb{V}}/\Gamma\\ \mathrm{rk}\,V=r}} \mathbb{Z}_p[\![T_P(\mathbb{Z}_p)]\!]\otimes_{\mathbb{Z}_p[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]} \mathcal{V}^{SP_{n-r},0}_{V}\longrightarrow 0. \tag{1.7.10}$$

1.8. **The q-expansions.** Later our analysis of the action of the $\mathbb{U}_p^P$-operators on $V^{SP,r,\flat}_{m,l}$, $\mathcal{V}^{SP,r,\flat}$ will mostly rely on q-expansions.

Specializing the construction in §1.4 to the case $r=n$, for $\gamma_N\in\mathrm{Sp}(2n,\mathbb{Z}/N\mathbb{Z})$ and $a_p\in\mathrm{GL}(n,\mathbb{Z}_p)$, the evaluation at the testing object

$$\left(\mathcal{G}_{/\widehat{\mathcal{M}}_{X_n^*,m,l,\sigma}},\, \psi_{N,\mathrm{can}}\circ\gamma_N,\, \phi_{p,\mathrm{can}}\circ a_p\right),\quad \sigma\in\Sigma,$$

defines the q-expansion map

$$\varepsilon^{\gamma_N,a_p}_{q\text{-exp},m,l}: V_{m,l}\longrightarrow \bigcap_{\sigma\in\Sigma}\mathcal{O}_F/p^m[\![N^{-1}S^2(X_n)\cap\sigma^\vee]\!]=\mathcal{O}_F/p^m[\![N^{-1}S^2(X_n)_{\geq 0}]\!].$$

These $\varepsilon^{\gamma_N,a_p}_{q\text{-exp},m,l}$'s glue to the q-expansion map on $\mathcal{V}$,

$$\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}:\mathcal{V}\longrightarrow F/\mathcal{O}_F[\![N^{-1}S^2(X_n)_{\geq 0}]\!]=F/\mathcal{O}_F[\![N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]\!]. \tag{1.8.1}$$

With our fixed basis $x_1,\cdots,x_n$ of X_n, we will freely identify $S^2(X_n)$ with $\mathrm{Sym}^2(n,\mathbb{Z})^*$, the set of symmetric $n\times n$ matrices with integers as diagonal entries and half-integers as off-diagonal entries, by identifying $\beta\in\mathrm{Sym}^2(n,\mathbb{Z})$ with $\sum\limits_{1\leq i,j\leq n}\beta_{ij}x_i\otimes x_j$. For $\beta\in N^{-1}S^2(X_n)$ and $f\in\mathcal{V}$, write $\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}(\beta,f)$ for the β-th Fourier coefficient of f, *i.e.* the coefficient associated with β in $\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}(f)$. One can check that given $a\in\mathrm{GL}(n,\mathbb{Z})$

$$\varepsilon^{m(a)\gamma_N,aa_p}_{q\text{-exp}}(\beta,f)=\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}({}^{\mathrm{t}}a\beta a,f), \tag{1.8.2}$$

where $m(a)=\begin{pmatrix}a&0\\0&{}^{\mathrm{t}}a^{-1}\end{pmatrix}$.

As illustrated in [FC90, V Lemma 1.4 and its proof], since the closure of every stratum associated with a cone in the toroidal compactification is irreducible and contains a stratum corresponding to a top dimensional cone, many properties of (p-adic) Siegel modular forms can be verified by examining the q-expansion. The following two propositions give a characterization of the space $\mathcal{V}^{SP,r}$, $\mathcal{V}^{SP,r,\flat}$ in terms of q-expansions.

Proposition 1.8.1. *Given $f\in\mathcal{V}$, it belongs to $\mathcal{V}^{SP,r}$ if and only if $\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}(\beta,f)$ vanishes for all $\gamma_N\in\mathrm{Sp}(2n,\mathbb{Z})$, $a_p\in\mathrm{GL}(n,\mathbb{Z}_p)$ and $\beta\in N^{-1}S^2(X_n)_{\geq 0}$ of rank less or equal to $n-r-1$.*

Proof. Given $\widetilde{\sigma} \in \Sigma_{\mathcal{C}_{\mathbb{V},p^l}}$, pick a top dimensional cone $\widetilde{\tau} \in \Sigma_{\mathcal{C}_{\mathbb{V},p^l}}$ with $\widetilde{\sigma}$ as a face. Fix an isomorphism of Spf $\left(\mathcal{O}_F/p^m\mathcal{O}_F[\![N^{-1}S^2(X_n) \cap \widetilde{\tau}^\vee]\!]\right)$ with the completion of $\mathscr{T}^{\Sigma}_{SP,m,l}$ along the point $Z_{\widetilde{\tau},m,l}$ (here for a cone in $\Sigma_{\mathcal{C}_{\mathbb{V},p^l}}$ we use the same notation to denote a corresponding cone in $C(X_n)$). Then the embedding of the completion of the Zariski closure of $Z_{\widetilde{\sigma},m,l}$ along the point $Z_{\widetilde{\tau},m,l}$ to the completion of $\mathscr{T}^{\Sigma}_{SP,m,l}$ along $Z_{\widetilde{\tau},m,l}$ corresponds to the quotient map from $\mathcal{O}_F/p^m[\![N^{-1}S^2(X_n) \cap \widetilde{\tau}^\vee]\!]$ onto $\mathcal{O}_F/p^m[\![N^{-1}S^2(X_n) \cap \widetilde{\tau}^\vee \cap \widetilde{\sigma}^\perp]\!]$, sending all $\beta \in N^{-1}S^2(X_n) \cap \widetilde{\tau}^\vee$ that does not belong to $\widetilde{\sigma}^\perp$ to 0. This description shows that the vanishing condition in the proposition implies the vanishing of f along $Z_{\widetilde{\sigma},m,l}$, and the proposition follows. □

In the following, by the radical of $\beta \in N^{-1}S^2(X_n)_{\geq 0}$, we mean the sub-$\mathbb{Z}$-module of X_n^* consisting of elements that pair trivially with β via the natural map $X_n^* \times S^2(X_n) \to X_n$, and by a primitive vector in X_n^*, we mean an element not divisible by p in X_n^*.

Proposition 1.8.2. *Given $f \in \mathscr{V}^{SP,r}$, it belongs to $\mathscr{V}^{SP,r,\flat}$ if and only if $\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}(\beta, f)$ vanishes for all β of corank r such that the radical of ${}^{\mathrm{t}}a_p\beta a_p$ contains a primitive vector inside $\mathbb{Z}\cdot x_1^* + \cdots + \mathbb{Z}\cdot x_{N_{d-1}}^* + p\mathbb{Z}\cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z}\cdot x_n^*$.*

Proof. We use the description of the completion of the Zariski closure of $Z_{\widetilde{\sigma},m,l}$ along the point $Z_{\widetilde{\tau},m,l}$ given in the proof of the previous proposition, and assume that $\widetilde{V}_{\widetilde{\sigma}}$ is of rank r. Identify V_n and X_n^* (together with standard basis). Take a $\gamma \in \Gamma_0(p^l)$ such that $\widetilde{V}_{\widetilde{\tau}} = \gamma^{-1}\cdot V_n$ and use it to fix an isomorphism between $\mathcal{O}_F/p^m\mathcal{O}_F[\![N^{-1}S^2(X_n)\cap\widetilde{\tau}^\vee]\!]$ and the formal completion of the structure sheaf at the point $Z_{\widetilde{\tau},m,l}$. Then the evaluation of f at the formal neighborhood of $Z_{\widetilde{\tau},m,l}$ corresponds to the q-expansion $\varepsilon^{\gamma,a_\gamma}_{q\text{-exp}}(\beta, f)$, and the $E_{\widetilde{V}_{\widetilde{\tau}},d-1}$ corresponds to the $\mathbb{Z}$-span of $a_\gamma(x_1^*)/p^l, \ldots, a_\gamma(x_{N_{d-1}}^*)/p^l$ (recall that $E_{\widetilde{V}_{\widetilde{\tau}},d-1}$ is the $d-1$-th step of the filtration in the level structure of the SP-Igusa tower). On the other hand, the canonical two-step filtration of the semi-abelian scheme over $Z_{\widetilde{\sigma},m,l}$ corresponds to $\widetilde{V}_{\widetilde{\sigma}} \subset \widetilde{V}_{\widetilde{\tau}}$. Therefore the vanishing condition in the definition of $\mathscr{V}^{SP,r,\flat}$ requires the vanishing of $\varepsilon^{\gamma,a_\gamma}_{q\text{-exp}}(\beta, f)$ for all $\beta \in N^{-1}S^2(X_n) \cap \widetilde{\tau}^\vee \cap \widetilde{\sigma}^\perp$ with $\widetilde{V}_{\widetilde{\sigma}}$ containing a primitive element in $a_\gamma\left(\mathbb{Z}\cdot x_1^* + \cdots + \mathbb{Z}\cdot x_{N_{d-1}}^* + p\mathbb{Z}\cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z}\cdot x_n^*\right)$. Also, for a semi-positive definite β inside $\widetilde{\sigma}^\perp$, the radical of β equals $\widetilde{V}_{\widetilde{\sigma}}$. Hence the vanishing condition in the proposition agrees with that for defining $\mathscr{V}^{SP,r,\flat}$. □

1.9. **The $\mathbb{U}_p^P$-operators.** To each matrix

$$\gamma_{p,i} = \begin{pmatrix} pI_i & 0 & 0 & 0 \\ 0 & I_{n-i} & 0 & 0 \\ 0 & 0 & p^{-1}I_i & 0 \\ 0 & 0 & 0 & I_{n-i} \end{pmatrix}, \qquad 1 \leq i \leq n,$$

corresponds a Hecke operator $U_{p,i}^P$ acting on $\mathscr{V}^{SP}$. The ordinarity condition for P requires the eigenvalues of $U_{p,N_1}^P, U_{p,N_2}^P, \ldots, U_{p,N_d}^P$ to be p-adic units (recall that $N_i = \sum_{j=1}^{i} n_j$). In [Pil12, §5.1.4], only these $U_{p,N_1}^P, U_{p,N_2}^P, \ldots, U_{p,N_d}^P$ are introduced as they are sufficient for defining the ordinary projection in order to establish Hida theory. However, given an automorphic representation π of $\mathrm{Sp}(2n, \mathbb{A})$ generated by a holomorphic Siegel modular form ordinary for the parabolic P, in order to retrieve the full information on π_p, one needs to consider the action of all the $U_{p,i}^P$, $1 \leq i \leq n$ (see §2.3 for details). If $i \neq N_1, \ldots, N_d$, the eigenvalue of $U_{p,i}^P$ on P-ordinary forms is not necessarily a p-adic unit.

Let $\mathscr{T}^{\circ}_{SP,m,l}$ be the restriction of $\mathscr{T}_{SP,m,l}$ to $Y_m^{\mathrm{ord}} \subset X_m^{\Sigma,\mathrm{ord}}$. The algebraic correspondence inside $\mathscr{T}^{\circ}_{SP,m,l} \times \mathscr{T}^{\circ}_{SP,m,l}$ associated to $\gamma_{p,i}$ is defined as follows. For $N_j \leq i < N_{j+1}$, let $C_{i,m,l}$ be the moduli scheme over $\mathcal{O}_F/p^m$ parametrizing the quintuple $\left(A,\lambda,\psi_N,(E_i,\varepsilon_i)_{1\leq i\leq d,p^l},L\right)$, where (A,λ,ψ_N) is an ordinary abelian scheme of genus n with principal polarization λ and principal level structure $\psi_N:(\mathbb{Z}/N)^{2n}\xrightarrow{\sim}A[N]$, defined over an $\mathcal{O}_F/p^m$-algebra, $(E_r,\varepsilon_r)_{1\leq r\leq d,p^l}$ is the structure used to define $\mathscr{T}_{SP,m,l}$ in §1.2, and $L\subset A[p^2]$ is a Lagrangian subgroup such that $\mathrm{rank}_{\mathbb{Z}/p}L[p]=2n-i$, $L[p]\cap E_j[p]=0$, $L[p]+E_{j+1}[p]=A[p]$. Denote by p_1 the projection from $C_{i,m,l}$ to $\mathscr{T}_{SP,m,l}$ which forgets L. There is another projection p_2 sending $\left(A,\lambda,\psi_N,(E_r,\varepsilon_r)_{1\leq r\leq d,p^l},L\right)$ to $\left(A/L,\lambda',p\circ\pi\circ\psi_N,(E'_r,\varepsilon'_r)_{1\leq r\leq d,p^l}\right)$, where $\pi:A\to A/L$ is the natural isogeny, λ' is defined by $\pi^*\lambda'=p^2\lambda$, and

$$\begin{aligned}
&E'_r=\pi(E_r), && \varepsilon'_r=\pi\circ\varepsilon, && 1\leq r\leq j\\
&E'_r=\pi\left(p^{-1}(E_r\cap p^{-l+1}L)\right), && \varepsilon'_r=p^{-\min\{N_r-i,n_r\}}\pi\circ\varepsilon_r, && j+1\leq r\leq d.
\end{aligned}$$

For $N_j\leq i<N_{j+1}$, we have the following composition

$$H^0\left(\mathscr{T}^\circ_{SP,m,l},\mathcal{O}_{\mathscr{T}^\circ_{SP,m,l}}\right)\xrightarrow{p_2^*}H^0\left(C_{i,m,l},\mathcal{O}_{C_{i,m,l}}\right)\xrightarrow{\mathrm{Tr}p_1}p^{i(n+1)}H^0\left(\mathscr{T}^\circ_{SP,m,l},\mathcal{O}_{\mathscr{T}^\circ_{SP,m,l}}\right).$$

The image of $\mathrm{Tr}p_1$ belongs to $p^{i(n+1)}H^0\left(\mathscr{T}^\circ_{SP,m,l},\mathcal{O}_{\mathscr{T}^\circ_{SP,m,l}}\right)$ because the pure inseparability degree of p_1 is $p^{i(n+1)}$ [Pil12, Appendice]. One can also check (for example by q-expansions) that such defined $U^P_{p,i}$ preserves various kinds of growth conditions along the boundary, i.e. the above map restricts to a map from $V^{SP}_{m,l}$ to $p^{i(n+1)}V^{SP}_{m,l}$. If $m>i(n+1)$, there is a well defined map $p^{-i(n+1)}:p^{i(n+1)}V^{SP}_{m,l}\to V^{SP}_{m-i(n+1),l}$. Now given $f\in V^{SP}_{m,l}$, thanks to (1.5.6), we can take $\widetilde{f}\in V^{SP}_{m+i(n+1)}$ such that $f\equiv\widetilde{f}\mod p^{i(n+1)}$, and we define

$$U^P_{p,i}(f)=p^{-i(n+1)}\circ\mathrm{Tr}p_1\circ p_2^*(\widetilde{f}).$$

In this section, only $U^P_{p,N_1},U^P_{p,N_2},\dots,U^P_{p,N_d}$ will be used. In order to show the desired properties of their action on $\mathcal{V}^{SP,r,\flat}$, $\mathcal{V}^{SP,r}$, we use the following proposition and Proposition 1.8.1, 1.8.2 to reduce to computations on q-expansions.

Proposition 1.9.1 (cf. [Hid02, Proposition 3.5])**.** *For $f\in\mathcal{V}^{SP}$, $\gamma_N\in\mathrm{Sp}(2n,\mathbb{Z})$ and $a_p\in T(\mathbb{Z}_p)\subset\mathrm{GL}(n,\mathbb{Z}_p)$, the formula on q-expansions for the action of the $\mathbb{U}^P_p$-operators on f is given by*

$$\varepsilon^{\gamma_N,a_p}_{q\text{-exp}}(\beta,U^P_{p,N_i}f)=\sum_{x\in M_{N_i,n-N_i}(\mathbb{Z}/p\mathbb{Z})}\varepsilon^{(\gamma^P_{p,i})^{-1}\gamma_N,a_p}_{q\text{-exp}}\left(\begin{pmatrix}pI_{N_i}&0\\N{}^{\mathrm{t}}x&I_{n-N_i}\end{pmatrix}\beta\begin{pmatrix}pI_{N_i}&Nx\\0&I_{n-N_i}\end{pmatrix},f\right),$$

for $\beta\in N^{-1}S^2(X_n)$ and $1\leq i\leq d$.

One can also write down the formula for general $a_p\in\mathrm{GL}(n,\mathbb{Z}_p)$ which is a little bit more complicated. We omit it here because the case a_p being diagonal suffices for our purpose thanks to (1.8.2).

Proposition 1.9.2. *All the spaces $\mathcal{V}^{SP,r}$, $0\leq r\leq n$, and $\mathcal{V}^{r,SP,\flat}$, $0\leq r\leq n_d$, are stable under the $\mathbb{U}^P_p$-operators.*

Proof. The statement for $\mathcal{V}^{SP,r}$ follows immediately from Proposition 1.8.1, 1.9.1. By Proposition 1.8.2, 1.9.1, in order to show the statement for $\mathcal{V}^{SP,r,\flat}$, it is enough to show that if the radical of β contains a primitive vector inside $\mathbb{Z}\cdot x^*_1+\cdots+\mathbb{Z}\cdot x^*_{N_{d-1}}+p\mathbb{Z}\cdot x^*_{N_{d-1}+1}+\cdots+p\mathbb{Z}\cdot x^*_n$, then the same holds for $\begin{pmatrix}pI_{N_i}&0\\N{}^{\mathrm{t}}x&I_{n-N_i}\end{pmatrix}\beta\begin{pmatrix}pI_{N_i}&Nx\\0&I_{n-N_i}\end{pmatrix}$. In fact, it is not difficult to check that if $v \in \mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z} \cdot x_n^*$ is a primitive vector, then for all $x \in M_{N_i, n-N_i}(\mathbb{Z})$,

$$\mathbb{Q} \cdot \begin{pmatrix} pI_{N_i} & Nx \\ 0 & I_{n-N_i} \end{pmatrix}^{-1} v_\beta \cap X_n \subset \mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z} \cdot x_n^*.$$

□

Now we want to define a $\mathbb{U}_p^P$-action on the quotient of the exact sequences in Proposition 1.7.1, and verify that the exact sequences are $\mathbb{U}_p^P$-equivariant.

For $V \in \mathfrak{C}_{\mathbb{V}}$ with rank $r \leq n_d$, we define the $\mathbb{U}_p^P$-action on $\mathcal{V}_V^{SP_{n-r},0}$ as follows. For $\gamma = \mathrm{diag}\left(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\right) \in \mathrm{Sp}(2n)$, set $\gamma' = \mathrm{diag}\left(a_1, \ldots, a_{n-r}, a_1^{-1}, \ldots, a_{n-r}^{-1}\right) \in \mathrm{Sp}(2n-2r)$. We make U_{p,N_i}^P act on $\mathcal{V}_V^{SP_{n-r},0}$ (the space of p-adic Siegel modular forms of degree $n-r$ for the parabolic P_{n-r}) by the $\mathbb{U}_p^{P_{n-r}}$-operator attached to γ'_{p,N_i}.

Let us denote by $\mathbb{U}_p^{P,[N]} \subset \mathbb{U}_p^P$ the subalgebra generated by the $\varphi(N)$-powers of U_{p,N_i}^P, $1 \leq i \leq d$. Here $\varphi(N) = N \cdot \prod_{q \text{ prime factors of } N} (1 - \frac{1}{q})$. Rather than showing the $\mathbb{U}_p^P$-equivariance of the exact sequences in Proposition 1.7.1, we are only able to show the $\mathbb{U}_p^{P,[N]}$-equivariance. However, this suffices for establishing Hida theory for $\mathcal{V}^{SP,r}$.

Proposition 1.9.3. *The exact sequences in Proposition 1.7.1 are* $\mathbb{U}_p^{P,[N]}$*-equivariant.*

Proof. We show the $\mathbb{U}_p^{P,[N]}$-equivariance of the projection $\mathfrak{p}_{\widetilde{V}} : V_{m,l}^{SP,r,\flat} \to V_{V,m,L}^{SP_{n-r},0}$, from $V_{m,l}^{SP,r,\flat}$ to the summand of $V_{m,l}^{SP,r,\flat} \big/ V_{m,l}^{SP,r-1}$ indexed by $\widetilde{V} \in \mathfrak{p}_{\mathfrak{C},l}^{-1}(V)^\flat$, by computing the q-expansions. Pick $\gamma \in \Gamma_0(p^l)$ such that $\gamma^{-1} V_n$ contains $\widetilde{V}$ (where V_n is identified with X_n^* with standard basis), and we view $\widetilde{V}$ as a subspace of X_n^* via γ. Then $\widetilde{V} \in \mathfrak{p}_{\mathfrak{C},l}^{-1}(V)^\flat$ implies that $\widetilde{V}$ is spanned by

$$(x_1, \ldots, x_n) \begin{pmatrix} I_{N_{d-1}} & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

with $w \in \mathrm{SL}(n_d, \mathbb{Z})$, $w \equiv I_{n_d} \mod N$, $\alpha_1 \in M_{n-r,r}(\mathbb{Z})$ and $\alpha_2 \in M_{r,r}(\mathbb{Z}) \cap \mathrm{GL}(r, \mathbb{Z}_p)$.

Take $s \geq \max\{l, \varphi(N)\}$. There exists

$$\begin{array}{c} \\ n-r \\ r \end{array} \begin{array}{c} \begin{array}{cc} n-r & \;\; r \end{array} \\ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \end{array} \in \mathrm{GL}(n, \mathbb{Z})$$

with $A \equiv \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mod p^s$, $A_1 \in \mathrm{GL}(N_i, \mathbb{Z})$, $A_2 \in \mathrm{GL}(n-r-N_i, \mathbb{Z})$, $C \equiv 0 \mod p^s$, such that

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0.$$

Define

$$\begin{aligned} \mathfrak{i}_{\widetilde{V}} : \mathrm{Sym}(n-r, \mathbb{Q}) &\longrightarrow \mathrm{Sym}(n, \mathbb{Q}) \\ \beta' &\longmapsto \begin{pmatrix} I_{N_{d-1}} & 0 \\ 0 & {}^{\mathrm{t}}w^{-1} \end{pmatrix} \begin{pmatrix} {}^{\mathrm{t}}A \\ {}^{\mathrm{t}}B \end{pmatrix} \beta' \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_{N_{d-1}} & 0 \\ 0 & w^{-1} \end{pmatrix}. \end{aligned}$$

Then for $\gamma'_N \in \mathrm{Sp}(2n-2r, \mathbb{Z})$ and $a'_p \in T_{n-r}(\mathbb{Z}_p)$, there exists $\gamma_N \in \mathrm{Sp}(2n, \mathbb{Z})$, $a_p \in T(\mathbb{Z}_p)$ such that

$$\varepsilon_{q\text{-exp}, \widetilde{V}}^{\gamma'_N, a'_p} \left(\beta', \mathfrak{p}_{\widetilde{V}}(f)\right) = \varepsilon_{q\text{-exp}}^{\gamma_N, a_p} \left(\mathfrak{i}_{\widetilde{V}}(\beta'), f\right).$$

To prove the proposition, it suffices to check that for $\beta' \in \mathrm{Sym}(n-r,\mathbb{Q})_{>0}$ and $f \in V^{SP,r,\flat}_{m,l}$,

$$\varepsilon^{(\gamma^{P'}_{p,i})^{\varphi(N)}\gamma'_N,\,a'_p}_{q\text{-exp},\widetilde{V}}\left(\beta',\,(U^P_{p,N_i})^{\varphi(N)}\mathfrak{p}_{\widetilde{V}}(f)\right)=\varepsilon^{(\gamma^{P}_{p,i})^{\varphi(N)}\gamma_N,\,a_p}_{q\text{-exp}}\left(\mathfrak{i}_{\widetilde{V}}(\beta'),\,(U^P_{p,N_i})^{\varphi(N)}f\right). \tag{1.9.1}$$

We have

LHS of (1.9.1)

$$=\sum_{x\in M_{N_i,n-r-N_i}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z})}\varepsilon^{\gamma'_N,\,a'_p}_{q\text{-exp},\widetilde{V}}\left(\begin{pmatrix}p^{\varphi(N)}I_{N_i} & 0\\ N{}^{\mathrm{t}}x & I_{n-r-N_i}\end{pmatrix}\beta'\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx\\ 0 & I_{n-r-N_i}\end{pmatrix},\,\mathfrak{p}_{\widetilde{V}}(f)\right)$$

(1.9.2)

$$=\sum_{x\in M_{N_i,n-r-N_i}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z})}\varepsilon^{\gamma_N,\,a_p}_{q\text{-exp}}\left(\begin{pmatrix}{}^{\mathrm{t}}A\\ {}^{\mathrm{t}}B\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & 0\\ N{}^{\mathrm{t}}x & I_{n-r-N_i}\end{pmatrix}\beta'\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx\\ 0 & I_{n-r-N_i}\end{pmatrix}\begin{pmatrix}A & B\end{pmatrix},\,f\right).$$

Set

$$x_A=A_1^{-1}xA_2\in M_{N_i,n-r-N_i}(\mathbb{Z}/p^{\varphi(N)}),\qquad y(x)=N^{-1}A_1^{-1}\begin{pmatrix}-I_{N_i} & Nx\end{pmatrix}B\in M_{N_i,r}(\mathbb{Z}/p^{\varphi(N)}).$$

The map $x\mapsto x_A$ is a bijection from $M_{N_i,n-r-N_i}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z})$ to itself. One can check that, by the definition of x_A, $y(x)$,

(1.9.3)

$$\left[\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx & 0\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}^{-1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & Ny(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\right]^{-1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma(N)\cap\Gamma_{SP}(p^s),$$

where $x,x_A,y(x)$ can be taken to be any lift to $\mathbb{Z}$. Then

$$\begin{aligned}
&\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx\\ 0 & I_{n-r-N_i}\end{pmatrix}\begin{pmatrix}A & B\end{pmatrix}=\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx\\ 0 & I_{n-r-N_i}\end{pmatrix}\begin{pmatrix}I_{n-r} & 0\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\
&=\begin{pmatrix}I_{n-r} & 0\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx & 0\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\
&=\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & Ny(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & Ny(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}^{-1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx & 0\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\
&=\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & y(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\left[\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx & 0\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}^{-1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & Ny(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\right]^{-1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\
&\in\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & Ny(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix}\cdot\Gamma(N)\cap\Gamma_{SP}(p^s).
\end{aligned}$$

Plugging into (1.9.2), we get

(1.9.4)

LHS of (1.9.1)

$$=\sum_{x\in M_{N_i,n-r-N_i}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z})}\varepsilon^{\gamma_N,\,a_p}_{q\text{-exp}}\left(\begin{pmatrix}p^{\varphi(N)}I_{N_i} & 0 & 0\\ N{}^{\mathrm{t}}x_A & I_{n-r-N_i} & 0\\ N{}^{\mathrm{t}}y(x) & 0 & I_r\end{pmatrix}\begin{pmatrix}{}^{\mathrm{t}}A\\ {}^{\mathrm{t}}B\end{pmatrix}\beta'\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}p^{\varphi(N)}I_{N_i} & Nx_A & Ny(x)\\ 0 & I_{n-r-N_i} & 0\\ 0 & 0 & I_r\end{pmatrix},\,f\right).$$

Next we need to use the condition $f \in V_{m,l}^{SP,r,\flat}$ to show that its Fourier coefficient in $\varepsilon_{q\text{-exp}}^{\gamma_N, a_p}(f)$ indexed by

$$\begin{pmatrix} p^{\varphi(N)} I_{N_i} & 0 & 0 \\ N{}^{\mathrm{t}}x_A & I_{n-r-N_i} & 0 \\ N{}^{\mathrm{t}}y & 0 & I_r \end{pmatrix} \begin{pmatrix} {}^{\mathrm{t}}A \\ {}^{\mathrm{t}}B \end{pmatrix} \beta' \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} p^{\varphi(N)} I_{N_i} & N x_A & N y \\ 0 & I_{n-r-N_i} & 0 \\ 0 & 0 & I_r \end{pmatrix} \tag{1.9.5}$$

is nonzero only if $y = y(x)$ in $M_{N_i,r}(\mathbb{Z}/p^{\varphi(N)})$. By Proposition 1.8.2, the coefficient indexed by (1.9.5) is nonzero only if the radical of (1.9.5) does not contain a primitive vector inside $\mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z} \cdot x_n^*$. The radical tensored with $\mathbb{Q}$ is spanned by the columns of

$$\begin{pmatrix} p^{\varphi(N)} I_{N_i} & N x_A & N y \\ 0 & I_{n-r-N_i} & 0 \\ 0 & 0 & I_r \end{pmatrix}^{-1} \begin{pmatrix} -A^{-1}B \\ I_r \end{pmatrix} = \left(-\begin{pmatrix} p^{-\varphi(N)} N y \\ 0 \end{pmatrix} - \begin{pmatrix} p^{-\varphi(N)} I_{N_i} & -p^{-\varphi(N)} N x_A \\ 0 & I_{n-r-N_i} \\ I_r & \end{pmatrix} A^{-1}B \right),$$

which contains no primitive vector in $\mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z} \cdot x_n^*$ only if

$$Ny + \begin{pmatrix} I_{N_i} & -N x_A \end{pmatrix} A^{-1} B \equiv 0 \mod p^{\varphi(N)},$$

and this equation is satisfied exactly when $y = y(x)$. Therefore, the coefficient indexed by (1.9.5) is nonzero only if $y = y(x)$ in $M_{N_i,r}(\mathbb{Z}/p^{\varphi(N)})$, and from (1.9.4) we get

LHS of (1.9.1)

$$= \sum_{\substack{x \in M_{N_i, n-r-N_i}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z}) \\ y \in M_{N_i,r}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z})}} \varepsilon_{q\text{-exp}}^{\gamma_N, a_p} \left(\begin{pmatrix} p^{\varphi(N)} I_{N_i} & 0 & 0 \\ N{}^{\mathrm{t}}x_A & I_{n-r-N_i} & 0 \\ N{}^{\mathrm{t}}y & 0 & I_r \end{pmatrix} \begin{pmatrix} {}^{\mathrm{t}}A \\ {}^{\mathrm{t}}B \end{pmatrix} \beta' \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} p^{\varphi(N)} I_{N_i} & N x_A & N y \\ 0 & I_{n-r-N_i} & 0 \\ 0 & 0 & I_r \end{pmatrix}, f \right)$$

$$= \sum_{x \in M_{N_i, n-N_i}(\mathbb{Z}/p^{\varphi(N)}\mathbb{Z})} \varepsilon_{q\text{-exp}}^{\gamma_N, a_p} \left(\begin{pmatrix} p^{\varphi(N)} I_{N_i} & 0 \\ N{}^{\mathrm{t}}x & I_{n-N_i} \end{pmatrix} \mathfrak{i}_{\widetilde{V}}(\beta') \begin{pmatrix} p^{\varphi(N)} I_{N_i} & N x \\ 0 & I_{n-N_i} \end{pmatrix}, f \right)$$

=RHS of (1.9.1).

□

Proposition 1.9.4. *Let $s \geq m, l$. Then* $\left(U_{p,N_{d-1}}^P\right)^{2s} V_{m,l}^{SP,r} \subset V_{m,l}^{SP,r,\flat}$.

Proof. By Proposition 1.8.2, what we need to show is that for all $\gamma_N \in \mathrm{Sp}(2n, \mathbb{Z})$, $a_p \in T(\mathbb{Z}_p)$, $f \in V_{m,l}^{SP,r}$ and $\beta \in N^{-1}S^2(X_n)_{\geq 0}$ whose radical is of rank r and contains a primitive vector v_β inside $\mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p\mathbb{Z} \cdot x_n^*$,

$$\varepsilon_{q\text{-exp}}^{\gamma_N, a_p} \left(\beta, \left(U_{p,N_{d-1}}^P\right)^{2s} f \right) = 0.$$

One can easily check that for all $x \in M_{N_{d-1}, n_d}(\mathbb{Z})$,

$$\mathbb{Q} \cdot \begin{pmatrix} p^s I_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix}^{-1} v_\beta \cap X_n^* \subset \mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p^s\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p^s\mathbb{Z} \cdot x_n^*,$$

i.e. the radical of $\begin{pmatrix} p^s I_{N_{d-1}} & 0 \\ N{}^{\mathrm{t}}x & I_r \end{pmatrix} \beta \begin{pmatrix} p^s I_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix}$ contains a primitive vector inside $\mathbb{Z} \cdot x_1^* + \cdots + \mathbb{Z} \cdot x_{N_{d-1}}^* + p^s\mathbb{Z} \cdot x_{N_{d-1}+1}^* + \cdots + p^s\mathbb{Z} \cdot x_n^*$. Since by Proposition 1.9.1,

$$\varepsilon_{q\text{-exp}}^{\gamma_N, a_p} \left(\beta, \left(U_{p,N_{d-1}}^P\right)^{2s} f \right) = \sum_{x \in M_{N_{d-1}, n_d}(\mathbb{Z})} \varepsilon_{q\text{-exp}}^{\gamma_{p,n-1}^{-s}\gamma_N, a_p} \left(\begin{pmatrix} p^s I_{N_{d-1}} & 0 \\ N{}^{\mathrm{t}}x & I_r \end{pmatrix} \beta \begin{pmatrix} p^s I_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix}, \left(U_{p,N_{d-1}}^P\right)^s f \right),$$

we reduce to showing

$$\varepsilon_{q\text{-exp}}^{\gamma_N,a_p}\left(\beta,\, \left(U^P_{p,N_{d-1}}\right)^s f\right) = 0,$$

for β whose radical contains a primitive vector v_β inside $\mathbb{Z}\cdot x_1^* + \cdots + \mathbb{Z}\cdot x_{N_{d-1}}^* + p^s\mathbb{Z}\cdot x_{N_{d-1}+1}^* + \cdots + p^s\mathbb{Z}\cdot x_n^*$.

Write $v_\beta = {}^{\mathrm{t}}(v_{\beta,1},\dots,v_{\beta,n})$. Then $p^s \mid v_{\beta,i}$, $N_{d-1}+1 \leq i \leq n$, and there exists $1\leq j\leq N_{d-1}$ such that $v_{\beta,j}$ is not divisible by p. Put $w_\beta = {}^{\mathrm{t}}(\underbrace{0,\dots,0}_{j-1}, v_{\beta,n}, 0,\dots,0,-v_{\beta,j}) \in \mathbb{Z}^n$. Then $I_n - N\eta\cdot v_\beta{}^{\mathrm{t}}w_\beta$ belongs to $\mathrm{GL}(n,\mathbb{Z})$ for all integer η. Moreover,

$$I_n - N\eta\cdot v_\beta{}^{\mathrm{t}}w_\beta \equiv I_n + \begin{pmatrix} 0 & \cdots & 0 & N\eta v_{\beta,j}\cdot v_{\beta,1} \\ & \ddots & & \vdots \\ 0 & \cdots & 0 & N\eta v_{\beta,j}\cdot v_{\beta,j} \\ & \ddots & & \vdots \\ 0 & \cdots & 0 & N\eta v_{\beta,j}\cdot v_{\beta,N_{d-1}} \\ 0 & \cdots & 0 & 0 \end{pmatrix} \mod Np^s.$$

Let $x_\beta = v_{\beta,j}\cdot {}^{\mathrm{t}}(v_{\beta,1},\cdots,v_{\beta,N_{d-1}}) \in \mathbb{Z}^{N_{d-1}}$. Then

$$\begin{pmatrix} I_{n-1} & N\eta x_\beta \\ 0 & I_r \end{pmatrix}^{-1}\left(I_n - \eta\cdot v_\beta{}^{\mathrm{t}}w_\beta\right) \equiv I_n \mod Np^s$$

and

(1.9.6)
$$\begin{pmatrix} p^sI_{n-1} & Nx + N\eta x_\beta \\ 0 & I_r \end{pmatrix}^{-1}\left(I_n - \eta\cdot v_\beta{}^{\mathrm{t}}w_\beta\right)\begin{pmatrix} p^sI_{n-1} & Nx \\ 0 & I_r \end{pmatrix} \in \mathrm{Im}\left(\Gamma\cap\Gamma_{SP}(p^s)\to \mathrm{GL}(n,\mathbb{Z})\right).$$

By definition the vector $x_\beta \in \mathbb{Z}^{N_{d-1}}$ is not divisible by p. Thus we can pick $C \subset M_{N_{d-1},n_d}(\mathbb{Z}/p^s)$ such that

$$M_{N_{d-1},n_d}(\mathbb{Z}/p^s) = (\mathbb{Z}/p^s)\cdot x_\beta \oplus C.$$

We have

$$\begin{aligned} &\varepsilon_{q\text{-exp}}^{\gamma_N,a_p}\left(\beta,\, \left(U^P_{p,N_{d-1}}\right)^s f\right) \\ &= \sum_{x\in M_{N_{d-1},n_d}(\mathbb{Z}/p^s)} \varepsilon_{q\text{-exp}}^{(\gamma^P_{p,d-1})^{-s}\gamma_N,a_p}\left(\begin{pmatrix} p^sI_{N_{d-1}} & 0 \\ N{}^{\mathrm{t}}x & I_r \end{pmatrix}\beta\begin{pmatrix} p^sI_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix},\, f\right) \\ &= \sum_{x\in C}\sum_{\eta\in\mathbb{Z}/p^s\mathbb{Z}} \varepsilon_{q\text{-exp}}^{(\gamma^P_{p,d-1})^{-s}\gamma_N,a_p}\left(\begin{pmatrix} p^sI_{N_{d-1}} & 0 \\ N{}^{\mathrm{t}}x + N\eta{}^{\mathrm{t}}x_\beta & I_r \end{pmatrix}\beta\begin{pmatrix} p^sI_{N_{d-1}} & Nx + N\eta x_\beta \\ 0 & I_r \end{pmatrix},\, f\right). \end{aligned}$$

Applying (1.9.6), we get

$$\begin{aligned} &\varepsilon_{q\text{-exp}}^{\gamma_N,a_p}\left(\beta,\, \left(U^P_{p,N_{d-1}}\right)^s f\right) \\ &= \sum_{x\in C}\sum_{\eta\in\mathbb{Z}/p^s\mathbb{Z}} \varepsilon_{q\text{-exp}}^{(\gamma^P_{p,d-1})^{-s}\gamma_N,1}\left(\begin{pmatrix} p^sI_{N_{d-1}} & 0 \\ N{}^{\mathrm{t}}x & I_r \end{pmatrix}\left(I_n - \eta\cdot w_\beta{}^{\mathrm{t}}v_\beta\right)\beta\left(I_n - \eta\cdot v_\beta{}^{\mathrm{t}}w_\beta\right)\begin{pmatrix} p^sI_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix},\, f\right). \end{aligned}$$

Since v_β belongs to the radical of β, we know that $\left(I_n - \eta \cdot w_\beta {}^{\mathrm{t}}v_\beta\right) \beta \left(I_n - \eta \cdot v_\beta {}^{\mathrm{t}}w_\beta\right) = \beta$, and

$$\begin{aligned}
&\varepsilon_{q\text{-exp}}^{\gamma_N, a_p}\left(\beta, \left(U_{p,N_{d-1}}^P\right)^s f\right) \\
=&\sum_{x \in C} \sum_{\eta \in \mathbb{Z}/p^s\mathbb{Z}} \varepsilon_{q\text{-exp}}^{(\gamma_{p,d-1}^P)^{-s}\gamma_N, a_p}\left(\begin{pmatrix} p^s I_{N_{d-1}} & 0 \\ N {}^{\mathrm{t}}x & I_r \end{pmatrix} \beta \begin{pmatrix} p^s I_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix}, f\right) \\
=&\sum_{x \in C} p^s \cdot \varepsilon_{q\text{-exp}}^{(\gamma_{p,d-1}^P)^{-s}\gamma_N, a_p}\left(\begin{pmatrix} p^s I_{N_{d-1}} & 0 \\ N {}^{\mathrm{t}}x & I_r \end{pmatrix} \beta \begin{pmatrix} p^s I_{N_{d-1}} & Nx \\ 0 & I_r \end{pmatrix}, f\right) \\
=&0.
\end{aligned}$$

□

1.10. **Hida families of p-adic Siegel modular forms vanishing along strata with cusp labels of rank $> r$.**

Set $U_p^P = \prod_{i=1}^{d} U_{p,N_i}^P$. We first show the existence of an ordinary projector on $\mathcal{V}^{r,SP}$ by applying induction on r and using $\mathcal{V}^{r,SP,\flat}$ plus Proposition 1.9.4.

Proposition 1.10.1. *For each $f \in \mathcal{V}^{SP,r}$, the limit $\lim_{j\to\infty}\left(U_p^P\right)^{j!} f$ exists in $\mathcal{V}^{SP,r}$.*

Proof. We remark that for any endomorphism of finitely generated $\mathcal{O}_F/p^m$-modules, its $j!$-th power stabilizes when j is large enough.

Given f inside an $\mathcal{O}_F/p^m$-module with an action by U_p^P, we define the following finiteness property for f.

(F) The submodule generated by $\left(U_p^P\right)^{n\varphi(N)} f$, $n \geq 0$, is finitely generated over $\mathcal{O}_F/p^m$.

It suffices to show that (F) holds for all elements in $V_{m,l}^{SP,r}$. For $r = 0$ this is known thanks to [Hid02, Pil12]. Now assume (F) holds for $V_{m,l}^{SP,r-1}$. Due to Proposition 1.9.4, we only need to show that (F) for all $f \in V_{m,l}^{SP,r,\flat}$. Take $f \in V_{m,l}^{SP,r,\flat}$, since (F) holds for the quotient in (1.7.9), there exists $a_0, a_1, \dots, a_{j_1} \in \mathcal{O}_F$ such that

$$g = \left(U_p^P\right)^{(j_1+1)\varphi(N)} f - \sum_{i=0}^{j_1} a_i \left(U_p^P\right)^{i\varphi(N)} f \in V_{m,l}^{SP,r-1,\flat}. \tag{1.10.1}$$

Then we apply (F) for (1.10.1). There exists $b_0, b_1, \dots, b_{j_2} \in \mathcal{O}_F$ such that

$$\begin{aligned}
&\left(U_p^P\right)^{(j_2+1)\varphi(N)}\left(\left(U_p^P\right)^{(j_1+1)\varphi(N)} f - \sum_{i=0}^{j_1} a_i \left(U_p^P\right)^{i\varphi(N)} f\right) \\
=&\sum_{s=0}^{j_2} b_s \left(U_p^P\right)^{s\varphi(N)}\left(\left(U_p^P\right)^{(j_1+1)\varphi(N)} f - \sum_{i=0}^{j_1} a_i \left(U_p^P\right)^{i\varphi(N)} f\right).
\end{aligned}$$

Therefore $\left(U_p^P\right)^{(j_1+j_2+2)\varphi(N)} f$ belongs to the $\mathcal{O}_F/p^m$-span of f, $\left(U_p^P\right)^{\varphi(N)} f, \dots, \left(U_p^P\right)^{(r_1+r_2+1)\varphi(N)} f$, and (F) holds for f. □

The above proposition shows that $\lim_{j\to\infty}\left(U_p^P\right)^{j!}$ can be well defined on $\mathcal{V}^{SP,r}$. Define the P-ordinary projector on $\mathcal{V}^{SP,r}$ as

$$e_P = \lim_{j\to\infty}\left(U_p^P\right).$$

It is an idempotent projecting the spaces into the subspaces spanned by generalized $\mathbb{U}_p^P$-eigenvectors with eigenvalues being p-adic units. Similarly a P_{n-r}-ordinary projector can be defined for the quotient terms in the exact sequences in Proposition 1.7.1.

Set

$$\mathcal{V}^r_{P\text{-ord}} = e_P \mathcal{V}^{SP,r} = e_P \mathcal{V}^{SP,r,\flat}, \qquad \mathcal{V}^0_{V,P_{n-r}\text{-ord}} = e_{P_{n-r}} \mathcal{V}_V^{SP_{n-r},0},\ V \in \mathfrak{C}_{\mathbb{V}},\ \operatorname{rk} V = r \leq n_d.$$

Applying e_P, $e_{P_{n-r}}$ to (1.7.10), we get

$$0 \longrightarrow \mathcal{V}^{r-1}_{P\text{-ord}} \longrightarrow \mathcal{V}^r_{P\text{-ord}} \longrightarrow \bigoplus_{\substack{V \in \mathfrak{C}_{\mathbb{V}}/\Gamma \\ \operatorname{rk} V = r}} \mathbb{Z}_p[\![T_P(\mathbb{Z}_p)]\!] \otimes_{\mathbb{Z}_p[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]} \mathcal{V}^0_{V,P_{n-r}\text{-ord}} \longrightarrow 0. \tag{1.10.2}$$

Define $\mathcal{V}^{r,*}_{P\text{-ord}}$ to be the Pontryagin dual of $\mathcal{V}^r_{P\text{-ord}}$, *i.e.* $\operatorname{Hom}_{\mathbb{Z}_p}(\mathcal{V}^r_{P\text{-ord}}, \mathbb{Q}_p/\mathbb{Z}_p)$, and similarly define $\mathcal{V}^{0,*}_{V,P_{n-r}\text{-ord}}$. Then (1.10.2) gives

$$0 \longrightarrow \bigoplus_{\substack{V \in \mathfrak{C}_{\mathbb{V}}/\Gamma \\ \operatorname{rk} V = r}} \mathcal{V}^{0,*}_{V,P_{n-r}\text{-ord}} \otimes_{\mathbb{Z}_p[\![T_{P_{n-r}}(\mathbb{Z}_p)]\!]} \mathbb{Z}_p[\![T_P(\mathbb{Z}_p)]\!] \longrightarrow \mathcal{V}^{r,*}_{P\text{-ord}} \longrightarrow \mathcal{V}^{r-1,*}_{P\text{-ord}} \longrightarrow 0. \tag{1.10.3}$$

Let $\Lambda_P = \mathcal{O}_F[\![T_P(\mathbb{Z}_p)^\circ]\!]$ (resp. $\Lambda_{P_{n-r}} = \mathcal{O}_F[\![T_{P_{n-r}}(\mathbb{Z}_p)^\circ]\!]$), where $T_P(\mathbb{Z}_p)^\circ$ is the maximal p-profinite subgroup of $T_P(\mathbb{Z}_p)$ (resp. $T_{P_{n-r}}(\mathbb{Z}_p)$).

Proposition 1.10.2. *$\mathcal{V}^{r,*}_{P\text{-ord}}$, $0 \leq r \leq n_d$, is a free Λ_P-module of finite rank.*

Proof. We prove the proposition by induction. For $r = 0$ the control theorem in [Pil12, Théorème 1.1 (7)] for $\mathcal{V}^{0,*}_{P\text{-ord}}$ (resp. $\mathcal{V}^{0,*}_{V,P_{n-r}\text{-ord}}$) says that it is a free Λ_P-module (rep. $\Lambda_{P_{n-r}}$-module) of finite rank. Suppose that $\mathcal{V}^{r-1,*}_{P\text{-ord}}$ is a free Λ_P-module of finite rank. Then the terms at the two ends of (1.10.3) are free Λ_P-modules of finite rank. Since $\operatorname{Ext}^1_{\Lambda_P}(M,N)$ vanishes if M is a free Λ_P-module, $\mathcal{V}^{r,*}_{P\text{-ord}}$ is isomorphic, as a Λ_P-module, to the direct sum the terms at the two ends of (1.10.3). ☐

Now we have established *(i)* in Theorem 1.3.1.

For $0 \leq r \leq n_d$, the $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-module of Hida families of p-adic Siegel modular forms ordinary for the parabolic P vanishing along the strata associated with cusp labels of rank $> r$, is defined as

$$\mathcal{M}^r_{P\text{-ord}} := \operatorname{Hom}_{\Lambda_P}\left(\mathcal{V}^{r,*}_{P\text{-ord}}, \Lambda_P\right).$$

Similarly, define

$$\mathcal{M}^0_{P_{n-r}\text{-ord}} := \operatorname{Hom}_{\Lambda_{P_{n-r}}}\left(\mathcal{V}^{0,*}_{V,P_{n-r}\text{-ord}}, \Lambda_{P_{n-r}}\right).$$

Applying $\operatorname{Hom}_{\Lambda_P}(\cdot, \Lambda_P)$ to (1.10.3) gives *(iv)* of Theorem 1.3.1.

Let $\underline{\tau}^P \in \operatorname{Hom}_{\text{cont}}\left(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right)$ be an arithmetic dominant weight. Attached to it is a prime ideal $\mathcal{P}_{\underline{\tau}^P}$ of $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$. Then unfolding the definitions, one gets the following isomorphisms,

$$\mathcal{M}^r_{P\text{-ord}} \otimes \mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]/\mathcal{P}_{\underline{\tau}^P} \xrightarrow{\sim} \operatorname{Hom}\left((\mathcal{V}^r_{P\text{-ord}}[\underline{\tau}^P])^*, \mathcal{O}_F\right) \xrightarrow{\sim} \varprojlim_m \varinjlim_l e_P V^{SP,r}_{m,l}[\underline{\tau}^P] \tag{1.10.4}$$

equivariant under the action of the unramfied Hecke algebra away from Np and the $\mathbb{U}_p^P$-operators. Combining (1.10.4) with the embedding (1.2.1) proves *(ii)* in Theorem 1.3.1.

The proof of *(iii)* relies on the uniform boundedness with respect to $k \geq 0$ of the dimension of ordinary forms of weight $\underline{t} + k$ [TU99], and the argument proceeds in the same way as [Hid02, §3.7].

For applications in §2.6, define $\mathbb{T}^{r,N}_{P\text{-ord}}$ as the $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-algebra generated by all the unramified Hecke operators away from $Np\infty$ and the $\mathbb{U}_p$-operators $U^P_{p,1}, U^P_{p,2}, \ldots, U^P_{p,n}$ acting on $\mathcal{M}^r_{P\text{-ord}}$. The algebra $\mathbb{T}^{r,N}_{P\text{-ord}}$ is finite and torsion free over $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$. Also, the uniqueness of the P-ordinary vectors (the last statement in Proposition 2.3.2) plus the Zariski density of arithmectic points in $\mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\right)$ with $t_1^P \gg t_2^P \gg \cdots \gg t_d^P \gg 0$ implies that $\mathbb{T}^{0,N}_{P\text{-ord}}$ and $\mathbb{T}^{n_d,N}_{P\text{-ord}}$ are reduced.

x

2. p-ADIC L-FUNCTIONS

In this section, for a given geometrically irreducible component of $\mathrm{Spec}\left(\mathbb{T}^{0,N}_{P\text{-ord}}\otimes F\right)$, we construct the $(d+1)$-variable p-adic standard L-function and its d-variable improvement as called in [GS93] (missing the cyclotomic variable). The construction uses the doubling method formula as the integral representation for the standard L-function. The d-variable improvement will be used to employ the Greenberg–Stevens method to prove Theorem 3.3.5 on the derivatives of cyclotomic p-adic L-functions at the so-called semi-stable trivial zeros. The Hida theory for non-cuspidal Siegel modular forms developed in the previous section will be used for the construction of the d-variable improved p-adic L-function.

Before starting the construction, we briefly mention several works on constructing p-adic L-functions using the doubling method. It is Böcherer and Schmidt [BS00] who first carried out such a construction in the special case where π is fixed and is $\mathrm{GL}(n)$-ordinary with π_∞ isomorphic to a scalar weight holomorphic discrete series. Later, the case where π varies in a cuspidal Hida family which is ordinary for the Borel subgroup is treated in [Liu16b] for symplectic groups and in [EW14, EHLS16] for unitary groups. Here we look at the more general case of P-ordinary Hida families for a general parabolic P. Moreover, we also construct its improvement as an important input for applying the Greenberg–Stevens method.

2.1. **Generalities on standard L-functions for symplectic groups.**

Let $\pi\subset\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ and $\xi:\mathbb{Q}^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$ be a finite order Dirichlet character. Take S to be a finite set of places of $\mathbb{Q}$ containing the archimedean place and all the finite places where π_v or ξ_v is ramified.

For $v\notin S$, there exist unramified characters $\theta_i:\mathbb{Q}_v^\times\to\mathbb{C}^\times$, $1\leq i\leq n$, such that π_v is isomorphic to the normalized induction $\mathrm{Ind}^{G(\mathbb{Q}_v)}_{B_G(\mathbb{Q}_v)}(\theta_1,\dots,\theta_n)$ as $G(\mathbb{Q}_v)$-representations. Put $\alpha_{v,i}=\theta_i(q_v)$ where q_v is the cardinality of the residue field of $\mathbb{Q}_v$. Then $\alpha^\pm_{v,1},\dots,\alpha^\pm_{v,n}$ are the Satake parameters of π_v, and the unramified local L-factor (for the standard representation ${}^LG^\circ=\mathrm{SO}(2n+1,\mathbb{C})\to\mathrm{GL}(2n+1,\mathbb{C})$) is defined as

$$L_v(s,\pi_v\times\xi)=(1-\xi(q_v)q_v^{-s})^{-1}\prod_{i=1}^n(1-\xi(q_v)\alpha_{v,i}q_v^{-s})^{-1}(1-\xi(q_v)\alpha_{v,i}^{-1}q_v^{-s})^{-1}.$$

The analytic properties (meromorphic continuation, functional equation, location of possible poles) of the partial standard L-function

$$L^S(s,\pi\times\xi)=\prod_{v\notin S}L_v(s,\pi_v\times\xi)$$

are established in [GPSR87, KR90].

Assuming $\pi_\infty\cong\mathcal{D}_{\underline{t}}$, the holomorphic discrete series of weight $\underline{t}=(t_1,\dots,t_n)$ (so $t_1\geq\cdots\geq t_n\geq n+1$), the critical points of $L^S(s,\pi\times\xi)$ are integers s_0 such that

$$1\leq s_0\leq t_n-n,\ (-1)^{s_0+n}=\xi(-1),\ \text{ or } n+1-t_n\leq s_0\leq 0,\ (-1)^{s_0+n+1}=\xi(-1).$$

The algebracity of these critical L-values divided by certain automorphic periods (expressed in terms of Petersson inner product) is obtained in [Har81, Shi00, BS00].

2.2. **The doubling method for symplectic groups.** One standard way to study the standard L-function $L^S(s,\pi\times\xi)$ and its critical values is to apply the doubling method developed by Piatetski-Shapiro–Rallis [PSR87], Garrett [Gar84] and Shimura [Shi97].

For the convenience of the reader, we briefly recall the setting for the doubling method used in [Liu16b]. Let $\mathbb{V}'$ be another copy of $\mathbb{V}$ with standard basis $e'_1,\dots,e'_n,f'_1,\dots,f'_n$. Put $\mathbb{W}=\mathbb{V}\oplus\mathbb{V}'$, for which we fix the basis $e_1,\dots,e_n,e'_1,\dots,e'_n,f_1,\dots,f_n,f'_1,\dots,f'_n$. Then $\mathbb{W}$ is endowed with a symplectic pairing induced from that of $\mathbb{V}$ and $\mathbb{V}'$. Let $H=\mathrm{Sp}(\mathbb{W})=\mathrm{Sp}(4n)$. There is the (holomorphic) embedding ι of $G\times G$ into H given by

$$\begin{aligned} \iota: G\times G &\longhookrightarrow H \\ \begin{pmatrix} a & b \\ c & d\end{pmatrix}\times\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d'\end{pmatrix}. \end{aligned}$$

The space $W=\sum_{i=1}^n \mathbb{Z}e_i+\mathbb{Z}e'_i$ is a maximal isotropic subspace of $\mathbb{W}$. Its stabilizer Q_H is the standard Siegel parabolic subgroup of H. Besides W, there is another maximal isotropic subspace relevant to us, which is $W^\diamondsuit=\{(v,\vartheta(v)):v\in V\}$, where $\vartheta:\mathbb{V}\to\mathbb{V}$ is the involution given by the matrix $\begin{pmatrix} 0 & I_n \\ I_n & 0\end{pmatrix}$ with respect to our fixed basis. Note that ϑ dose not preserve the symplectic pairing but has similitude -1. The space $W^\diamondsuit$ is spanned by $e_i+f'_i$, $f_i+e'_i$, $1\le i\le n$. The doubling Siegel parabolic $Q_H^\diamondsuit$ is defined to be the stabilizer of $W^\diamondsuit$. We have

$$Q_H^\diamondsuit=\mathcal{S}Q_H\mathcal{S}^{-1} \qquad \text{with} \qquad \mathcal{S}=\begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & I_n & I_n & 0 \\ I_n & 0 & 0 & I_n\end{pmatrix}.$$

For an element $g\in G$, define g^ϑ to be $\vartheta g\vartheta\in G$. This conjugation by ϑ is called the MVW involution. The MVW involution of an irreducible smooth representation of $G(\mathbb{Q}_v)$ is isomorphic to its contragredient [MgVW87, p. 91]. For $\varphi\in\pi$ we define its MVW involution φ^ϑ as $\varphi^\vartheta(g)=\varphi(g^\vartheta)$. Thanks to the multiplicity one theorem [Art13], φ^ϑ lies inside $\overline{\pi}\subset\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Remark 2.2.1. Our formulation here aligns with those in [Gar84, Shi00] but differs from [GPSR87] in that the embedding used in [GPSR87] corresponds to the above defined ι composite with a conjugation by ϑ on the second copy of G. Hence in our later computation using formulas from [GPSR87], the involution ϑ shows up a lot.

Let s be a complex variable. Denote by ξ_s (resp. $\xi_s^\diamondsuit$) the character of $Q_H(\mathbb{A})$ (resp. $Q_H^\diamondsuit(\mathbb{A})$) sending $\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix}$ (resp. $\mathcal{S}\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix}\mathcal{S}^{-1}$) to $\xi(\mathrm{detA})|\mathrm{det}A|^s$. Let $I_{Q_H}(s,\xi)=\mathrm{Ind}_{Q_H(\mathbb{A})}^{H(\mathbb{A})}\xi_s$ (resp. $I_{Q_H^\diamondsuit}(s,\xi)=\mathrm{Ind}_{Q_H^\diamondsuit(\mathbb{A})}^{H(\mathbb{A})}\xi_s^\diamondsuit$) be the normalized induction consisting of smooth functions f on $H(\mathbb{A})$ that satisfy $f(qh)=\xi_s(q)\delta_{Q_H}^{1/2}(q)f(h)$ (resp. $f(qh)=\xi_s^\diamondsuit(q)\delta_{Q_H^\diamondsuit}^{1/2}(q)f(h)$) for all $h\in H(\mathbb{A})$ and $q\in Q_H(\mathbb{A})$ (resp. $q\in Q_H^\diamondsuit(\mathbb{A})$). Recall that the modulus character δ_{Q_H} takes value $|\mathrm{det}A|^{\frac{2n+1}{2}}$ at $\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix}$. The local degenerate principal series $I_{Q_H,v}(s,\xi)$, $I_{Q_H^\diamondsuit,v}(s,\xi)$ for all places of $\mathbb{Q}$ are defined similarly. There is the simple isomorphism

$$\begin{aligned} I_{Q_H}(s,\xi) &\longrightarrow I_{Q_H^{\diamondsuit}}(s,\xi) \\ f(s,\xi) &\longmapsto f^{\diamondsuit}(s,\xi)(h) = f(s,\xi)(\mathcal{S}^{-1}h). \end{aligned}$$

Given $f(s,\xi) \in I_{Q_H}(s,\xi)$, the associated Siegel Eisenstein series is defined as

$$E(h, f(s,\xi)) = \sum_{\gamma \in Q_H(\mathbb{Q})\backslash H(\mathbb{Q})} f(s,\xi)(\gamma h) = \sum_{\gamma \in Q_H^{\diamondsuit}(\mathbb{Q})\backslash H(\mathbb{Q})} f^{\diamondsuit}(s,\xi)(\gamma h).$$

This sum is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation.

For a finite place v we fix the Haar measure on $G(\mathbb{Q}_v)$ such that the maximal compact subgroup $G(\mathbb{Z}_v)$ has volume 1. For the archimedean place we fix for $G(\mathbb{R})$ the product measure of the one on the maximal compact subgroup $K_{G,\infty} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a+bi \in U(n,\mathbb{R}) \right\}$ which has total volume 1 with the one on $G(\mathbb{R})/K_{G,\infty} = \mathbb{H}_n = \{z \in \mathrm{Sym}(n,\mathbb{C}) : \mathrm{Im} z > 0\}$ given by $\det(y)^{-n-1} \prod_{1\le i\le j\le n} dx_{ij}\, dy_{ij}$. The Haar measures on $G(\mathbb{A})$ is taken to be the product of the local ones.

For a given irreducible cuspidal automorphic representation $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and its complex conjugation $\overline{\pi} \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$, which is isomorphic to the contragredient of π, we fix isomorphisms $\pi \cong \bigotimes_v' \pi_v$ and $\overline{\pi} \cong \bigotimes_v' \tilde{\pi}_v$ such that for factorizable $\varphi_1, \varphi_2 \in \pi$ with images $\bigotimes_v \varphi_{1,v} \in \bigotimes_v' \pi_v$ and $\bigotimes_v \overline{\varphi}_{2,v} \in \bigotimes_v' \tilde{\pi}_v$, we have

$$\langle \varphi_1, \overline{\varphi}_2 \rangle = \prod_v \left\langle \varphi_{1,v}, \overline{\varphi}_{2,v} \right\rangle_v,$$

where the pairing on the left hand side is the bi-$\mathbb{C}$-linear Petersson inner product with respect to our fixed Haar measure on $G(\mathbb{A})$ and the pairing on the right hand side is the natural pairing between π_v and its contragredient $\tilde{\pi}_v$.

For a local section $f_v(s,\xi) \in I_{Q_H,v}(s,\xi)$, define

$$\begin{aligned} &T_{f_v(s,\xi)} : \pi \longrightarrow \pi \\ &\qquad \varphi \longmapsto \left(T_{f_v(s,\xi)}\varphi\right)(g) = \int_{G(\mathbb{Q}_v)} f_v^{\diamondsuit}(s,\xi)(\iota(g_v',1))\varphi(gg_v')d_v g_v'. \end{aligned}$$

We need to be careful with the convergence issue here, especially for $v = p, \infty$. The doubling local zeta integral is defined as

$$Z_v(f_v(s,\xi),\cdot,\cdot) : \pi_v \times \tilde{\pi}_v \longrightarrow \mathbb{C}$$

$$(v_1, \tilde{v}_2) \longmapsto Z_v(f_v(s,\xi), v_1, \tilde{v}_2) = \int_{G(\mathbb{Q}_v)} f_v^{\diamondsuit}(s,\xi)(\iota(g_v,1)) \left\langle \pi_v(g_v)v_1, \tilde{v}_2 \right\rangle_v d_v g_v. \tag{2.2.1}$$

For factorizable $\varphi_1, \varphi_2 \in \pi$, we have

$$\left\langle T_{f_v(s,\xi)}\varphi_1, \overline{\varphi}_2 \right\rangle = \frac{Z_v(f_v(s,\xi), \overline{\varphi}_{1,v}, \varphi_{2,v})}{\left\langle \overline{\varphi}_{1,v}, \varphi_{2,v} \right\rangle_v} \left\langle \overline{\varphi}_1, \varphi_2 \right\rangle.$$

Given $\varphi \in \pi$, we define the linear form

$$\begin{aligned} &\mathscr{L}_{\overline{\varphi}} : \mathcal{A}(G(\mathbb{Q}) \times G(\mathbb{Q})\backslash G(\mathbb{A}) \times G(\mathbb{A})) \longrightarrow \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \\ &\qquad\qquad F \longmapsto \mathscr{L}_{\overline{\varphi}}(F)(g) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} F(g', g)\overline{\varphi}(g')\, dg'. \end{aligned}$$

The doubling method formula is a formula on

$$\mathscr{L}_{\overline{\varphi}}\left(E(\cdot, f(s,\xi))|_{G\times G}\right),$$

involving the partial standard L-function of π and some local zeta integrals.

For a finite place v where ξ is unramified, we denote by $f_v^{\mathrm{ur}}(s,\xi)$ the unique section in $I_{Q_H,v}(s,\xi)$ that is fixed by the maximal compact subgroup $H(\mathbb{Z}_v) \subset H(\mathbb{Q}_v)$ and takes value 1 at the identity.

Theorem 2.2.2 ([GPSR87, Gar84, Shi97])**.** *Suppose* $f(s,\xi) = \bigotimes_{s\notin S} f_v^{\mathrm{ur}}(s,\xi) \otimes \bigotimes_{v\in S} f_v(s,\xi)$ *is a section inside to* $I_{Q_H}(s,\xi)$*. If* $\varphi \in \pi^{K_G^S}$ *with* $K_G^S = \prod_{v\notin S} G(\mathbb{Z}_v)$*, then*

$$\mathscr{L}_{\overline{\varphi}}\left(E(\cdot, f(s,\xi))|_{G\times G}\right) = d^S(s,\xi)^{-1} \cdot L^S(s+\frac{1}{2}, \pi\times\xi) \cdot \left(\prod_{v\in S} T_{f_v(s,\xi)}\overline{\varphi}\right)^{\vartheta}. \tag{2.2.2}$$

Equivalently for all factorizable $\varphi_1, \varphi_2 \in \pi^{K_G^S}$*,*

$$\left\langle E(\cdot, f(s,\xi))|_{G\times G}, \overline{\varphi}_1 \otimes \varphi_2^{\vartheta}\right\rangle = d^S(s,\xi)^{-1} \cdot L^S(s+\frac{1}{2}, \pi\times\xi) \cdot \prod_{v\in S} \frac{Z_v(f_v(s,\xi), \overline{\varphi}_{1,v}, \varphi_{2,v})}{\left\langle \overline{\varphi}_{1,v}, \varphi_{2,v}\right\rangle_v} \left\langle \overline{\varphi}_1, \varphi_2\right\rangle.$$

Here $d^S(s,\xi) = \prod_{v\notin S} d_v(s,\xi)$ *with*

$$d_v(s,\xi) := L_v(s+\frac{2n+1}{2}, \xi) \prod_{j=1}^{n} L_v(2s+2n+1-2j, \xi^2).$$

For later use we also define the normalized Siegel Eisenstein series

$$E^*(h, f(s,\xi)) = d^S(s,\xi) E(h, f(s,\xi)).$$

Then the identity (2.2.2) from the above theorem becomes

$$\mathscr{L}_{\overline{\varphi}}\left(E^*(\cdot, f(s,\xi))|_{G\times G}\right) = L^S(s+\frac{1}{2}, \pi\times\xi) \cdot \left(\prod_{v\in S} T_{f_v(s,\xi)}\overline{\varphi}\right)^{\vartheta}. \tag{2.2.3}$$

The identities provided by the doubling method reduce the study of the standard L-function $L^S(s, \pi\times\xi)$ to that of the Siegel Eisenstein series $E(\cdot, f(s,\xi))$, or more precisely its restriction to $G\times G$, and local zeta integrals at places $v \in S$.

2.3. The modified Euler factor at p.

Before starting the construction of p-adic L-functions, we first recall some basic theory of Jacquet modules and unfold the definition in [Coa91] in our case to write down explicitly the expected modified Euler factor at p in the interpolation formula. We also define the modified Euler factor at p for the improved p-adic L-function, and see that when restricting to the leftmost critical points with $\chi = \epsilon_d^P$, the difference of the two factors lies inside a finite extension of $\mathcal{O}_F[[T_P(\mathbb{Z}_p)^\circ]]$.

2.3.1. *Jacquet modules and* $\mathbb{U}_p^P$*-operators.* Suppose π_p is the component at the place p of an irreducible automorphic representation π generated by a P-ordinary Siegel modular form. Our discussion on Jacquet modules aims to: (1) show the uniqueness of the P-ordinary forms inside π, or more precisely that the space of P-ordinary Siegel modular forms projects into a one dimensional subspace inside π_p; (2) explain how to retrieve the information on π_p from the eigenvalues of the $\mathbb{U}_p^P$-operators. The uniqueness result will also play an important role in our later computation of the local zeta integral at p.

Let P_G (resp. SP_G, U_{P_G}) be the inverse image of P (resp. SP, U_P) of the projection (0.0.3). The Jacquet module of π_p associated to the parabolic P_G is defined as

$$\mathcal{J}_{P_G}(\pi_p) = V_{\pi_p} / \left\{\pi_p(u)\cdot v - v : u \in U_{P_G}(\mathbb{Q}_p),\, v \in V_{\pi_p}\right\}.$$

It follows from Jacquet's lemma [Cas, Theorem 4.1.2, Proposition 4.1.4] that $\mathcal{J}_{P_G}(\pi_p)$ is naturally isomorphic to the following subspace of V_{π_p},

$$\bigcap_{\substack{\underline{a}=(\underbrace{a_1,\dots,a_1}_{n_1},\dots,\underbrace{a_d,\dots,a_d}_{n_d}) \\ a_1\geq\cdots\geq a_d\geq 0}} \left\{ \int_{U_{P_G}(\mathbb{Q}_p)} \pi_p(up^{\underline{a}})\cdot v\, du : v\in V_{\pi_p} \right\}, \tag{2.3.1}$$

where $p^{\underline{a}}=\mathrm{diag}(p^{a_1},\dots,p^{a_1},\dots,p^{a_d},\dots,p^{a_d},p^{-a_1},\dots,p^{-a_1},\dots,p^{-a_d},\dots,p^{-a_d})$. Denote by M_P the Levi subgroup of P and we identify it with the Levi subgroup of P_G via (0.0.3). Both $\mathcal{J}_{P_G}(\pi_p)$ and the space in (2.3.1) are equipped with a natural action of $M_P(\mathbb{Q}_p)$, and the isomorphism between them is $M_P(\mathbb{Q}_p)$-equivariant.

Given irreducible smooth admissible representations σ_i of $\mathrm{GL}(n_i,\mathbb{Q}_p)$, $1\leq i\leq d$, Frobenius reciprocity gives
(2.3.2)

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\pi_p,\mathrm{Ind}_{P_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\sigma_1\boxtimes\sigma_2\boxtimes\cdots\boxtimes\sigma_d\right)\cong\mathrm{Hom}_{M_P(\mathbb{Q}_p)}\left(\mathcal{J}_{P_G}(\pi_p),(\sigma_1\boxtimes\sigma_2\boxtimes\cdots\boxtimes\sigma_d)\otimes\delta_{P_G}^{1/2}\right),$$

where δ_{P_G} is the modulus character sending $\mathrm{diag}(b_1,b_2,\dots,b_d)\in M_P(\mathbb{Q}_p)$, $b_i\in\mathrm{GL}(n_i,\mathbb{Q}_p)$, to $\prod_{i=1}^d|\det(b_i)|_p^{2n+1+n_i-2N_i}$.

Suppose that the P-ordinary Siegel modular form generating π is of weight $\underline{t}=\imath(\underline{t}^P)$ with $t_1^P\geq\cdots\geq t_d^P\geq n+1$, so $\pi_\infty\cong\mathcal{D}_{\imath(\underline{t}^P)}$. Denote by $\pi_{\imath(\underline{t}^P)}$ the subspace of π consisting of forms whose projection to π_∞ belongs to the lowest K_∞-type in $\mathcal{D}_{\imath(\underline{t}^P)}$. There is the canonical embedding
(2.3.3)

$$H^0\left(X^{\Sigma}_{\Gamma\cap\Gamma_{SP}(p^l)},\omega_{\imath(\underline{t}^P)}\right)\hookrightarrow M_{\imath(\underline{t}^P)}\left(\mathbb{H}_n,\Gamma\cap\Gamma_{SP}(p^l)\right)\overset{\varphi_G(\cdot,\mathfrak{e}_{\mathrm{can}})}{\hookrightarrow}\mathcal{A}\left(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma}\cap\widehat{\Gamma}_{SP}(p^l)\right)_{\imath(\underline{t}^P)}$$

from Siegel modular forms defined as global sections of the automorphic sheaf $\omega_{\underline{t}}$ into automorphic forms on $G(\mathbb{A})$ of K_∞-type $\underline{t}=\imath(\underline{t}^P)$ (see, for example, [Liu16b, (2.3.1)(2.4.1)] for precise definition of this embedding).

Under the embedding (2.3.3), the $\mathbb{U}_p^P$-operator $U_{p,\underline{a}}^P=\prod_{i=1}^n\left(\mathbb{U}_{p,i}^P\right)^{a_i}$, for $\underline{a}=(a_1,a_2,\dots,a_n)$ with $a_1\geq a_2\geq\cdots\geq a_n\geq 0$, on the left hand side corresponds to the following operator on the right hand side,

$$U_{p,\underline{a}}^P=p^{\langle\underline{t}+2\rho_{G,c},\,\underline{a}\rangle}\int_{SP_G(\mathbb{Z}_p)}\pi_p(up^{\underline{a}})\,du, \tag{2.3.4}$$

where $\rho_{G,c}=(\frac{n-1}{2},\frac{n-3}{2},\dots,\frac{1-n}{2})$ is the half sum of positive compact roots of G.

We have assumed that π contains a P-ordinary Siegel modular. It follows immediately from the definition of the P-ordinarity and (2.3.1), (2.3.4) that

$$\mathcal{J}_{P_G}(\pi_p)^{\mathrm{SL}(n_1,\mathbb{Z}_p)\times\cdots\times\mathrm{SL}(n_d,\mathbb{Z}_p)}\neq\{0\},$$

which combined with the Frobenius reciprocity (2.3.2) implies that there exists spherical representations σ_i of $\mathrm{GL}(n_i,\mathbb{Q}_p)$, $1\leq i\leq d$, and continuous characters $\eta_1,\dots,\eta_d$ of $\mathbb{Q}_p^\times$ taking value 1 at p, such that

$$\pi_p\hookrightarrow\mathrm{Ind}_{P_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\sigma_1\otimes\eta_1\circ\det)\boxtimes(\sigma_2\otimes\eta_2\circ\det)\boxtimes\cdots\boxtimes(\sigma_d\otimes\eta_d\circ\det). \tag{2.3.5}$$

In particular, π_p embeds into a principal series. In general, π_p being isomorphic a subquotient of a principal series is equivalent to π containing a finite slope form.

Remark 2.3.1. It does not make sense to say P-ordinarity for a purely local representation π_p as the normalization in the definition of the $\mathbb{U}_p^P$-operators depends on the weight of the holomorphic discrete series at the archimedean space. However, being of finite slope is a purely local property.

Next, we say more about the relation between the Satake parameters of the $\sigma_i' s$ in (2.3.5) and the eigenvalues of the $\mathbb{U}_p^P$-operators. Let $\underline{\theta} = (\theta_1, \dots, \theta_n)$ be an n-tuple of continuous characters of $\mathbb{Q}_p^\times$ (valued in $\mathbb{C}^\times$), viewed as a character of $T_G(\mathbb{Q}_p)$ via our fixed isomorphism of $\mathbb{G}_m^n$ with T_G, such that π_p is isomorphic to a subquotient of the principal series $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$. We consider the eigenvalues for the $\mathbb{U}_p^P$-action on $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$.

Denote by W_G (resp. W_{P_G}) the Weyl group with respect to T_G (resp. the subgroup of W_G that maps P_G to itself). Define $[W_G/W_{P_G}]$ to be the subset of W_G consisting of representatives of smallest lengths of elements in W_G/W_{P_G}. An element $w \in W_G$ acts on $\underline{\theta}$ by sending it to $\underline{\theta}^w(t) = \underline{\theta}(w^{-1}tw)$, $t \in T_G$. Like $\underline{\theta}$, via our fixed isomorphism between T_G and $\mathbb{G}_m^n$, we can write $\underline{\theta}^w$ as an n-tuple of characters $(\theta_1^w, \dots, \theta_n^w)$.

It follows from [Cas, Proposition 6.3.1, 6.3.3] that the $M_P(\mathbb{Q}_p)$-representation $\mathscr{J}_{P_G}\left(\mathrm{Ind}_{P_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right)$ has a filtration with graded pieces as

$$\boxtimes_{i=1}^{d} \mathrm{Ind}_{B_{n_i}(\mathbb{Q}_p)}^{\mathrm{GL}(n_i,\mathbb{Q}_p)} \left(\theta_{N_{i-1}+1}^w, \dots, \theta_{N_i}^w\right) \cdot \delta_{B_{n_i}}^{-1/2} \, \delta_{B_G}^{1/2}\Big|_{B_{n_i}}, \quad w \in [W_G/W_{P_G}],$$

where B_{n_i} is the standard Borel subgroup of $\mathrm{GL}(n_i)$ with modulus character $\delta_{B_{n_i}}$.

Thus, the dimension of the $\mathrm{SL}(n_1, \mathbb{Z}_p) \times \cdots \times \mathrm{SL}(n_d, \mathbb{Z}_p)$-invariant space inside $\mathscr{J}_{P_G}(\pi_p)$ is at most $|W_G/W_{P_G}| = 2^n \cdot |\mathfrak{S}_n / (\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_d})|$. Each $w \in [W_G/W_{P_G}]$ corresponds to an eigensystem of the $\mathbb{U}_p^P$-operators, and the existence of a P-ordinary Siegel modular form in π indicates that there exists $w \in W_G$ satisfying

$$\sum_{j=1}^{r} v_p\left(\theta_{N_{i-1}+j}^w(p)\right) \geq -r\left(t_i^P - N_{i-1} - \frac{r+1}{2}\right), \qquad 1 \leq r \leq n_i,\ 1 \leq i \leq d, \tag{2.3.6}$$

$$\sum_{j=1}^{n_i} v_p\left(\theta_{N_{i-1}+j}^w(p)\right) = -n_i\left(t_i^P - \frac{N_{i-1}+N_i+1}{2}\right) \qquad 1 \leq i \leq d. \tag{2.3.7}$$

These conditions on the p-adic valuation of θ_i^w, $1 \leq i \leq n$, imply

$$-\left(t_i - N_{i-1} + 1\right) \leq v_p\big(\theta_{N_{i-1}+1}^w(p)\big), \dots, v_p\left(\theta_{N_i}^w(p)\right) \leq -(t_i - N_i), \qquad 1 \leq i \leq d. \tag{2.3.8}$$

It is easily seen that given $\underline{\theta}$, there is at most one $w \in [W_G/W_{P_G}]$ to make (2.3.8) hold. By rearranging the $\theta_i^{\pm 1}$'s, we can assume that $w = 1$ in (2.3.6), (2.3.7), (2.3.8), and that $v_p(\theta_1(p)) \leq \cdots \leq v_p(\theta_n(p)) \leq 0$.

The above discussion proves the following proposition.

Proposition 2.3.2. *Suppose that π is an irreducible automorphic representation of $G(\mathbb{A})$ containing a nonzero P-ordinary holomorphic Siegel modular form of weight $\imath(\underline{t}^P)$, $t_d^P \geq n+1$, and p-nebentypus $\underline{\epsilon}^P$.*

- *There exists unramified characters $\theta_1, \dots, \theta_n : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ satisfying*

$$\underline{\theta}|_{\mathbb{Z}_p^\times} = (\underbrace{\epsilon_1^{P-1}, \dots, \epsilon_1^{P-1}}_{n_1}, \dots, \underbrace{\epsilon_d^{P-1}, \dots, \epsilon_d^{P-1}}_{n_d}),$$

$$\underline{\theta}(p) = (\alpha_1, \alpha_2, \dots, \alpha_n), \qquad \begin{aligned} &\cdots \leq -(t_i - N_{i-1} + 1) \leq v_p(\alpha_{N_{i-1}+1}) \leq \cdots \leq v_p(\alpha_{N_i}) \\ &\leq -(t_i - N_i) \leq -(t_1 - N_i + 1) \leq v_p(\alpha_{N_i+1}) \leq \dots \end{aligned}$$

such that $\pi_p \hookrightarrow \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$.

- *Let* $\mathfrak{a}_i$ *be the eigenvalue for the action of* $\mathbb{U}_{p,i}^P$ *on the* P*-ordinary form in* π*. Then* $\mathfrak{a}_{N_1}, \ldots, \mathfrak{a}_{N_d}$ *are* p*-adic units given by*

$$\mathfrak{a}_{N_i} = \prod_{j=1}^{i} p^{n_j\left(t_j^P - \frac{N_{j-1}+N_j+1}{2}\right)} \alpha_{N_{j-1}+1}\alpha_{N_{j-1}+2}\cdots\alpha_{N_j}, \tag{2.3.9}$$

  More generally, for $N_j \leq i \leq N_{j+1}$*, the eigenvalue* $\mathfrak{a}_i$ *is an* p*-adic integer given as*

$$\mathfrak{a}_i = \mathfrak{a}_{N_j} \cdot p^{(i-N_j)\left(t_{i+1}^P - \frac{N_j+i+1}{2}\right)} \sum_{\varrho \in \mathfrak{S}_{n_{j+1}}/\mathfrak{S}_{i-N_j}\times\mathfrak{S}_{N_{j+1}-i}} \alpha_{N_j+\varrho(1)}\alpha_{N_j+\varrho(2)}\cdots\alpha_{N_j+\varrho(r)}. \tag{2.3.10}$$

- *Inside* $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$*, there is a unique generalized eigenvector (up to scalar) for the operator* $U_p^P = \prod_{i=1}^d U_{p,N_i}^P$ *with eigenvalue being a* p*-adic unit. In particular, under the projection* $\pi \to \pi_p$*, the image of* P*-ordinary Siegel modular forms is one dimensional.*

Remark 2.3.3. Let $\underline{\theta}$ be as in the above proposition. If $\epsilon_d^P = \mathrm{triv}$ and $\alpha_n = p^{-1}$, since the P-ordinary condition implies that $\pi_p \hookrightarrow \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$, in the Weil–Deligne representation attached to π_p, there should be a nontrivial monodromy between the eigenspaces with Frobenious eigenvalues 1 and $\alpha_n = p^{-1}$.

2.3.2. *The modified Euler factor at p for p-adic interpolation.* If we consider the Weil–Deligne representation attached to π_p, the eigenvalues of Frobenius are $1, \alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}$. Meanwhile, for the p-adic representation associated to π [Art13, CHLN11, Shi11, CH13], the Hodge–Tate weights are $0, \pm(t_1^P - 1), \ldots, \pm(t_1^P - n_1), \ldots, \pm(t_d^P - (N_{d-1}+1)), \ldots, \pm(t_d^P - n)$. Thus, (2.3.6) and (2.3.7) essentially say that the Newton polygon is above the Hodge polygon and the two polygons meet at the points with horizontal coordinates $0, N_1, N_2, \ldots, N_d, 2n+1-N_d, \ldots, 2n+1-N_2, 2n+1-N_1, 2n+1$.

Since the definition of the modified Euler factor in [Coa91], formulated in terms of the Weil–Deligne representation, does not depend on the monodromy operator, our above description of the Weil–Deligne representation associated to π_p is enough for us to unfold the definition in this case to obtain the explicit modified Euler factor at p in terms of the $\mathbb{U}_p^P$-eigenvalues.

From now on we fix a (tame) finite order Dirichlet character $\phi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ unramified away from $N\infty$. Suppose $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ is of finite order. We also view it also as a $\mathbb{C}^\times$-valued character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ which sends the uniformizer in $\mathbb{Q}_v$ to $\chi(q_v)$ for finite places $v \neq p$. In the same way, we view the finite order characters $\epsilon_1^P, \ldots, \epsilon_d^P$ as adelic characters. Let $\underline{\theta}$ be as in Proposition 2.3.2. Denote by σ_i the unramified representation of $\mathrm{GL}(n, \mathbb{Q}_p)$ such that $\sigma_i \otimes \epsilon_{i,p}^P \circ \det = \mathrm{Ind}_{B_{n_i}(\mathbb{Q}_p)}^{\mathrm{GL}(n_i,\mathbb{Q}_p)} \left(\theta_{N_{i-1}+1}, \ldots, \theta_{N_i}\right)$. Denote by $\widetilde{\sigma}_i$ the contragredient of σ_i.

The modified Euler factor at p for p-adically interpolating the critical values of $L^S(s, \pi \times \phi\chi)$ to the left of the center is

$$E_p(s, \pi \times \phi\chi) = \prod_{i=1}^{d} \gamma_p\left(1 - s, \widetilde{\sigma}_i \otimes \phi_p^{-1}\chi_p^{-1}\epsilon_{i,p}^P\right). \tag{2.3.11}$$

Here we omit our fixed additive character $\mathbf{e}_p$ from the usual notation for gamma factors. One can also write the gamma factors in terms of the Satake parameters as

(2.3.12)
$$\gamma_p\left(1-s,\widetilde{\sigma}_i\otimes\phi_p^{-1}\chi_p^{-1}\epsilon_{i,p}^P\right)=\begin{cases}\prod\limits_{j=N_{i-1}+1}^{N_i}\frac{1-\phi_p(p)^{-1}\alpha_j^{-1}p^{s-1}}{1-\phi_p(p)\alpha_j p^{-s}}, & \text{if } \chi\epsilon_i^{P-1} \text{ is trivial,}\\ G(\chi_p\epsilon_{i,p}^{P-1})^{n_i}\prod\limits_{j=N_{i-1}+1}^{N_i}\left(\phi_p(p)^{-1}\alpha_j^{-1}p^{s-1}\right)^{c(\chi\epsilon_i^{P-1})}, & \text{otherwise,}\end{cases}$$

where $p^{c(\chi\epsilon_i^{P-1})}$ is the conductor of $\chi\epsilon_i^{P-1}$, and the Gauss sum is defined as

$$G(\chi_p\epsilon_{i,p}^{P-1})=\int_{p^{-m}\mathbb{Z}_p}\chi_p\epsilon_{i,p}^{P-1}(x)\mathbf{e}_p(x)\,dx,\quad m\gg 0.$$

We also define the improved modified Euler factor at p for the d-variable improved p-adic L-function. The improved p-adic L-function is supposed to interpolate the leftmost critical L-values with $\chi=\epsilon_d^P$. Define

$$E_p^{P\text{-imp}}(s,\pi\times\phi\epsilon_d^P)=\prod_{i=1}^{d-1}\gamma_p\left(1-s,\widetilde{\sigma}_i\otimes\phi_p^{-1}\epsilon_{d,p}^{P-1}\epsilon_{i,p}^P\right)\cdot L_p(s,\sigma_d\otimes\phi_p)$$

It is easy to see that by (2.3.9)(2.3.10) both the $E_p(s,\pi\times\phi\chi)$ and $E_p^{P\text{-imp}}(s,\pi\times\phi\epsilon_d^P)$ can be written in term of the $\mathbb{U}_p^P$-eigenvalues of the P-ordinary Siegel modular form contained in π.

We have

$$E_p(n+1-t_d^P,\pi\times\phi\epsilon_d^P)=\mathcal{A}^P(\pi\times\phi\epsilon_d^P)\cdot E_p^{P\text{-imp}}(n+1-t_d^P,\pi\times\phi\epsilon_d^P)\tag{2.3.13}$$

with

$$\begin{aligned}\mathcal{A}^P(\pi\times\phi\epsilon_d^P)&=\prod_{j=N_{d-1}+1}^{n}\left(1-\phi_p(p)^{-1}\alpha_j^{-1}p^{n-t_d^P}\right)\\&=1+\mathfrak{a}_n^{-1}\mathfrak{a}_{N_{d-1}}\left(-\phi_p(p)^{-1}p^{\frac{n_d-1}{2}}\right)^{n_d}+\mathfrak{a}_n^{-1}\sum_{r=1}^{n_d-1}\mathfrak{a}_{N_{d-1}+r}\left(-\phi_p(p)^{-1}p^{\frac{n_d-1-r}{2}}\right)^{n_d-r}.\end{aligned}$$

Since all the $\mathfrak{a}_i$'s are the $\mathbb{U}_p^P$-eigenvalues of the P-ordinary Siegel modular forms, when (the eigensystem of) π varies in a P-ordinary Hida family, $\mathcal{A}^P(\pi,\phi)$ becomes a d-variable p-adic analytic function lying inside a finite extension of $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)^\circ]\!]$. This explains why when restricting to the leftmost critical values with $\chi=\epsilon_d^P$, one expects the existence of the improved d-variable p-adic L-function with the improved modified Euler factor $E_p^{P\text{-imp}}(\pi,\phi)$ at p (improved in the sense of saving part of the numerator from $E_p(n+1-t_d^P,\pi\times\phi\epsilon_d^P)$).

2.4. **The choices of local sections for the Siegel Eisenstein series.** Our choices of local sections for Siegel Eisenstein series on Sp$(4n)$, and the formulae for local Fourier coefficients as well as the doubling local zeta integrals corresponding to those selected sections are summarized in the two tables in §2.4.8. This section explains the strategy for the section selections. The computation of the zeta integrals at the place p is done in §2.8.

2.4.1. *Criteria for selecting sections.* We first describe the context and criteria for choosing the sections for Siegel Eisenstein series on Sp$(4n)$. Recall that $\mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\right)$ is the weight space for Hida families which are ordinary for the parabolic for P. An arithmetic point in it corresponds to a character $\underline{\tau}^P=\underline{\tau}_{\mathrm{alg}}^P\cdot\underline{\tau}_{\mathrm{f}}^P\in\mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\right)$, a product of the algebraic part $\underline{\tau}^P_{\mathrm{alg}} = \underline{t}^P = (t^P_1, \dots, t^P_d)$ and the finite order part $\underline{\tau}^P_{\mathrm{f}} = \underline{\epsilon}^P = (\epsilon^P_1, \dots, \epsilon^P_d)$. Similarly, the parameterization space for the cyclotomic variable is $\mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times\right)$, and an arithmetic point in it is a character $\kappa = \kappa_{\mathrm{alg}} \cdot \kappa_{\mathrm{f}}$, with algebraic part $\kappa_{\mathrm{alg}} = k$ and finite order part $\kappa_{\mathrm{f}} = \chi$. We call an arithmetic point $\underline{\tau}^P$ (resp. $(\kappa, \underline{\tau}^P)$) admissible if $t^P_1 \geq \cdots \geq t^P_d \geq n+1$ and (resp. $t^P_1 \geq \cdots \geq t^P_d \geq k \geq n+1$).

Let $\mathscr{C}_P$ be a geometrically irreducible component of $\mathrm{Spec}\left(\mathbb{T}^{0,N}_{P\text{-ord}} \otimes_{\mathcal{O}_F} F\right)$. The projection of $\mathscr{C}_P$ to the weight space is one of its $|T_P(\mathbb{Z}/p)|$ connected components. We say the parity of $\mathscr{C}_P$ is compatible with ϕ if all the points $\underline{\tau}^P$ in that connected component satisfy $\tau^P_d(-1) = \phi(-1)$.

A point $x \in \mathscr{C}_P(\overline{\mathbb{Q}}_p)$ is called arithmetic if its projection $\underline{\tau}^P$ inside the weight space is arithmetic, and an arithmetic pair (κ, x) (resp. an arithmetic point x) is called admissible if $(\kappa, \underline{\tau}^P)$ (resp. $\underline{\tau}^P$) is admissible. If the Hecke eigensystem parametrized by x appears in an irreducible cuspidal automorphic representation $\pi_x \in \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ with $\pi_{x,\infty} \cong \mathcal{D}_{\imath(\underline{t}^P)}$, we call such an x classical, and one can define the corresponding $L^{Np\infty}(s, \pi_x \times \phi\chi)$, $E_p(s, \pi_x \times \phi\chi)$, $E_p^{P\text{-imp}}(\pi_x, \phi)$. Note that because of the lack of strong multiplicity one, π_x may not be unique, but the partial L-functions and the modified Euler factors at p do not depend on the choice of π_x.

The $(d+1)$-variable p-adic L-function is intended to interpolate the critical values

$$E_p(n+1-k, \pi_x \times \phi\chi) \cdot L^{Np\infty}(n+1-k, \pi_x \times \phi\chi)$$

divided by a Petersson inner product period for (κ, x) admissible with $\kappa(-1) = \phi(-1)$ and x classical (by our construction, if (κ, x) is admissible but x is not classical, one can see that the evaluation of our p-adic L-function there is 0). Its d-variable improvement (assuming the parity of $\mathscr{C}_P$ is compatible with ϕ) is supposed to interpolate

$$E_p^{P\text{-imp}}(n+1-t^P_d, \pi_x \times \phi\epsilon^P_d) \cdot L^{Np\infty}(n+1-t^P_d, \pi_x \times \phi\epsilon^P_d)$$

divided by a Petersson inner product period for classical x.

From Theorem 2.2.2, we see that in order to get the above L-values, we need to pick a Siegel Eisenstein series on $\mathrm{Sp}(4n)$ with nice properties for each admissible $(\kappa, \underline{\tau}^P) \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p, \overline{\mathbb{Q}}_p^\times))$ with $\kappa(-1) = \phi(-1)$ as well as for each admissible $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p, \overline{\mathbb{Q}}_p^\times))$ with $\tau^P_d(-1) = \phi(-1)$, so that we can deduce the desired congruences among the L-values from those of the Siegel Eisenstein series. Picking the Siegel Eisenstein series amounts to selecting sections in the degenerate principal series.

More precisely, for $(\kappa, \underline{\tau}^P)$ (resp. $\underline{\tau}^P$) as above and each place v of $\mathbb{Q}$, we need to pick a section $f_{\kappa, \underline{\tau}^P, v}$ (resp. $f_{\underline{\tau}^P, v}$) from $I_{Q_H, v}(\frac{2n+1}{2} - k, \phi\chi)$ (resp. $I_{Q_H, v}(\frac{2n+1}{2} - t^P_d, \phi\epsilon^P_d)$), such that

- We have enough control of the local zeta integrals at places dividing $Np\infty$. In particular, we are able guarantee the nonvanishing of the archimedean zeta integrals and compute the zeta integrals at p.
- The collection of the Eisenstein series $E(\cdot, f_{\kappa, \underline{\tau}^P})|_{G \times G}$ (resp. $E(\cdot, f_{\underline{\tau}^P})|_{G \times G}$) (after suitable normalizations) assembles to a p-adic family.

The way we treat the second requirement is via looking at their Fourier coefficients and invoking the q-expansion principle. Also, the second requirement provides us a hint for making the choices of the sections at p based on our selection of archimedean sections.

2.4.2. *The Fourier coefficients for Siegel Eisenstein series.* For $\boldsymbol{\beta} \in \mathrm{Sym}(2n, \mathbb{Q})$, the $\boldsymbol{\beta}$-th Fourier coefficient of $E\big(\cdot, f(s, \xi)\big)$ is defined as

$$E_{\boldsymbol{\beta}}(h, f(s, \xi)) := \int_{\mathrm{Sym}(2n, \mathbb{Q})\backslash \mathrm{Sym}(2n, \mathbb{A})} E\left(\begin{pmatrix} I_{2n} & \varsigma \\ 0 & I_{2n} \end{pmatrix} h, f(s, \xi)\right) \mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\boldsymbol{\beta}\varsigma)\, d\varsigma.$$

Suppose $f(s,\xi)=\otimes_v f_v(s,\xi)$ is factorizable. If $\det(\boldsymbol{\beta})\neq 0$ or there exists a finite place v such that $f_v(s,\xi)$ is supported on the "big cell" $Q_H(\mathbb{Q}_v)\begin{pmatrix}0 & -I_{2n}\\ I_{2n} & 0\end{pmatrix}Q_H(\mathbb{Q}_v)$, then

$$E_{\boldsymbol{\beta}}(h,f(s,\xi))=\prod_v W_{\boldsymbol{\beta},v}(h,f(s,\xi)) \tag{2.4.1}$$

with

$$W_{\boldsymbol{\beta},v}(h_v,f_v(s,\xi))=\int_{\mathrm{Sym}(2n,\mathbb{Q}_v)} f_v(s,\xi)\left(\begin{pmatrix}0 & -I_{2n}\\ I_{2n} & \varsigma\end{pmatrix}h_v\right)\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}\varsigma)\,d_v\varsigma.$$

For $\boldsymbol{z}=\boldsymbol{x}+\sqrt{-1}\boldsymbol{y}$, a point in the Siegel upper half space $\mathbb{H}_{2n}$, set $h_{\boldsymbol{z}}=1_{\mathrm{f}}\cdot\begin{pmatrix}\sqrt{\boldsymbol{y}} & \boldsymbol{x}\sqrt{\boldsymbol{y}}^{-1}\\ 0 & \sqrt{\boldsymbol{y}}^{-1}\end{pmatrix}\in H(\mathbb{A})$. It is a standard fact that if $E(h_{\boldsymbol{z}},f(s_0,\xi))$ is nearly holomorphic as a function in $\boldsymbol{z}$ for some $s_0\in 2^{-1}\cdot\mathbb{Z}$, then $E_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f(s,\xi))$ gives the $\boldsymbol{\beta}$-th coefficient of the q-expansion associated to $E(h,f(s_0,\xi))$ viewed as a p-adic form by the maps (1.2.1)(2.3.3).

2.4.3. *The unramified places.* For $v\nmid Np\infty$, we simply take

$$f_{\kappa,\underline{\tau}^P,v}=f_v^{\mathrm{ur}}(s,\phi\chi)\big|_{s=\frac{2n+1}{2}-k}, \qquad f_{\underline{\tau}^P,v}=f_v^{\mathrm{ur}}(s,\phi\epsilon_d^P)\big|_{s=\frac{2n+1}{2}-t_d^P}.$$

The formulae for $W_{\boldsymbol{\beta}}\left(1_v,f_{\kappa,\underline{\tau}^P,v}\right)$ and $W_{\boldsymbol{\beta}}\left(1_v,f_{\underline{\tau}^P,v}\right)$ are computed by Shimura [Shi97, Theorem 13.6, Proposition 14.9] and are listed in the tables in §2.4.8. The formulae for the local zeta integrals are part of Theorem 2.2.2.

2.4.4. *The archimedean place.* For an integer $k\geq n+1$ satisfying $\xi(-1)=(-1)^k$, the classical section of weight k in $I_{Q_H,\infty}(s,\xi)$ is defined as

$$f_\infty^k(s,\mathrm{sgn}^k)(h)=j(h,i)^{-k}|j(h,i)|^{k-(s+\frac{2n+1}{2})}$$

where $j(h,i)=\det\left(\mu(h,i)\right)=\det(Ci+D)$ for $h=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$.

Let $\widehat{\mu}_0^+=\left(\widehat{\mu}_{0,ij}^+\right)_{1\leq i,j\leq n}$, where the entries are elements inside $(\mathrm{Lie}H)_{\mathbb{C}}$ given as

$$\widehat{\mu}_{0,ij}^+=\begin{pmatrix}I_{2n} & \sqrt{-1}\cdot I_{2n}\\ \sqrt{-1}\cdot I_{2n} & I_{2n}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & E_{ij}\\ 0 & 0 & E_{ji} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}I_{2n} & \sqrt{-1}\cdot I_{2n}\\ \sqrt{-1}\cdot I_{2n} & I_{2n}\end{pmatrix}^{-1},$$

where E_{ij} is the $n\times n$ matrix with 1 as the (i,j)-entry and 0 elsewhere.

The $\widehat{\mu}_{0,ij}^+$'s act on $\mathcal{A}(H(\mathbb{Q})\backslash H(\mathbb{A}))$ by differentiating the right translation of $H(\mathbb{R})$. Their realizations on the Siegel upper half space are the Maass–Shimura differential operators (see [Liu16b, §2.4]).

For admissible $(\kappa,\underline{\tau}^P)$ (resp. $\underline{\tau}^P$) with $\phi\chi(-1)=(-1)^k$ (resp. $\phi\epsilon_d^P(-1)=(-1)^{t_d^P}$), set

$$f_{\kappa,\underline{\tau}^P,\infty}=\left(\prod_{i=1}^{d-1}\det\left(\frac{(\widehat{\mu}_0^+)_{N_i}}{4\pi\sqrt{-1}}\right)^{t_i^P-t_{i+1}^P}\det\left(\frac{\widehat{\mu}_0^+}{4\pi\sqrt{-1}}\right)^{t_d^P-k}\cdot f_\infty^k(s,\mathrm{sgn}^k)\right)\Bigg|_{s=\frac{2n+1}{2}-k},$$

$$f_{\underline{\tau}^P,\infty}=\left(\prod_{i=1}^{d-1}\det\left(\frac{(\widehat{\mu}_0^+)_{N_i}}{4\pi\sqrt{-1}}\right)^{t_i^P-t_{i+1}^P}\cdot f_\infty^{t_d^P}(s,\mathrm{sgn}^{t_d^P})\right)\Bigg|_{s=\frac{2n+1}{2}-t_d^P}.$$

The formulae for the corresponding Fourier coefficients (listed in tables in §2.4.8) are deduced from Shimura's computation [Shi82, Theorem 4.2] for the classical scalar weight section and formulae for the action of differential operators on p-adic expansions (see the proof of [Liu16b, Proposition

4.4.1]). The proof of the nonvanishing of the corresponding archimedean zeta integrals is postponed to §2.7.

2.4.5. *The "big cell" section at a finite place.* We choose our sections at $v|Np$ from a special type of sections, the so-called "big cell" sections. Given a finite place v and a compactly supported locally constant function α_v on $\mathrm{Sym}(2n,\mathbb{Q}_v)$, the "big cell" section inside $I_{Q_H,v}(s,\xi)$ associated to α_v is defined as

$$f_v^{\alpha_v}(s,\xi)\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{cases} \xi^{-1}(\det C)|\det C|^{-(s+\frac{2n+1}{2})}\alpha_v(C^{-1}D) & \text{if } \det C \neq 0, \\ 0 & \text{if } \det C = 0. \end{cases} \tag{2.4.2}$$

An easy computation shows that

$$W_{\boldsymbol{\beta},v}(1_v, f_v^{\alpha_v}(s,\xi)) = \int_{\mathrm{Sym}(2n,\mathbb{Q}_v)} \alpha_v(\varsigma)\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}\varsigma)\,d_v\varsigma = \widehat{\alpha}_v(\boldsymbol{\beta}). \tag{2.4.3}$$

2.4.6. *The volume sections at places dividing N.* For a positive integer N and a place $v|N$, the volume section $f_v^{\mathrm{vol}}(s,\xi)$ inside $I_{Q_H,v}(s,\xi)$ is defined as the "big cell" section associated to the characteristic function of the open compact subset

$$-\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + N\,\mathrm{Sym}(2n,\mathbb{Z}_v) \subset \mathrm{Sym}(2n,\mathbb{Q}_v). \tag{2.4.4}$$

We set

$$f_{\kappa,\underline{\tau}^P,v} = f_v^{\mathrm{vol}}(s,\phi\chi)\Big|_{s=\frac{2n+1}{2}-k}, \qquad f_{\underline{\tau}^P,v} = f_v^{\mathrm{vol}}(s,\phi\epsilon_d^P)\Big|_{s=\frac{2n+1}{2}-t_d^P}.$$

The Fourier coefficients associated to the volume sections are easily computed by computing the Fourier transform of the characteristic function of (2.4.4). The computation of local zeta integrals is also easy (the same as [Liu16b, Proposition 4.2.1]). See the tables for the formulae.

2.4.7. *The place p.* It remains to pick Schwartz functions $\alpha_{\kappa,\underline{\tau}^P}$ and $\alpha_{\underline{\tau}^P}$ on $\mathrm{Sym}(2n,\mathbb{Q}_p)$, and our choices for $f_{\kappa,\underline{\tau}^P,p}$ (resp. $f_{\underline{\tau}^P,p}$) will the "big cell" section attached to $\alpha_{\kappa,\underline{\tau}^P}$ (resp. $\alpha_{\underline{\tau}^P}$). The criterion for picking them is to make the (p-adic) q-expansions of the resulting Siegel Eisenstein series p-adically interpolable. In fact we will first pick $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ and $\widehat{\alpha}_{\underline{\tau}^P}$, and then apply inverse Fourier transform to get $\alpha_{\kappa,\underline{\tau}^P}$ and $\alpha_{\underline{\tau}^P}$ for computing the local zeta integrals.

The theory of nearly holomorphic forms and Maass–Shimura differential operators formulated in terms of automorphic sheaves and their interpretations as p-adic Siegel modular forms are needed in our construction. We will freely use the formulation and notation in [Liu16a, §2] and [Liu16b, §2].

Recall some notation *loc. cit.*; denote by $\mathcal{V}_{\underline{t}}^r$ the automorphic bundle of degree r and weight $\underline{t}$ nearly holomorphic forms over the Siegel variety defined as in [Liu16b, §2.2], and by $\mathrm{N}_{\underline{t}}^r\left(\mathbb{H}_n,\Gamma\cap\Gamma_{SP}(p^l)\right)$ the space of vector-valued nearly holomorphic Siegel modular forms on the Siegel upper half space $\mathbb{H}_n$ of degree r, weight $\underline{t}$ and level $\Gamma\cap\Gamma_{SP}(p^l)$ in the sense of Shimura. There is the embedding

(2.4.5)

$$\begin{aligned}\mathrm{H}^0\left(X^{\Sigma}_{G,\Gamma\cap\Gamma_{SP}(p^l)}\times X^{\Sigma}_{G,\Gamma\cap\Gamma_{SP}(p^l)},\mathcal{V}_{\underline{t}}^r\boxtimes\mathcal{V}_{\underline{t}}^r\right) &\longhookrightarrow \mathrm{N}_{\underline{t}}^r\left(\mathbb{H}_n,\Gamma\cap\Gamma_{SP}(p^l)\right)\otimes\mathrm{N}_{\underline{t}}^r\left(\mathbb{H}_n,\Gamma\cap\Gamma_{SP}(p^l)\right)\\ &\xhookrightarrow{\varphi(\cdot,\mathfrak{e}_{\mathrm{can}})} \mathcal{A}(G(\mathbb{Q})\times G(\mathbb{Q})\backslash G(\mathbb{A})\times G(\mathbb{A})).\end{aligned}$$

Generalizing the embedding (1.2.1), as explained in [Liu16a, Proposition 3.2.1], the space of nearly holomorphic Siegel modular forms of level $\Gamma\cap\Gamma_{SP}(p^l)$ also embeds into the space of p-adic

Siegel modular forms,

$$\mathrm{H}^0\left(X^{\Sigma}_{G,\Gamma\cap\Gamma_{SP}(p^l)}\times X^{\Sigma}_{G,\Gamma\cap\Gamma_{SP}(p^l)},\mathcal{V}^r_{\imath(\underline{t}^P)}\boxtimes\mathcal{V}^r_{\imath(\underline{t}^P)}\right)\hookrightarrow\left(\varprojlim_m\varinjlim_l V^{SP}_{m,l}\otimes_{\mathcal{O}_F}V^{SP}_{m,l}\right)[1/p]. \tag{2.4.6}$$

When choosing $\alpha_{\kappa,\underline{\tau}^P}$ and $\alpha_{\underline{\tau}^P}$, we want to ensure that the restrictions to $G\times G$ of the resulting adelic Siegel Eisenstein series lie inside the image of the embedding (2.4.5), so that we can study them as p-adic Siegel modular forms via (2.4.6). Hence we require:

(1) $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ and $\widehat{\alpha}_{\underline{\tau}^P}$ take values in a finite extension of $\mathbb{Q}$.
(2) $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ and $\widehat{\alpha}_{\underline{\tau}^P}$ are supported on $\mathrm{Sym}(2n,\mathbb{Z}_p)^*$, and for $x\in\mathrm{Sym}(2n,\mathbb{Q}_p)^*$, $a_1,a_2\in SP(\mathbb{Z}_p)$,

$$\widehat{\alpha}_{\kappa,\underline{\tau}^P}(x)=\widehat{\alpha}_{\kappa,\underline{\tau}^P}\left(\left(\begin{smallmatrix}{}^{\mathrm{t}}a_1&0\\0&{}^{\mathrm{t}}a_2\end{smallmatrix}\right)x\left(\begin{smallmatrix}a_1&0\\0&a_2\end{smallmatrix}\right)\right),\qquad \widehat{\alpha}_{\underline{\tau}^P}(x)=\widehat{\alpha}_{\underline{\tau}^P}\left(\left(\begin{smallmatrix}{}^{\mathrm{t}}a_1&0\\0&{}^{\mathrm{t}}a_2\end{smallmatrix}\right)x\left(\begin{smallmatrix}a_1&0\\0&a_2\end{smallmatrix}\right)\right).$$

With $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ and $\widehat{\alpha}_{\underline{\tau}^P}$ satisfying these conditions, we can define

$$\mathcal{E}_{\kappa,\underline{\tau}^P},\,\mathcal{E}_{\underline{\tau}^P}\in\varinjlim_r\varinjlim_l\mathrm{H}^0\left(X^{\Sigma}_{G,\Gamma\cap\Gamma_{SP}(p^l)}\times X^{\Sigma}_{G,\Gamma\cap\Gamma_{SP}(p^l)},\mathcal{V}^r_{\imath(\underline{t}^P)}\boxtimes\mathcal{V}^r_{\imath(\underline{t}^P)}\right),$$

as the preimage of the adelic forms

$$(-1)^{nk}2^{-n+2n^2-2nk}\pi^{-n-2n^2}\Gamma_{2n}\left(\frac{2n+1}{2}\right)\cdot E^*\left(\cdot,f_{\kappa,\underline{\tau}^P}\right)\big|_{G\times G}, \tag{2.4.7}$$

$$(-1)^{nt^P_d}2^{-n+2n^2-2nt^P_d}\pi^{-n-2n^2}\Gamma_{2n}\left(\frac{2n+1}{2}\right)\cdot E^*\left(\cdot,f_{\underline{\tau}^P}\right)\big|_{G\times G}. \tag{2.4.8}$$

Here for a positive integer m,

$$\Gamma_m(s):=\pi^{\frac{m(m-1)}{4}}\prod_{j=0}^{m-1}\Gamma\left(s-\frac{j}{2}\right).$$

In the following we will not distinguish $\mathcal{E}_{\kappa,\underline{\tau}^P}$, $\mathcal{E}_{\underline{\tau}^P}$ from their images under the embedding (2.4.6).

Set

$$\varepsilon_{q\text{-exp}}=(\varprojlim_m\varinjlim_l\varepsilon^{1,1}_{q\text{-exp},m,l},\varprojlim_m\varinjlim_l\varepsilon^{1,1}_{q\text{-exp},m,l}):\varprojlim_m\varinjlim_l V^{SP}_{m,l}\otimes_{\mathcal{O}_F}V^{SP}_{m,l}\longrightarrow\mathcal{O}_F[\![N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{\geq 0}]\!]$$

with

$$\varprojlim_m\varinjlim_l\varepsilon^{1,1}_{q\text{-exp},m,l}:\varprojlim_m\varinjlim_l V_{m,l}\longrightarrow\mathcal{O}_F[\![N^{-1}S^2(X_n)_{\geq 0}]\!]=\mathcal{O}_F[\![N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]\!].$$

being the p-adic q-expansion map at infinity.

Regarding the q-expansion of $\mathcal{E}_{\kappa,\underline{\tau}^P}$ and $\mathcal{E}_{\underline{\tau}^P}$, we have the following proposition.

Proposition 2.4.1. *Let (β_1,β_2) be a pair of elements in $N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}$. For admissible $(\kappa,\underline{\tau}^P)\in\mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times\times T_P(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\right)$ with $\phi\chi(-1)=(-1)^k$, we have*

$$\varepsilon_{q\text{-exp}}\left(\beta_1,\beta_2,\mathcal{E}_{\kappa,\underline{\tau}^P}\right)=\sum_{\boldsymbol{\beta}=\left(\begin{smallmatrix}\beta_1&\beta_0\\{}^{\mathrm{t}}\beta_0&\beta_2\end{smallmatrix}\right)\in N^{-1}\,\mathrm{Sym}(2n,\mathbb{Z})^*_{\geq 0}}\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta}),$$

where for $\boldsymbol{\beta} > 0$, the coefficient $\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})$ is given as

$$\begin{aligned}\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta}) =& N^{-n(2n+1)}\prod_{v|N}\mathbf{e}_v(2\mathrm{Tr}\beta_0)\cdot L^{Np\infty}(n+1-k,\phi\chi\lambda_{\boldsymbol{\beta}})\cdot\prod_{\substack{v|\det(2\boldsymbol{\beta})\\ v\nmid Np\infty}} g_{\boldsymbol{\beta},v}\left(\phi\chi(q_v)q_v^{k-2n-1}\right)\\ &\times\ \widehat{\alpha}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}\det(2\beta_0)^{t_d^P-k}.\end{aligned}$$

For admissible $\underline{\tau}\in\mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ with $\phi\epsilon_d^P(-1)=(-1)^{t_d^P}$, we have

$$\varepsilon_{q\text{-exp}}\left(\beta_1,\beta_2,\mathcal{E}_{\underline{\tau}^P}\right)=\sum_{\boldsymbol{\beta}=\left(\begin{smallmatrix}\beta_1&\beta_0\\ {}^{\mathrm{t}}\beta_0&\beta_2\end{smallmatrix}\right)\in N^{-1}\,\mathrm{Sym}(2n,\mathbb{Z})^*_{\geq 0}}\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta}),$$

where for $\boldsymbol{\beta}\geq 0$, the coefficient $\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta})$ is given as

$$\begin{aligned}\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta}) =& N^{-n(2n+1)}\prod_{v|N}\mathbf{e}_v(2\mathrm{Tr}\beta_0)\cdot L^{Np\infty}(n+1-t_d^P-\frac{r}{2},\phi\chi\lambda_{\boldsymbol{\beta}})\cdot\prod_{j=1}^{\frac{\mathrm{corank}(\boldsymbol{\beta})}{2}} L^{NP\infty}(2n+3-2t_d^P-2j,(\phi\epsilon_d^P)^2)\\ &\times\prod_{\substack{v|\det^*(2\boldsymbol{\beta})\\ v\nmid Np\infty}} g_{\boldsymbol{\beta},v}\left(\phi\chi(q_v)q_v^{t_d^P-2n-1}\right)\cdot\widehat{\alpha}_{\underline{\tau}^P}(\boldsymbol{\beta})\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}, \qquad \textit{if } \mathrm{rank}(\boldsymbol{\beta}) \textit{ is even,}\end{aligned}$$

$$\begin{aligned}\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta}) =& N^{-n(2n+1)}\prod_{v|N}\mathbf{e}_v(2\mathrm{Tr}\beta_0)\cdot\prod_{j=1}^{\frac{\mathrm{corank}(\boldsymbol{\beta})+1}{2}} L^{NP\infty}(2n+3-2t_d^P-2j,(\phi\epsilon_d^P)^2)\\ &\times\prod_{\substack{v|\det^*(2\boldsymbol{\beta})\\ v\nmid Np\infty}} g_{\boldsymbol{\beta},v}\left(\phi\chi(q_v)q_v^{t_d^P-2n-1}\right)\cdot\widehat{\alpha}_{\underline{\tau}^P}(\boldsymbol{\beta})\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}, \qquad \textit{if } \mathrm{rank}(\boldsymbol{\beta}) \textit{ is odd,}\end{aligned}$$

Here for $\boldsymbol{\beta}\in\mathrm{Sym}(2n,\mathbb{Q})\cap\mathrm{Sym}(2n,\mathbb{Z}_v)^*$, $\det^*(2\boldsymbol{\beta})$ denotes the product of all the nonzero eigenvalues of $2\boldsymbol{\beta}$. If $\mathrm{rank}(\boldsymbol{\beta})$ is even, the quadratic character $\lambda_{\boldsymbol{\beta}}$ is defined as $\lambda_{\boldsymbol{\beta}}(q_v)=\left(\frac{(-1)^{\mathrm{rank}(\boldsymbol{\beta})/2}\det^*(\boldsymbol{\beta})}{q_v}\right)$. The $g_{\boldsymbol{\beta},v}(\cdot)$ appearing in above formulae is a polynomial with coefficients in $\mathbb{Z}$. For an integer m, $(2\beta_0)_m$ denotes the upper left $m\times m$-minor of $2\beta_0$.

Proof. The proof is similar as [Liu16b, Proposition 4.4.1]. It relies on the formulae for local Fourier coefficients as listed in the two tables in §2.4.8, and uses formulae of differential operators on p-adic q-expansions. ☐

It is not difficult to observe that all the terms in the above formulae for $\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})$, $\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta})$ are ready for p-adic interpolation with respect to $(\kappa,\underline{\tau}^P)$, $\underline{\tau}^P$, except the last terms

$$\widehat{\alpha}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}\det(2\beta_0)^{t_d^P-k}, \tag{2.4.9}$$

$$\widehat{\alpha}_{\underline{\tau}^P}(\boldsymbol{\beta})\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}. \tag{2.4.10}$$

(The p-adic interpolation of the Dirichlet L-values in the formulae follows from the existence of the Kubota–Leopoldt p-adic L-function [Hid93, Theorem 4.4.1].)

In order to make (2.4.9) and (2.4.10) p-adically interpolable, one needs to require that if $\boldsymbol{\beta}$ belongs to the support of the Schwartz function $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ (resp. $\widehat{\alpha}_{\underline{\tau}^P}$), then $\det\left((2\beta_0)_{N_i}\right)$ is a p-adic unit for $1 \leq i \leq d$ (resp. $1 \leq i \leq d-1$). The very natural choices of $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ and $\widehat{\alpha}_{\underline{\tau}^P}$ are

$$\begin{aligned}\widehat{\alpha}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta}) =&\mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\prod_{i=1}^{d}\mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}\left((2\beta_0)_{N_i}\right)\\ &\times\prod_{i=1}^{d-1}\epsilon_i^P\epsilon_{i+1}^{P-1}(\det((2\beta_0)_{N_i}))\cdot\epsilon_d^P\chi^{-1}(\det(2\beta_0)),\end{aligned}\tag{2.4.11}$$

and

$$\begin{aligned}\widehat{\alpha}_{\underline{\tau}^P}(\boldsymbol{\beta}) =&\mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\mathbb{1}_{M_n(\mathbb{Z}_p)}(\beta_0)\prod_{i=1}^{d-1}\mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}\left((2\beta_0)_{N_i}\right)\\ &\times\prod_{i=1}^{d-1}\epsilon_i^P\epsilon_{i+1}^{P-1}(\det((2\beta_0)_{N_i})).\end{aligned}\tag{2.4.12}$$

Here M_n denotes the space of $n \times n$ matrices.

Since $W_{\boldsymbol{\beta},p}(1_p, f_{\kappa,\underline{\tau}^P,p}) = \widehat{\alpha}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})$, our choice (2.4.11) makes $\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})$ vanish unless $\boldsymbol{\beta}$ is invertible. The semi-positivity of $\boldsymbol{\beta}$ implies that both β_1 and β_2 are positive definite. Thus $\varepsilon_{q\text{-exp}}\left(\beta_1,\beta_2,\mathcal{E}_{\kappa,\underline{\tau}^P}\right)$ is nonzero only if $\beta_1, \beta_2 > 0$. Similarly, $\varepsilon_{q\text{-exp}}\left(\beta_1,\beta_2,\mathcal{E}_{\underline{\tau}^P}\right)$ is nonzero only if $\beta_1, \beta_2 \geq 0$ and their ranks are at least $n - n_d$.

With $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$, $\widehat{\alpha}_{\underline{\tau}^P}$ being set as in (2.4.11), (2.4.12), we have

$$\begin{aligned}\varepsilon_{q\text{-exp}}\left(\mathcal{E}_{\kappa,\underline{\tau}^P}\right) &= \sum_{\beta_1,\beta_2\in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}}\ \sum_{\boldsymbol{\beta}=\begin{pmatrix}\beta_1&\beta_0\\ {}^{\mathrm{t}}\beta_0&\beta_2\end{pmatrix}\in N^{-1}\,\mathrm{Sym}(2n,\mathbb{Z})^*_{>0}}\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})\,q^{\beta_1}q^{\beta_2},\\ \varepsilon_{q\text{-exp}}\left(\mathcal{E}_{\underline{\tau}^P}\right) &= \sum_{\substack{\beta_1,\beta_2\in\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}\\ \mathrm{rk}(\beta_1),\mathrm{rk}(\beta_2)\geq n-n_d}}\ \sum_{\boldsymbol{\beta}=\begin{pmatrix}\beta_1&\beta_0\\ {}^{\mathrm{t}}\beta_0&\beta_2\end{pmatrix}\in N^{-1}\,\mathrm{Sym}(2n,\mathbb{Z})^*_{\geq 0}}\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta})\,q^{\beta_1}q^{\beta_2},\end{aligned}$$

and each $\mathfrak{c}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})$ (resp. $\mathfrak{c}_{\underline{\tau}^P}(\boldsymbol{\beta})$) appearing here admits p-adic interpolation with respect to $(\kappa, \underline{\tau}^P)$ (resp. $\underline{\tau}^P$).

If we look at the q-expansions of $\mathcal{E}_{\kappa,\underline{\tau}^P}$ and $\mathcal{E}_{\underline{\tau}^P}$ at other cusps in the ordinary locus (which is equivalent to look at $W_{\boldsymbol{\beta}}\left(\iota(g_1,g_2), f_{\kappa,\underline{\tau}^P}\right)$, $W_{\boldsymbol{\beta}}\left(\iota(g_1,g_2), f_{\underline{\tau}^P}\right)$ for $g_i \in G(\mathbb{A})$ with $g_{i,p} = 1$, $g_{i,\infty} = g_{z_i}$, $i = 1, 2$), the support of $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ (resp. $\widehat{\alpha}_{\underline{\tau}^P}$) again makes the term indexed by degenerate (β_1, β_2) (resp. β_1 or β_2 of rank $< n - n_d$) vanish. Hence

$$\mathcal{E}_{\kappa,\underline{\tau}^P} \in \varprojlim_m \varinjlim_l V^{SP,0}_{m,l} \otimes_{\mathcal{O}_F} V^{SP,0}_{m,l}, \qquad \mathcal{E}_{\underline{\tau}^P} \in \varprojlim_m \varinjlim_l V^{SP,n_d}_{m,l} \otimes_{\mathcal{O}_F} V^{SP,n_d}_{m,l}.\tag{2.4.13}$$

Remark 2.4.2. Compared to $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$, the support of $\widehat{\alpha}_{\underline{\tau}^P}$ is enlarged. As the cyclotomic variable κ is fixed to be equal to τ_d^P, the term $\det(2\beta_0)$ does not appear in (2.4.10), and one does not need to require $2\beta_0 \in \mathrm{GL}(n, \mathbb{Z}_p)$ for the support of $\widehat{\alpha}_{\underline{\tau}^P}(\boldsymbol{\beta})$. Later, we will see that it is this relaxation on the support that saves us the factor $\mathcal{A}^P(\pi, \phi\epsilon_d^P)$ in the local zeta integral for $f_p^{\alpha_{\underline{\tau}^P}}(\frac{2n+1}{2} - t_d^P, \phi\epsilon_d^P)$ compared to that for $f_p^{\alpha_{\kappa,\underline{\tau}^P}}(\frac{2n+1}{2} - k, \phi\chi)\Big|_{\kappa=\tau_d^P}$. This relaxation also means that the resulting $\mathcal{E}_{\underline{\tau}^P}$ is not necessarily cuspidal as p-adic forms. Thus, the Hida theory for non-cuspidal Siegel modular forms developed in §1 is needed to construct the improved p-adic L-function from the $\mathcal{E}_{\underline{\tau}^P}$'s.

2.4.8. *The two tables.* In the tables on the next two pages, we summarize our choices of sections for Siegel Eisenstein series, the formulae of the corresponding local Fourier coefficients, and the local zeta integrals.

We explain some notation. In the tables, $\varphi \in \pi$ is a P-ordinary cuspidal holomorphic Siegel modular form of weight $\imath(\underline{t}^P)$ fixed by $\widehat{\Gamma} \cap SP_G(\mathbb{Z}_p)$ with p-nebentypus $\underline{\epsilon}^P$.

The operator $\mathcal{W}$ is defined as

$$\mathcal{W}(\varphi)(g) = \int_{SP_G(\mathbb{Z}_p)} \overline{\varphi}^{\vartheta}(gu)\, du. \tag{2.4.14}$$

As explained in §2.2, the form $\mathcal{W}(\varphi)$ belongs to π. If φ is holomorphic of weight $\underline{t}^P$, then so is $\mathcal{W}(\varphi)$. The operator $\mathcal{W}$ should be viewed as an analogue of the operator sending a modular form f of level $\Gamma_0(N)$ to $f^c \Big| \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. In the same way as [Liu16b, Proposition 5.7.2], one can show that the P-ordinary projection $e_P\mathcal{W}(\varphi)$ is nonzero if $e_P\varphi$ is nonzero.

We denote by $v_{\imath(\underline{t}^P)} \in \mathcal{D}_{\imath(\underline{t}^P)}$ the highest weight vector inside the lowest $K_{G,\infty}$-type, and $v^{\vee}_{\imath(\underline{t}^P)} \in \widetilde{\mathcal{D}}_{\imath(\underline{t}^P)}$ is the dual vector to $v_{\imath(\underline{t}^P)}$.

	$f_{\kappa,\underline{\tau}^P,v}$	$W_{\boldsymbol{\beta},v}\left(h_{\boldsymbol{z},v}, f_{\kappa,\underline{\tau}^P,v}\right)$ with $\boldsymbol{\beta} = \left(\begin{smallmatrix}\beta_1 & \beta_0 \\ \beta_0 & \beta_2\end{smallmatrix}\right)$	$\left(T_{f_{\kappa,\underline{\tau}^P,v}}\overline{\varphi}\right)^{\vartheta}$
$v \nmid Np\infty$	the standard unramified section $f_v^{\mathrm{ur}}(s,\phi\chi)\big\vert_{s=\frac{2n+1}{2}-k}$	$\mathbb{1}_{\mathrm{Sym}(2n,\mathbb{Z}_v)^*}(\boldsymbol{\beta})$ $\times\, d_v(n+1-k,\phi\chi)^{-1}L_v(n+1-k,\phi\chi\lambda_{\boldsymbol{\beta}})$ $\times\, g_{\boldsymbol{\beta},v}\left(\phi\chi(q_v)q_v^{k-2n-1}\right)$ for $\det(\boldsymbol{\beta}) \neq 0$, where $g_{\boldsymbol{\beta},v}(T) \in \mathbb{Z}[T]$ with $g_{\boldsymbol{\beta},v}(0) = 1$ and degree at most $4n \cdot \mathrm{val}_v(\det(2\boldsymbol{\beta}))$	$d_v(n+1-k,\phi\chi)^{-1}$ $\cdot L_v(n+1-k,\pi\times\phi\chi)\cdot\mathcal{W}(\varphi)$
$v\vert N$	the "big cell" section $f_v^{\mathrm{vol}}(s,\phi\chi)\Big\vert_{s=\frac{2n+1}{2}-k}$ associated to the characteristic function of $-\left(\begin{smallmatrix}0 & I_n \\ I_n & 0\end{smallmatrix}\right) + N\mathrm{Sym}(2n,\mathbb{Z}_v)$	$\vert N\vert_v^{n(2n+1)}\mathbf{e}_v(2\mathrm{Tr}\beta_0)\cdot\mathbb{1}_{N^{-1}\mathrm{Sym}(2n,\mathbb{Z}_v)^*}(\boldsymbol{\beta})$	$\phi_v\chi_v((-1)^n)\mathrm{vol}(\Gamma(N)_v)\cdot\mathcal{W}(\varphi)$
$v = p$	the "big cell" section $f_p^{\alpha_{\kappa,\underline{\tau}^P}}(s,\phi\chi)\Big\vert_{s=\frac{2n+1}{2}-k}$, where $\alpha_{\kappa,\underline{\tau}^P}$ is the inverse Fourier transform of the Schwartz function in the next column	$\widehat{\alpha}_{\kappa,\underline{\tau}^P}(\boldsymbol{\beta})$ $=\mathbb{1}_{p^2\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)$ $\times\prod_{i=1}^{d}\mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}((2\beta_0)_{N_i})$ $\times\prod_{i=1}^{d-1}\epsilon_i^P\epsilon_{i+1}^{P-1}(\det((2\beta_0)_{N_i}))$ $\times\,\epsilon_d^P\chi^{-1}(\det(2\beta_0))$	$\chi(-1)^n\dfrac{\prod_{i=1}^{d}\prod_{j=1}^{n_i}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-2j})}$ $\cdot\, E_p(n+1-k,\pi\times\phi\chi)\cdot e_P\mathcal{W}(\varphi)$ (after $f_{\kappa,\underline{\tau}^P,p}$ being modified by appropriate $\mathbb{U}_p^P$-operators in accordance with ordinary projection applied to $E^*(\cdot,f_{\kappa,\underline{\tau}^P})\big\vert_{G\times G}$)
$v = \infty$	$\left(\prod_{i=1}^{d-1}\det\left(\frac{(\widehat{\mu}_0^+)_{N_i}}{4\pi\sqrt{-1}}\right)^{t_i^P-t_{i+1}^P}\cdot\det\left(\frac{\widehat{\mu}_0^+}{4\pi\sqrt{-1}}\right)^{t_d^P-k}\cdot f_\infty^k(s,\mathrm{sgn}^k)\right)\Bigg\vert_{s=\frac{2n+1}{2}-k}$	vanishing unless $\boldsymbol{\beta}\geq 0$ and equals $(-1)^{nk}\dfrac{2^{n-2n^2+2nk}}{\Gamma_{2n}\left(\frac{2n+1}{2}\right)}\pi^{n+2n^2}$ $\times\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}\det(2\beta_0)^{t_d^P-k}$ $\times\prod_{i=1}^{d-1}\det((y_0)_{N_i})^{t_i^P-t_{i+1}^P}\det(y_0)^{t_d^P-k}$ $\times\det(\mathbf{y})^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\boldsymbol{\beta}\boldsymbol{z})$ $+$ terms irrelevant to $\varepsilon_{q\text{-exp}}(\mathcal{E}_{\kappa,\underline{\tau}^P})$ where $\mathbf{y} = \mathrm{Im}(\boldsymbol{z}) = \left(\begin{smallmatrix}y_1 & y_0 \\ {}^{t}y_0 & y_2\end{smallmatrix}\right)$	$\dfrac{Z_\infty(f_{\underline{\tau}^P,\infty},v_{\underline{t}^P}^\vee,v_{\underline{t}^P})}{\left\langle v_{\underline{t}^P}^\vee,v_{\underline{t}^P}\right\rangle}\cdot\mathcal{W}(\varphi)\neq 0$

TABLE 1. data for $(d+1)$-variable p-adic L-function

	$f_{\underline{\tau}^P,v}$	$W_{\boldsymbol{\beta},v}\left(h_{\boldsymbol{z}}, f_{\underline{\tau}^P,v}\right)$ with $\boldsymbol{\beta}=\left(\begin{smallmatrix}\beta_1 & \beta_0\\ \beta_0 & \beta_2\end{smallmatrix}\right)$	$\left(T_{f_{\underline{\tau}^P,v}}\overline{\varphi}\right)^{\vartheta}$
$v \nmid Np\infty$	the standard unramified section $f_v^{\mathrm{ur}}(s,\phi\epsilon_d^P)\big\vert_{s=\frac{2n+1}{2}-t_d^P}$	$\mathbb{1}_{\mathrm{Sym}(2n,\mathbb{Z}_v)^*}(\boldsymbol{\beta})\cdot d_v(n+1-k,\phi\epsilon_d^P)^{-1}$ $\times\begin{cases} L_v(n+1-t_d^P-\frac{r}{2},\phi\epsilon_d^P\lambda_{\boldsymbol{\beta}}) & r \text{ even,}\\ \times\prod_{j=1}^{r/2} L_v(2n+3-2t_d^P-2j,(\phi\epsilon_d^P)^2) & \\ \prod_{j=1}^{\frac{r+1}{2}} L_v(2n+3-2t_d^P-2j,(\phi\epsilon_d^P)^2) & r \text{ odd,}\end{cases}$ $\times g_{\boldsymbol{\beta},v}\left(\phi\chi(q_v)q_v^{k-2n-1}\right)$ for $\boldsymbol{\beta}\geq 0$ with rank $2n-r$, where $g_{\boldsymbol{\beta},v}(T)\in\mathbb{Z}[T]$ with $g_{\boldsymbol{\beta},v}(0)=1$ and degree at most $4n\cdot\mathrm{val}_v(\det^*(2\boldsymbol{\beta}))$	$d_v(n+1-k,\phi\epsilon_d^P)^{-1}$ $\cdot L_v(n+1-t_d^P,\pi\times\phi\epsilon_d^P)\cdot\mathcal{W}(\varphi)$
$v\vert N$	the "big cell" section $f_v^{\mathrm{vol}}(s,\phi\epsilon_d^P)\big\vert_{s=\frac{2n+1}{2}-t_d^P}$ associated to the characteristic function of $-\left(\begin{smallmatrix}0 & I_n\\ I_n & 0\end{smallmatrix}\right)+N\mathrm{Sym}(2n,\mathbb{Z}_v)$	$\vert N\vert_v^{n(2n+1)}\mathbf{e}_v(2\mathrm{Tr}\beta_0)\cdot\mathbb{1}_{N^{-1}\mathrm{Sym}(2n,\mathbb{Z}_v)^*}(\boldsymbol{\beta})$	$\phi_v(-1)^n\mathrm{vol}(\Gamma(N)_v)\cdot\mathcal{W}(\varphi)$
$v=p$	the "big cell" section $f_p^{\alpha_{\underline{\tau}^P}}(s,\phi\epsilon_d^P)\big\vert_{s=\frac{2n+1}{2}-t_d^P}$, where $\alpha_{\underline{\tau}^P}$ is the inverse Fourier transform of the Schwartz function in the next column	$\widehat{\alpha}_{\underline{\tau}^P}(\boldsymbol{\beta})=\mathbb{1}_{p^2\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)$ $\times\mathbb{1}_{M_n(\mathbb{Z}_p)}(\beta_0)\prod_{i=1}^{d-1}\mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}((2\beta_0)_{N_i})$ $\times\prod_{i=1}^{d-1}\epsilon_i^P\epsilon_{i+1}^{P-1}(\det((2\beta_0)_{N_i}))$ The major difference from the previous $\widehat{\alpha}_{\kappa,\underline{\tau}^P}$ is that here the support of $\widehat{\alpha}_{\underline{\tau}^P}$ has been enlarged and is no longer contained in $\mathrm{GL}(2n,\mathbb{Z}_p)$.	$\epsilon_d^P(-1)^n\frac{\prod_{i=1}^d\prod_{j=1}^{n_i}(1-p^{-j})}{\prod_{j=1}^n(1-p^{-2j})}$ $\cdot E_p^{P\text{-imp}}(n+1-t_d^P,\pi\times\phi\epsilon_d^P)$ $\cdot e_P\mathcal{W}(\varphi)$ (after $f_{\underline{\tau}^P,p}$ being modified by appropriate $\mathbb{U}_p^P$-operators in accordance with ordinary projection applied to $E^*(\cdot,f_{\underline{\tau}^P})\big\vert_{G\times G}$)
$v=\infty$	$\left(\prod_{i=1}^{d-1}\det\left(\frac{(\widehat{\mu}_0^+)_{N_i}}{4\pi\sqrt{-1}}\right)^{t_i^P-t_{i+1}^P}\right.$ $\left.\cdot f_\infty^{t_d^P}(s,\mathrm{sgn}^{t_d^P})\right)\Big\vert_{s=\frac{2n+1}{2}-t_d^P}$	vanishing unless $\boldsymbol{\beta}\geq 0$ and equals $(-1)^{nt_d^P}\frac{2^{n-2n^2+2nt_d^P}}{\Gamma_{2n}\left(\frac{2n+1}{2}\right)}\pi^{n+2n^2}$ $\times\prod_{i=1}^{d-1}\det((2\beta_0)_{N_i})^{t_i^P-t_{i+1}^P}$ $\times\prod_{i=1}^{d-1}\det((y_0)_{N_i})^{t_i^P-t_{i+1}^P}\det(\mathbf{y})^{\frac{t_d^P}{2}}\mathbf{e}_\infty(\mathrm{Tr}\boldsymbol{\beta}\boldsymbol{z})$ $+$ terms irrelevant to $\varepsilon_{q\text{-exp}}(\mathcal{E}_{\kappa,\underline{\tau}^P})$ where $\mathbf{y}=\mathrm{Im}(\boldsymbol{z})=\left(\begin{smallmatrix}y_1 & y_0\\ {}^{t}y_0 & y_2\end{smallmatrix}\right)$	$\frac{Z_\infty(f_{\underline{\tau}^P,\infty},v_{\underline{t}^P}^\vee,v_{\underline{t}^P})}{\left\langle v_{\underline{t}^P}^\vee,v_{\underline{t}^P}\right\rangle}\cdot\mathcal{W}(\varphi)$ $\neq 0$

TABLE 2. data for d-variable improved p-adic L-function

2.5. The construction of $\mu_{\mathcal{E},P\text{-ord}}$ and $\mathcal{E}^{\mathrm{imp}}_{P\text{-ord}}$. Assume $\phi^2 \neq \mathrm{triv}$. Then it follows from our choices of the sections $f_{\kappa,\underline{\tau}^P} \in I_{Q_H}(\frac{2n+1}{2} - k, \phi\chi)$ and $f_{\underline{\tau}^P} \in I_{Q_H}(\frac{2n+1}{2} - t^P_d, \phi\epsilon^P_d)$ that there exist p-adic measures

$$\mu_{\mathcal{E},q\text{-exp}} \in \mathscr{M}eas\left(\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p), \mathcal{O}_F[\![N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]\!]\right),$$
$$\mu_{\mathcal{E}^{\mathrm{imp}},q\text{-exp}} \in \mathscr{M}eas\left(T_P(\mathbb{Z}_p), \mathcal{O}_F[\![N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{\geq 0}]\!]\right)$$

satisfying the interpolation properties

$$\int_{\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p)} (\kappa, \underline{\tau}^P)\, d\mu_{\mathcal{E},q\text{-exp}} = \begin{cases} \varepsilon_{q\text{-exp}}\left(\mathcal{E}_{\kappa,\underline{\tau}^P}\right) & \text{if } (\kappa,\underline{\tau}^P) \text{ is admissible with } \phi\chi(-1) = (-1)^k, \\ 0 & \text{if } \phi(-1) \neq \kappa(-1), \end{cases}$$
$$\int_{T_P(\mathbb{Z}_p)} \underline{\tau}^P\, d\mu_{\mathcal{E}^{\mathrm{imp}},q\text{-exp}} = \begin{cases} \varepsilon_{q\text{-exp}}\left(\mathcal{E}_{\underline{\tau}^P}\right) & \text{if } \underline{\tau}^P \text{ is admissible with } \phi\epsilon^P_d(-1) = (-1)^{t^P_d}, \\ 0 & \text{if } \phi(-1) \neq \tau^P_d(-1), \end{cases}$$

(see [Liu16b, §5.1, 5.2] for more details on how to obtain these p-adic measures from the formulae for $\varepsilon_{q\text{-exp}}\left(\mathcal{E}_{\kappa,\underline{\tau}^P}\right)$, $\varepsilon_{q\text{-exp}}\left(\mathcal{E}_{\underline{\tau}^P}\right)$ in §2.4.7).

Remark 2.5.1. The assumption $\phi^2 \neq \mathrm{triv}$ is not essential. Without it, due to the pole of the Kubota–Leopoldt p-adic L-function, we need to make some modification accordingly to allow a possible pole in the constructed measures.

Let $V^{SP,r,\Delta}$ be the subspace of $\varprojlim_m \varinjlim_l V^{SP,r}_{m,l} \otimes_{\mathcal{O}_F/p^m} V^{SP,r}_{m,l}$ consisting of elements annihilated by $\gamma \otimes 1 - 1 \otimes \gamma$ for all $\gamma \in P(\mathbb{Z}_p)$. By definition and (2.4.13), we know that $\mathcal{E}_{\kappa,\underline{\tau}^P} \in V^{SP,0,\Delta}$ and $\mathcal{E}_{\underline{\tau}^P} \in V^{SP,n_d,\Delta}$. Then due to the Zariski density of the admissible points $(\kappa, \underline{\tau}^P)$ (resp. $\underline{\tau}^P$) inside $\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p)$ (resp. $T_P(\mathbb{Z}_p)$), the measure $\mu_{\mathcal{E},q\text{-exp}}$ (resp. $\mu_{\mathcal{E}^{\mathrm{imp}},q\text{-exp}}$) lies inside the image of the following embedding induced by p-adic q-expansion

$$\mathscr{M}eas\left(\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p), V^{SP,0,\Delta}\right) \hookrightarrow \mathscr{M}eas\left(\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p), \mathcal{O}_F[\![N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]\!]\right)$$
$$\left(\text{resp. } \mathscr{M}eas\left(T_P(\mathbb{Z}_p), V^{SP,n_d,\Delta}\right) \hookrightarrow \mathscr{M}eas\left(T_P(\mathbb{Z}_p), \mathcal{O}_F[\![N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{\geq 0}]\!]\right)\right).$$

Viewing $\mu_{\mathcal{E},q\text{-exp}}$ (resp. $\mu_{\mathcal{E}^{\mathrm{imp}},q\text{-exp}}$) as a p-adic measure valued in $V^{SP,0,\Delta}$ (resp. $V^{SP,n_d,\Delta}$) , Propositions 1.9.4 and 1.10.1 show that one can apply $e_P \times e_P$ to it and get

$$\mu_{\mathcal{E},P\text{-ord}} \in \mathscr{M}eas\left(\mathbb{Z}_p^\times \times T_P(\mathbb{Z}_p), e_P V^{SP,0,\Delta}\right) \quad \left(\text{resp. } \mu_{\mathcal{E}^{\mathrm{imp}},P\text{-ord}} \in \mathscr{M}eas\left(T_P(\mathbb{Z}_p), e_P V^{SP,n_d,\Delta}\right)\right).$$

For $\underline{\nu} \in \operatorname{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}/p), \mu_{p-1}\right)$ and an $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-module, we use a subscript $\underline{\nu}$ to denote its $\underline{\nu}$-isotypic part for the action of $T_P(\mathbb{Z}/p)$. Then like [Liu16b, (6.1.8)], for all $0 \leq r \leq n_d$ and $\underline{\tau}^P$ such that $\underline{\tau}^P|_{T_P(\mathbb{Z}/p)} = \underline{\nu}$, we have the commutative diagram

$$\begin{tikzcd}
\mathscr{M}eas\left(T_P(\mathbb{Z}_p), e_P V^{SP,r,\Delta}\right)^{\natural}_{\underline{\nu}} \arrow[r, "\Phi^{\Delta}_{\underline{\nu}}"] \arrow[dr, "\mu \mapsto \int_{T_P(\mathbb{Z}_p)} \underline{\tau}^P\, d\mu"'] & \mathcal{M}^r_{P\text{-ord},\underline{\nu}} \otimes_{\Lambda_P} \mathcal{M}^r_{P\text{-ord},\underline{\nu}} \arrow[d, "\mathrm{mod}\ \mathcal{P}_{\underline{\tau}^P}"] \\
& \varprojlim_m \varinjlim_l e_P V^{SP,r}_{m,l}[\underline{\tau}^P] \otimes_{\mathcal{O}_F} e_P V^{SP,r}_{m,l}[\underline{\tau}^P],
\end{tikzcd}$$

where $\mathscr{M}eas\left(T_P(\mathbb{Z}_p), e_P V^{SP,r,\Delta}\right)^{\natural}$ is the subspace of $\mathscr{M}eas\left(T_P(\mathbb{Z}_p), e_P V^{SP,r,\Delta}\right)$ consisting of measures μ satisfying

$$\int_{T_P(\mathbb{Z}_p)} \underline{\tau}^P\, d\mu \in e_P V^{SP,r,\Delta}[\underline{\tau}^P].$$

Set

$$\begin{aligned}\Phi^{\Delta} := \bigoplus_{\underline{\nu}} \Phi^{\Delta}_{\underline{\nu}} : \mathcal{M}eas\left(T_P(\mathbb{Z}_p), e_P V^{SP,r,\Delta}\right)^{\natural} &= \bigoplus_{\underline{\nu}} \mathcal{M}eas\left(T_P(\mathbb{Z}_p), e_P V^{SP,r,\Delta}\right)^{\natural}_{\underline{\nu}} \\ &\longrightarrow \mathcal{M}^{r}_{P\text{-ord}} \otimes_{\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]} \mathcal{M}^{r}_{P\text{-ord}} = \bigoplus_{\underline{\nu}} \mathcal{M}^{r}_{P\text{-ord},\underline{\nu}} \otimes_{\Lambda_P} \mathcal{M}^{r}_{P\text{-ord},\underline{\nu}},\end{aligned}$$

where $\underline{\nu}$ runs over $\mathrm{Hom}_{\mathrm{cont}}(T_P(\mathbb{Z}/p), \mu_{p-1})$. This Λ_P-module morphism Φ^{Δ} also induces

$$\Phi^{\Delta} : \mathcal{M}eas\left(\mathbb{Z}_p^{\times} \times T_P(\mathbb{Z}_p), e_P V^{SP,r,\Delta}\right)^{\natural} \longrightarrow \mathcal{M}eas\left(\mathbb{Z}_p^{\times}, \mathcal{M}^{r}_{P\text{-ord}} \otimes_{\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]} \mathcal{M}^{r}_{P\text{-ord}}\right).$$

We define

$$\begin{aligned}\mu_{\mathcal{E},P\text{-ord}} &= \Phi^{\Delta}(\mu_{\mathcal{E},P\text{-ord}}) \in \mathcal{M}eas\left(\mathbb{Z}_p^{\times}, \mathcal{M}^{0}_{P\text{-ord}} \otimes_{\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]} \mathcal{M}^{0}_{P\text{-ord}}\right), \\ \mathcal{E}^{\mathrm{imp}}_{P\text{-ord}} &= \Phi^{\Delta}(\mu_{\mathcal{E}^{\mathrm{imp}},P\text{-ord}}) \in \mathcal{M}^{n_d}_{P\text{-ord}} \otimes_{\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]} \mathcal{M}^{n_d}_{P\text{-ord}}.\end{aligned}$$

2.6. **The p-adic L-functions and their interpolation properties.** Let $\mathbb{T}^{0,N}_{P\text{-ord}}$ (resp. $\mathbb{T}^{n_d,N}_{P\text{-ord}}$) be the $\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]$-algebra as defined at the end of §1.10. Let $\mathscr{C}_P$ be a geometrically irreducible component of Spec $\left(\mathbb{T}^{0,N}_{P\text{-ord}}\right)$. Because of the natural quotient map $\mathbb{T}^{n_d,N}_{P\text{-ord}} \to \mathbb{T}^{0,N}_{P\text{-ord}}$, $\mathscr{C}_P$ can also be viewed as a geometrically irreducible component of Spec $\left(\mathbb{T}^{n_d,N}_{P\text{-ord}}\right)$. Denote by $F_{\mathscr{C}_P}$ the function field of $\mathscr{C}_P$ and by $\mathbb{I}_{\mathscr{C}_P}$ the integral closure of Λ_P inside $F_{\mathscr{C}_P}$. Let $\lambda_{\mathscr{C}_P} : \mathbb{T}^{n_d,N}_{P\text{-ord}} \to \mathbb{I}_{\mathscr{C}_P}$ be the homomorphism of Λ_P-algebra associated to $\mathscr{C}_P$.

As $F_{\mathscr{C}_P}$-algebras, we have the decomposition

$$\mathbb{T}^{n_d,N}_{P\text{-ord}} \otimes_{\Lambda_P} F_{\mathscr{C}_P} = F_{\mathscr{C}_P} \oplus R_{\mathscr{C}_P}$$

such that the projection onto $F_{\mathscr{C}_P}$ is given by $\lambda_{\mathscr{C}_P}$. Define $\mathbb{1}_{\mathscr{C}_P}$ as the element in $\mathbb{T}^{n_d,N}_{P\text{-ord}} \otimes_{\Lambda_P} F_{\mathscr{C}_P}$ which corresponds to $(\mathrm{id}, 0)$ in $F_{\mathscr{C}_P} \oplus R_{\mathscr{C}_P}$.

Proposition 2.6.1.

$$\mathbb{1}_{\mathscr{C}_P}\left(\mathcal{M}^{n_d}_{P\text{-ord}}\right) \subset \mathcal{M}^{0}_{P\text{-ord}} \otimes_{\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]} F_{\mathscr{C}_P}.$$

Proof. The cuspidality condition is about the vanishing of the restriction to the boundary, therefore an element in $\mathcal{M}^{n_d}_{P\text{-ord}}$ is cuspidal as long as its specializations at a Zariski dense subset in $\mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^{\times}\right)$ are cuspidal. We know that for all $t_1^P \gg t_2^P \gg \cdots \gg t_d^P \gg 0$, the specialization of $\mathcal{M}^{n_d}_{P\text{-ord}}$ at the algebraic point $\underline{t}^P \in \mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^{\times}\right)$ consists of classical holomorphic Siegel modular forms of weight $\imath(\underline{t}^P)$ and level $\Gamma(N) \cap \Gamma_P(p)$. We reduce to show that for all $t_1^P \geq t_2^P \geq \cdots \geq t_d^P \gg 0$, if a Hecke eigenform φ (for all unramified Hecke operators away from $Np\infty$) in $\mathrm{M}^{n_d}_{\imath(\underline{t}^P)}(\Gamma(N) \cap \Gamma_P(p); \mathbb{C})$ shares the same eigenvalues as a cuspidal Hecke eigenform in $\mathrm{M}^{0}_{\imath(\underline{t}^P)}(\Gamma(N) \cap \Gamma_P(p); \mathbb{C})$, then φ is cuspidal. When $t_d^P > 2n+1$ this is true according to [Har84, Theorem 2.5.6]. ☐

As explained in [Liu16b, §6.1.5], for each pair $(\beta_1, \beta_2) \in N^{-1}\,\mathrm{Sym}(n, \mathbb{Z})^{*\oplus 2}_{>0}$ there is a Λ_P-linear map

$$\varepsilon_{q\text{-exp},\beta_1,\beta_2} : \mathcal{M}^{0}_{P\text{-ord}} \otimes_{\mathcal{O}_F[\![T_P(\mathbb{Z}_p)]\!]} \mathcal{M}^{0}_{P\text{-ord}} \longrightarrow \Lambda_P,$$

which, when specialized at primes $\mathcal{P}_{\underline{\tau}^P}$, gives the map of taking the coefficients indexed by β_1, β_2 in the p-adic q-expansion $\varepsilon_{q\text{-exp}}$.

Define

$$\mu_{\mathscr{C}_P,\phi,\beta_1,\beta_2} = \varepsilon_{q\text{-exp},\beta_1,\beta_2} \circ \mathbb{1}_{\mathscr{C}_P}\left(\mu_{\mathcal{E},P\text{-ord}}\right) \in \mathscr{M}eas\left(\mathbb{Z}_p^\times, \mathcal{M}^0_{P\text{-ord}}\right) \otimes_{\Lambda_P} F_{\mathscr{C}_P},$$
$$\mathcal{L}^{\mathrm{imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2} = \varepsilon_{q\text{-exp},\beta_1,\beta_2} \circ \mathbb{1}_{\mathscr{C}_P}\left(\mathcal{E}^{\mathrm{imp}}_{P\text{-ord}}\right) \in F_{\mathscr{C}_P}.$$

Theorem 2.6.2. *For a Dirichlet character ϕ with conductor dividing N and $\phi^2 \neq 1$, a geometrically irreducible component $\mathscr{C}_P \subset \mathrm{Spec}\left(\mathbb{T}^{0,N}_{P\text{-ord}}\right)$, and $(\beta_1, \beta_2) \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$, the above constructed $\mu_{\mathscr{C}_P,\phi,\beta_1,\beta_2}$ and $\mathcal{L}^{P\text{-imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}$ satisfy the following interpolation properties. Let $x : \mathbb{I}_{\mathscr{C}_P} \to F'$ be an F'-point of $\mathscr{C}_P$ with F' being a finite extension of F. Suppose that the weight projection map $\Lambda_P \to \mathbb{T}^{n_d,N}_{P\text{-ord}}$ is étale at x mapping x to an arithmetic point $\underline{\tau}^P \in \mathrm{Hom}_{\mathrm{cont}}\left(T_P(\mathbb{Z}_p), F'^\times\right)$.*

If $(\kappa, \underline{\tau})$ is admissible, then

$$\left(\int_{\mathbb{Z}_p^\times} \kappa\, d\mu_{\mathscr{C}_P,\phi,\beta_1,\beta_2}\right)(x) = C_{k,\underline{t}^P} \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\mathfrak{c}(\beta_1,\varphi)\mathfrak{c}(\beta_2, e_P \mathscr{W}(\varphi))}{\langle \varphi, \overline{\varphi}\rangle}$$
$$\times E_p(n+1-k, \pi_x \times \phi\chi) \cdot L^{Np\infty}(n+1-k, \pi_x \times \phi\chi),$$

if $\kappa(-1) = \phi(-1)$, and otherwise vanishes.

Assume that the parity of $\mathscr{C}_P$ is compatible with ϕ. If $\underline{\tau}^P$ is admissible, then

$$\mathcal{L}^{\mathrm{imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(x) = C_{\underline{t}^P} \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\mathfrak{c}(\beta_1,\varphi)\mathfrak{c}(\beta_2, e_P \mathscr{W}(\varphi))}{\langle \varphi, \overline{\varphi}\rangle}$$
$$\times E_p^{P\text{-imp}}(n+1-t^P_d, \pi_x \times \phi\epsilon^P_d) \cdot L^{Np\infty}(n+1-t^P_d, \pi_x \times \phi\epsilon^P_d).$$

The following are some explanations of the interpolation formulae.

- *The constant $C_{k,\underline{t}^P}$ (resp. $C_{\underline{t}^P}$) only depends on $k, \underline{t}^P, \phi, N$ (resp. $\underline{t}^P, \phi, N$) and is defined as*

$$C_{k,\underline{t}^P} = \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \frac{\prod_{l=1}^d \prod_{j=1}^{n_l}(1-p^{-j})}{\prod_{j=1}^n (1-p^{-2j})} \cdot \frac{\Gamma_{2n}\left(\frac{2n+1}{2}\right)}{2^{-2n^2+n+2nk}\pi^{2n^2+n}} \cdot \frac{Z_\infty\left(f_{\kappa,\underline{\tau}^P,\infty}, v_{\imath(\underline{t}^P)}, v^\vee_{\imath(\underline{t}^P)}\right)}{\left\langle v_{\imath(\underline{t}^P)}, v^\vee_{\imath(\underline{t}^P)}\right\rangle},$$
$$C_{\underline{t}^P} = C_{t^P_d,\underline{t}^P}.$$

  The nonvanishing of $C_{k,\underline{t}^P}$ is shown in §2.7.
- *$\mathfrak{s}_x$ is a finite set consisting of an orthonormal basis of the eigenspace for the Hecke eigensystem parametrized by x inside $e_P \mathrm{M}^0_{\imath(\underline{t}^P)}\left(\Gamma(N) \cap \Gamma_{SP}(p^\infty), \underline{\epsilon}; \mathbb{C}\right)$. The set $\mathfrak{s}_x$ is empty and the evaluation is 0 if x is not classical, i.e. if there exists no cuspidal irreducible automorphic representation $\pi_x \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ with $\pi_{x,\infty} \cong \mathcal{D}_{\imath(\underline{t}^P)}$ such that the Hecke eigensystem parametrized by x appears in π_x.*
- *$\mathfrak{c}(\beta_i, \cdot)$, $i = 1, 2$, denotes the coefficient indexed by β_i in the q-expansion.*
- *The operator $\mathscr{W}$ is defined as in (2.4.14), and by the reasoning as in* [Liu16b, Proposition 5.7.2] *we know that $e_P \mathscr{W}(\varphi) \neq 0$ for $\varphi \in \mathfrak{s}_x$.*

Proof. By our construction, for a classical x as in the theorem, the evaluations of $\mathbb{1}_{\mathscr{C}_P}\left(\int_{\mathbb{Z}_p^\times} \kappa\, d\mu_{\mathcal{E},P\text{-ord}}\right)$ and $\mathbb{1}_{\mathscr{C}_P}\left(\mathcal{E}^{\mathrm{imp}}_{P\text{-ord}}\right)$ at x are classical cuspidal Siegel modular forms obtained by projecting the forms in (2.4.7) and (2.4.8) to the eigenspace associated to the Hecke eigensystem parametrized by x. Thus the interpolation formulae follow from the formulae for

$$\mathscr{L}_{\overline{\varphi}}\left(\left(e_P \times e_P\right) E^*(\cdot, f_{\kappa,\underline{\tau}^P})\big|_{G\times G}\right) \quad \text{and} \quad \mathscr{L}_{\overline{\varphi}}\left(\left(e_P \times e_P\right) E^*(\cdot, f_{\underline{\tau}^P})\big|_{G\times G}\right), \tag{2.6.1}$$

for $\varphi \in \mathfrak{s}_x$, which are proved later in Proposition 2.8.1. The nonvanishing of the archimedean zeta is proved in the next section. $\square$

Remark 2.6.3. For each $j \in (\mathbb{Z}/(p-1)$ such that $\phi\omega^j(-1) = 1$, applying the p-adic Mellin transform with respect to ω^j to the measure $\mu_{\mathscr{C}_P,\phi,\beta_1,\beta_2}$, one gets the p-adic L-function $\mathcal{L}_{\mathscr{C}_P,\phi\omega^j,\beta_1,\beta_2} \in \mathbb{I}_{\mathscr{C}_P}[[S]] \otimes_{\mathbb{I}_{\mathscr{C}_P}} F_{\mathscr{C}_P}$ as described in the theorem in the introduction.

2.7. **The nonvanishing of the archimedean zeta integral.** Since $f_{\underline{\tau}^P,\infty} = f_{\tau_d^P,\underline{\tau}^P,\infty}$, it suffices to show the nonvanishing of $Z_\infty\left(f_{\kappa,\underline{\tau}^P,\infty}, v_{\underline{\tau}^P}, v_{\underline{\tau}^P}^\vee\right)$ for all $(\kappa,\underline{\tau}^P)$ such that $t_1^P \geq \cdots \geq t_d^P \geq k \geq n+1$. This nonvanishing will follow from the proposition below.

Proposition 2.7.1. *With $f_{\kappa,\underline{\tau}^P,\infty} \in I_{Q_H,\infty}(\frac{2n+1}{2}-k,\mathrm{sgn}^k)$ as in (2.4.4), the map*

$$Z_\infty(f_{\kappa,\underline{\tau}^P,\infty},\cdot,\cdot) : \mathcal{D}_{\imath(\underline{t}^P)} \times \widetilde{\mathcal{D}}_{\imath(\underline{t}^P)} \longrightarrow \mathbb{C}$$

is nonzero.

Proof. Let $M(s,\mathrm{sgn}^k) : I_{Q_H,\infty}(s,\mathrm{sgn}^k) \to I_{Q_H,\infty}(-s,\mathrm{sgn}^k)$ be the intertwining operator defined as

$$\left(M(s,\mathrm{sgn}^k)f_\infty(s,\mathrm{sgn}^k)\right)(h) = \int_{U_{Q_H}(\mathbb{R})} f_\infty(s,\mathrm{sgn}^k)\left(\begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix} uh\right)\, du.$$

Set

$$f_\infty(s,\mathrm{sgn}^k) = \prod_{i=1}^{d-1} \det\left(\frac{(\widehat{\mu}_0^+)_{N_i}}{4\pi\sqrt{-1}}\right)^{t_i^P - t_{i+1}^P} \cdot \det\left(\frac{\widehat{\mu}_0^+}{4\pi\sqrt{-1}}\right)^{t_d^P - k} \cdot f_\infty^k(s,\mathrm{sgn}^k).$$

According to [LR05], the local zeta integral for π_∞ and $f_\infty(s,\mathrm{sgn}^k) \in I_{Q_{H,\infty}}(s,\mathrm{sgn}^k)$ satisfies the functional equation

$$Z_\infty\left(M(s,\mathrm{sgn}^k)f_\infty(s,\mathrm{sgn}^k),\cdot,\cdot\right) = \Gamma_\infty(s,\pi_\infty,\mathrm{sgn}^k) \cdot Z_\infty\left(f_\infty(s,\mathrm{sgn}^k),\cdot,\cdot\right), \tag{2.7.1}$$

with
(2.7.2)

$$\Gamma_\infty(s,\pi_\infty,\mathrm{sgn}^k) = \pi_\infty(-1)\cdot\gamma_\infty\left(s+\frac{1}{2},\pi_\infty\times\mathrm{sgn}^k\right)\cdot\left(\gamma_\infty\left(s-\frac{2n-1}{2},\mathrm{sgn}^k\right)\prod_{j=1}^{n}\gamma_\infty\left(2s-2n+2j,\mathrm{triv}\right)\right)^{-1}.$$

On the other hand, it follows from [Shi82, (1.31)(4.34K)] that

$$\begin{aligned} &M(s,\mathrm{sgn}^k)f_\infty(s,\mathrm{sgn}^k) \\ &\quad= (-1)^{nk} \cdot 2^{2n-2ns} \cdot \pi^{n(2n+1)} \cdot \frac{\Gamma_{2n}(s)}{\Gamma_{2n}\left(\frac{1}{2}(s+\frac{2n+1}{2})+\frac{k}{2}\right)\Gamma_{2n}\left(\frac{1}{2}(s+\frac{2n+1}{2})-\frac{k}{2}\right)} \cdot f_\infty(-s,\mathrm{sgn}^k). \end{aligned} \tag{2.7.3}$$

Combining (2.7.3), (2.7.1), and (2.7.2), we get for $\pi_\infty = \mathcal{D}_{\imath(\underline{t}^P)}$,

$$
\begin{aligned}
&Z_\infty\left(f_{\kappa,\underline{\tau}^P,\infty},\cdot,\cdot\right) = Z_\infty\left(f_\infty(s,\mathrm{sgn}^k),\cdot,\cdot\right)\Big|_{s=\frac{2n+1}{2}-k}\\
&=(-1)^{nk}\cdot 2^{2n-2ns}\cdot\pi^{n(2n+1)}\cdot\frac{\Gamma_{2n}(s)}{\Gamma_{2n}\left(\frac{1}{2}(s+\frac{2n+1}{2})+\frac{k}{2}\right)\Gamma_{2n}\left(\frac{1}{2}(s+\frac{2n+1}{2})-\frac{k}{2}\right)}\\
&\quad\times\pi_\infty(-1)\cdot\frac{\gamma_\infty\left(s-\frac{2n-1}{2},\mathrm{sgn}^k\right)\prod_{j=1}^{n}\gamma_\infty\left(2s-2n+2j,\mathrm{triv}\right)}{\gamma_\infty\left(s+\frac{1}{2},\pi_\infty\times\mathrm{sgn}^k\right)}\cdot Z_\infty\left(f_\infty(-s,\mathrm{sgn}^k),\cdot,\cdot\right)\Big|_{s=\frac{2n+1}{2}-k}\\
&=(-1)^{nk}\cdot 2^{2nk-2n^2+n}\cdot\pi^{n(2n+1)}\cdot\pi_\infty(-1)\cdot\frac{\gamma_\infty\left(1-k,\mathrm{sgn}^k\right)\prod_{j=1}^{n}\gamma_\infty\left(1+2j-2k,\mathrm{triv}\right)}{\Gamma_{2n}\left(\frac{2n+1}{2}\right)\cdot\gamma_\infty\left(n+1-k,\pi_\infty\times\mathrm{sgn}^k\right)}\\
&\quad\times Z_\infty\left(f_\infty(k-\frac{2n+1}{2},\mathrm{sgn}^k),\cdot,\cdot\right).
\end{aligned}
$$

The nonvanishing of $Z_\infty\left(f_\infty(k-\frac{2n+1}{2},\mathrm{sgn}^k),\cdot,\cdot\right)$ for $\mathcal{D}_{\imath(\underline{t}^P)}$ is shown in [Liu16b, Proposition 4.3.1] using results in [Li90, JV79]. The γ_∞-factors appearing here has neither poles nor zeros at the relevant points because the condition $n+1 \leq k \leq t_d^P$ guarantees that those points are critical. $\square$

2.8. **Computing the zeta integrals at p.** The goal of this section is to prove the following proposition, which will give the interpolation properties for our p-adic L-functions.

Proposition 2.8.1. *Let $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreduicble cuspidal automorphic representation with $\pi_\infty \cong \mathcal{D}_{\imath(\underline{t}^P)}$. Also, assume that π contains a P-ordinary Siegel modular form φ of weight $\imath(\underline{t}^P)$ and p-nebentypus $\underline{\epsilon}^P$, invariant under the translation of $(\Gamma(N)\cap\Gamma_{SP}(p^\infty))^\wedge \subset G(\mathbb{A}_\mathrm{f})$. Then*

$$
\begin{aligned}
&\mathscr{L}_{\overline{\varphi}}\left(\left(e_P\times e_P\right)E^*(\cdot,f_{\kappa,\underline{\tau}^P})\big|_{G\times G}\right)\\
&=\phi\chi(-1)^n\mathrm{vol}\left(\widehat{\Gamma}(N)\right)\frac{\prod_{l=1}^{d}\prod_{j=1}^{n_l}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-2j})}\cdot\frac{Z_\infty\left(f_{\kappa,\underline{\tau}^P,\infty},v_{\imath(\underline{t}^P)},v^\vee_{\imath(\underline{t}^P)}\right)}{\left\langle v_{\imath(\underline{t}^P)},v^\vee_{\imath(\underline{t}^P)}\right\rangle}\\
&\qquad\times E_p(n+1-k,\pi\times\phi\chi)\cdot L^{Np\infty}(n+1-k,\pi\times\phi\chi)\cdot e_P\mathcal{W}(\varphi),
\end{aligned}
\tag{2.8.1}
$$

and

$$
\begin{aligned}
&\mathscr{L}_{\overline{\varphi}}\left(\left(e_P\times e_P\right)E^*(\cdot,f_{\underline{\tau}^P})\big|_{G\times G}\right)\\
&=\phi\epsilon_d^P(-1)^n\mathrm{vol}\left(\widehat{\Gamma}(N)\right)\frac{\prod_{l=1}^{d}\prod_{j=1}^{n_l}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-2j})}\cdot\frac{Z_\infty\left(f_{\underline{\tau}^P,\infty},v_{\imath(\underline{t}^P)},v^\vee_{\imath(\underline{t}^P)}\right)}{\left\langle v_{\imath(\underline{t}^P)},v^\vee_{\imath(\underline{t}^P)}\right\rangle}\\
&\qquad\times E_p^{P\text{-imp}}(n+1-t_d^P,\pi\times\phi\epsilon_d^P)\cdot L^{Np\infty}(n+1-t_d^P,\pi\times\phi\epsilon_d^P)\cdot e_P\mathcal{W}(\varphi).
\end{aligned}
\tag{2.8.2}
$$

The proof of the proposition is mainly about computing the zeta integrals at p. There is a little bit of subtlety here. One needs to first apply the ordinary projection $e_P\times e_P$ to $E^*(\cdot,f_{\kappa,\underline{\tau}^P})\big|_{G\times G}$ and $E^*(\cdot,f_{\underline{\tau}^P})\big|_{G\times G}$ before pairing it with $\overline{\varphi}$. Thus, what we need to compute at p for proving the proposition is not $T_{f_{\kappa,\underline{\tau}^P}}\overline{\varphi}$, $T_{f_{\underline{\tau}^P}}\overline{\varphi}$, but the zeta integrals with $f_{\kappa,\underline{\tau}^P}$ and $f_{\underline{\tau}^P}$ replaced by $R(U_p^P)\times R(U_p^P)f_{\kappa,\underline{\tau}^P}$ and $R(U_p^P)\times R(U_p^P)f_{\underline{\tau}^P}$, where R is a polynomial depending on π_p and a sufficiently small open compact subgroup $K_p\subset G(\mathbb{Z}_p)$ satisfying $e_P = R(U_p^P)$ on π^{K_p}. Recall that with a fixed $\underline{t}^P$, we make $\mathbb{U}_p^P$-operators act on smooth $G(\mathbb{Q}_p)$-representations by (2.3.4).

Our computation will essentially use Proposition 2.3.2 and is similar as the computation in [Liu16b, p.39-42].

Proof. Since the projections to π_p of P-ordinary forms inside π span a one-dimensional subspace, we know that $\mathscr{L}_{\overline{\varphi}}\left((e_P \times e_P)\ E^*(\cdot, f_{\kappa,\underline{\tau}^P})\big|_{G\times G}\right)$ (resp. $\mathscr{L}_{\overline{\varphi}}\left((e_P \times e_P)\ E^*(\cdot, f_{\underline{\tau}^P})\big|_{G\times G}\right)$) equals $e_P\mathscr{W}(\varphi)$ up to a scalar, given by

$$\frac{\left\langle \mathscr{L}_{\overline{\varphi}}\left((e_P \times e_P)\ E^*(\cdot, f_{\kappa,\underline{\tau}^P})\big|_{G\times G}\right), \varphi'^{\vartheta}\right\rangle}{\langle e_P\mathscr{W}(\varphi), \varphi'^{\vartheta}\rangle}, \qquad (\text{resp. } \frac{\left\langle \mathscr{L}_{\overline{\varphi}}\left((e_P \times e_P)\ E^*(\cdot, f_{\underline{\tau}^P})\big|_{G\times G}\right), \varphi'^{\vartheta}\right\rangle}{\langle e_P\mathscr{W}(\varphi), \varphi'^{\vartheta}\rangle})$$

for an arbitrary $\varphi' \in \pi$ such that the denominator is nonzero. We will always assume that φ' is fixed by both K_p and $\vartheta SP_G(\mathbb{Z}_p)\vartheta$ and $e_P\varphi' \neq 0$. Then

$$\left\langle e_P\mathscr{W}(\varphi), \varphi'^{\vartheta}\right\rangle = \langle \overline{\varphi}, e_P\varphi'\rangle \neq 0.$$

Let $\underline{b} = (\underbrace{b_1, \dots, b_1}_{n_1}, \dots, \underbrace{b_d, \dots, b_d}_{n_d})$ for $b_1 > b_2 > \cdots > b_d > 0$ and denote $\mathrm{diag}(p^{b_1}, \dots, p^{b_1}, \dots, p^{b_d}, \dots, p^{b_d})$ by $p^{\underline{b}}$. A direct computation shows that

$$\begin{aligned}
&\left\langle \mathscr{L}_{\overline{\varphi}}\left((e_P \times U^P_{p,\underline{b}})\ E^*(\cdot, f_{\kappa,\underline{\tau}^P})\big|_{G\times G}\right), \varphi'^{\vartheta}\right\rangle \\
&= \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \cdot \frac{Z_\infty\left(f_{\kappa,\underline{\tau}^P,\infty}, v_{\imath(\underline{t}^P)}, v^\vee_{\imath(\underline{t}^P)}\right)}{\left\langle v_{\imath(\underline{t}^P)}, v^\vee_{\imath(\underline{t}^P)}\right\rangle} \cdot L^{Np\infty}(n+1-k, \pi \times \phi\chi) \\
&\quad \times p^{\langle \underline{t}+2\rho_{G,c}, \underline{b}\rangle} \left\langle \overline{\varphi}, R(U^P_p) \int_{G(\mathbb{Q}_p)} f_p^{\alpha_{\kappa,\underline{\tau}^P}}(s, \phi\chi)\left(\mathcal{S}^{-1}\iota(g^{-1}, 1)\right) \pi_p(gp^{\underline{b}})\varphi'\, dg \right\rangle\Bigg|_{s=\frac{2n+1}{2}-k}.
\end{aligned}$$

The ratio

$$\frac{\left\langle \overline{\varphi}, R(U^P_p) \int_{G(\mathbb{Q}_p)} f_p^{\alpha_{\kappa,\underline{\tau}^P}}(s, \phi\chi)\left(\mathcal{S}^{-1}\iota(g^{-1}, 1)\right) \pi_p(gp^{\underline{b}})\varphi'\, dg \right\rangle}{\langle \overline{\varphi}, e_P\varphi'\rangle} \tag{2.8.3}$$

is independent of the choice of φ. Let $\theta_1, \dots, \theta_n$, $\alpha_1, \dots, \alpha_n$ be as in Proposition 2.3.2, and $\sigma_1, \dots, \sigma_d$ be as in the definition (2.3.11). We know that the P-ordinary space (with respect to $\underline{t}^P$) inside $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ is one dimensional. Therefore, if we take from $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ a function $\mathcal{G} : G(\mathbb{Q}_p) \to \mathbb{C}$ invariant under the right translation by K_p and $\vartheta SP_G(\mathbb{Z}_p)\vartheta$, then

$$(2.8.3) = \frac{\left(R(U^P_p) \int_{G(\mathbb{Q}_p)} f_p^{\alpha_{\kappa,\underline{\tau}^P}}(s, \phi\chi)\left(\mathcal{S}^{-1}\iota(g^{-1}, 1)\right) \pi_p(gp^{\underline{b}})\mathcal{G}\, dg\right)(1)}{\left(R(U^P_p)\mathcal{G}\right)(1)},$$

as long as the denominator is nonzero. Now we further assume that $\mathcal{G}(1) = 1$. Then by our description of the $\mathbb{U}^P_p$-eigenvalues for the P-ordinary vector in terms of the α_i's in Proposition 2.3.2, we have $\left(R(U^P_p)\mathcal{G}\right)(1) = 1$.

Now let $\mathcal{G} \in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ be the smooth function on $G(\mathbb{Q}_p)$ supported on $B_G(\mathbb{Q}_p)\vartheta SP_G(\mathbb{Z}_p)\vartheta$ and taking the value 1 on $\vartheta SP_G(\mathbb{Z}_p)\vartheta$. We also put $\mathfrak{w}(a) = |\det a|_p^{-\frac{n+1}{2}}\mathcal{G}\left(\begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\right)$. Then $\mathfrak{w} \in \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}(n,\mathbb{Q}_p)}\underline{\theta}$ and is invariant under the right translation of u for ${}^{\mathrm{t}}u \in SP(\mathbb{Z}_p)$ and takes the value 1 at identity. Let $\Phi_{\chi,\underline{\epsilon}^P}$ be the Schwartz function on $M_n(\mathbb{Q}_p)$ whose Fourier transform is

$$\widehat{\Phi}_{\chi,\underline{\epsilon}^P}(\beta_0)=\prod_{i=1}^{d}\mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}((\beta_0)_{N_i})\times\prod_{i=1}^{d-1}\epsilon_i^P\epsilon_{i+1}^{P-1}(\det(-(\beta_0)_{N_i}))\cdot\epsilon_d^P\chi^{-1}(\det(-\beta_0)),\quad \beta_0\in M_n(\mathbb{Q}_p).$$

An easy computation shows that

$$\begin{aligned}f_p^{\alpha_{\kappa,\underline{\tau}^P}}(s,\phi_p\chi_p)\left(\mathcal{S}^{-1}\iota\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),1\right)\right)=&|\det a|_p^{-(s+\frac{2n+1}{2})}\phi\chi(\det(-a))^{-1}p^{-n(n+1)}\\&\times\mathbb{1}_{\mathrm{GL}(n,\mathbb{Q}_p)}(a)\cdot\mathbb{1}_{p^{-2}\,\mathrm{Sym}(n,\mathbb{Z}_p)}(a^{-1}b)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(ca^{-1})\cdot\Phi_{\chi,\underline{\epsilon}^P}(a^{-1}),\end{aligned}$$

and

$$\begin{aligned}&\left(U_{p,\underline{c}}^P\int_{G(\mathbb{Q}_p)}f_p^{\alpha_{\kappa,\underline{\tau}^P}}(s,\phi_p\chi_p)\left(\mathcal{S}^{-1}\iota(g^{-1},1)\right)\pi_p(gp^{\underline{b}})\mathcal{G}\,dg\right)(1)\\&=\frac{\prod_{j=1}^{n}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-2j})}p^{-\langle\rho_{G,n},\underline{b}\rangle}\prod_{i=1}^{d}\mathfrak{a}_{N_i}^{c_i}\int_{\mathrm{GL}(n,\mathbb{Q}_p)}|\det a|_p^{s+\frac{n}{2}}\phi\chi(\det(-a))\Phi_{\chi,\underline{\epsilon}^P}(a)\mathfrak{w}\left(ap^{\underline{b}}\right)da,\end{aligned}$$

for $\underline{c}=(\underbrace{c_1,\dots,c_1}_{n_1},\dots,\underbrace{c_d,\dots,c_d}_{n_d})$, $c_1>c_2>\dots>c_d>0$.

Therefore, in order to show (2.8.1), it suffices to show that with our chosen $\mathfrak{w}\in\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}(n,\mathbb{Q}_p)}\underline{\theta}$,

$$\begin{aligned}&\int_{\mathrm{GL}(n,\mathbb{Q}_p)}|\det a|_p^{s+\frac{n}{2}}\phi_p\chi_p(\det a)\Phi_{\chi,\underline{\epsilon}^P}(a)\mathfrak{w}\left(ap^{\underline{b}}\right)da\\&\qquad=\frac{\prod_{i=1}^{d}\prod_{j=1}^{n_i}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-j})}\prod_{i=1}^{d}\prod_{j=N_{i-1}+1}^{N_i}\alpha_j^{b_i}\cdot E_p(s,\pi\times\phi\chi).\end{aligned}\tag{2.8.4}$$

By the same reasoning, in order to show (2.8.2), it suffices to show

$$\begin{aligned}&\int_{\mathrm{GL}(n,\mathbb{Q}_p)}|\det a|_p^{s+\frac{n}{2}}\phi_p\epsilon_{d,p}^P(\det a)\Phi_{\underline{\epsilon}^P}(a)\mathfrak{w}\left(ap^{\underline{b}}\right)da\\&\qquad=\frac{\prod_{i=1}^{d}\prod_{j=1}^{n_l}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-j})}\prod_{i=1}^{d}\prod_{j=N_{i-1}+1}^{N_i}\alpha_j^{b_i}\cdot E_p^{P\text{-imp}}(s,\pi\times\phi\epsilon_d^P),\end{aligned}\tag{2.8.5}$$

where $\Phi_{\underline{\tau}^P}$ is the Schwartz function on $M_n(\mathbb{Q}_p)$ whose Fourier transform is

$$\widehat{\Phi}_{\underline{\epsilon}^P}(\beta_0)=\mathbb{1}_{M_n(\mathbb{Z}_p)}(\beta_0)\prod_{i=1}^{d-1}\mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}((2\beta_0)_{N_i})\times\prod_{i=1}^{d-1}\epsilon_i^P\epsilon_{i+1}^{P-1}(\det(-(2\beta_0)_{N_i})),\quad\beta_0\in M_n(\mathbb{Q}_p).$$

We will show (2.8.4) and (2.8.5) by induction. Write $n' = N_{d-1}$, $a = \begin{array}{c c} & \begin{array}{cc} n' & n_d \end{array} \\ \begin{array}{c} n' \\ n_d \end{array} & \begin{pmatrix} a' & \eta \\ 0 & \lambda \end{pmatrix} \end{array} \cdot \begin{pmatrix} I_{n'} & 0 \\ \lambda^{-1\,{}^{\mathrm{t}}}\mu & I_{n_d} \end{pmatrix}$.

Define $\mathfrak{w}' \in \mathrm{Ind}_{B_{n'}(\mathbb{Q}_p)}^{\mathrm{GL}(n',\mathbb{Q}_p)}(\theta_1, \ldots, \theta_{n'})$ by $\mathfrak{w}'(a') = |\det a'|_p^{-\frac{n_d}{2}} \mathfrak{w}\left(\begin{pmatrix} a' & 0 \\ 0 & I_{n_d} \end{pmatrix}\right)$, and the $\mathrm{SL}(n_d, \mathbb{Z}_p)$-fixed $\mathfrak{w}_d \in \mathrm{Ind}_{B_{n_d}(\mathbb{Q}_p)}^{\mathrm{GL}(n_d,\mathbb{Q}_p)}(\theta_{n'+1}, \ldots, \theta_n)$ by $\mathfrak{w}_d(\lambda) = |\det a|_p^{\frac{n'}{2}} \mathfrak{w}\left(\begin{pmatrix} I_{n'} & 0 \\ 0 & \lambda \end{pmatrix}\right)$. Also, let

$$\widehat{\Phi}'_{\chi,\underline{\epsilon}^P}(\beta'_0) = \prod_{i=1}^{d-1} \mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}\left((\beta'_0)_{N_i}\right) \times \prod_{i=1}^{d-2} \epsilon_i^P \epsilon_{i+1}^{P-1}(\det(-(\beta'_0)_{N_i})) \cdot \epsilon_{d-1}^P \chi^{-1}(\det(-\beta'_0)), \quad \beta'_0 \in M_{n'}(\mathbb{Q}_p),$$

$$\widehat{\Phi}'_{\underline{\epsilon}^P}(\beta'_0) = \prod_{i=1}^{d-1} \mathbb{1}_{\mathrm{GL}(N_i,\mathbb{Z}_p)}\left((\beta'_0)_{N_i}\right) \times \prod_{i=1}^{d-2} \epsilon_i^P \epsilon_{i+1}^{P-1}(\det(-(\beta'_0)_{N_i})) \cdot \epsilon_{d-1}^P \epsilon_d^{P-1}(\det(-\beta'_0)), \quad \beta'_0 \in M_{n'}(\mathbb{Q}_p),$$

and let $\Phi'_{\chi,\underline{\epsilon}^P}$, $\Phi'_{\underline{\epsilon}^P}$ be the inverse Fourier transform of $\widehat{\Phi}'_{\chi,\underline{\epsilon}^P}$, $\widehat{\Phi}'_{\underline{\epsilon}^P}$. Denote by $\mathcal{F}_{\underline{\epsilon}_d^P \chi^{-1}}$ the inverse Fourier transform of the Schwartz function $\lambda \mapsto \mathbb{1}_{\mathrm{GL}(n_d,\mathbb{Z}_p)}(\lambda) \cdot \underline{\epsilon}_d^P \chi^{-1}(\det(-\lambda))$ on $M_{n_d}(\mathbb{Q}_p)$. Then by a routine computation we get

(2.8.6)

$$\text{LHS of (2.7.3)} = \frac{\prod_{j=1}^{n'}(1-p^{-j})\prod_{j=1}^{n_d}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-j})} \int_{\mathrm{GL}(n',\mathbb{Q}_p)} |\det a'|_p^{s+\frac{n'}{2}} \phi_p\chi_p(\det a')\mathfrak{w}'\left(a' p^{\underline{b}'}\right) \Phi'_{\chi,\underline{\epsilon}^P}(a')\, da'$$
$$\times \prod_{i=N_{d-1}+1}^{n} \alpha_i^{b_d} \int_{\mathrm{GL}(n_d,\mathbb{Q}_p)} |\det \lambda|_p^{s+\frac{n_d}{2}} \phi_p\chi_p(\det \lambda)\mathfrak{w}_d(\lambda)\mathcal{F}_{\epsilon_d^P \chi^{-1}}(\lambda)\, d\lambda,$$

(2.8.7)

$$\text{LHS of (2.7.1)} = \frac{\prod_{j=1}^{n'}(1-p^{-j})\prod_{j=1}^{n_d}(1-p^{-j})}{\prod_{j=1}^{n}(1-p^{-j})} \int_{\mathrm{GL}(n',\mathbb{Q}_p)} |\det a'|_p^{s+\frac{n'}{2}} \phi_p\chi_p(\det a')\mathfrak{w}'\left(a' p^{\underline{b}'}\right) \Phi'_{\underline{\epsilon}^P}(a')\, da'$$
$$\times \prod_{i=N_{d-1}+1}^{n} \alpha_i^{b_d} \int_{\mathrm{GL}(n_d,\mathbb{Q}_p)} |\det \lambda|_p^{s+\frac{n_d}{2}} \phi_p\chi_p(\det \lambda)\mathfrak{w}_d(\lambda)\mathbb{1}_{M_{n_d}(\mathbb{Z}_p)}(\lambda)\, d\lambda.$$

For $\mathcal{F}_{\epsilon_d^P \chi^{-1}}$, we have the following formula [BS00, Proposition 6.1, Appendix of §6]. If $\mathrm{cond}(\epsilon_d^P \chi^{-1}) = p^c > 1$,

$$\mathcal{F}_{\epsilon_d^P \chi^{-1}}(\lambda) = p^{-\frac{n_d(n_d+1)c}{2}} G(\epsilon_d^P \chi^{-1})^{n_d} \cdot \mathbb{1}_{\mathrm{GL}(n_d,\mathbb{Z}_p)}(p^c\lambda)\, \epsilon_d^{P-1}\chi(\det(p^c\lambda)), \tag{2.8.8}$$

and if $\epsilon_d^P = \chi$,

$$\mathcal{F}_{\mathrm{triv}}(\lambda) = \sum_{j=0}^{n_d}(-1)^j p^{\frac{j(j-1-n_d)}{2}} \int_{\mathrm{GL}(n_d,\mathbb{Z}_p)\begin{pmatrix} pI_j & 0 \\ 0 & I_{n_d-j} \end{pmatrix}\mathrm{GL}(n_d,\mathbb{Z}_p)} \mathbb{1}_{M_{n_d}(\mathbb{Z}_p)}(\lambda g)\, dg. \tag{2.8.9}$$

We first look at the easier case (2.8.7), where for the integral in the second line of it, since $\mathfrak{w}_d$ is fixed by $\mathrm{SL}(n_d, \mathbb{Z}_p)$ and $\mathrm{GL}(n_d, \mathbb{Z}_p)$ acts on it by $\epsilon_{d,p}^{P-1} \circ \det$, we have

$$\int_{\mathrm{GL}(n_d,\mathbb{Q}_p)} |\det \lambda|_p^{s+\frac{n_d}{2}} \phi_p\epsilon_{d,p}^P(\det \lambda)\mathfrak{w}_d(\lambda)\mathbb{1}_{M_{n_d}(\mathbb{Z}_p)}(\lambda)\, d\lambda = L_p\left(s+\frac{1}{2}, \sigma_d \otimes \phi_p\right). \tag{2.8.10}$$

Next we treat the integral in the second line of (2.8.6) with $\epsilon^P\chi^{-1} \neq \text{triv}$. Plugging in (2.8.8), we get

(2.8.11)

$$\int_{\mathrm{GL}(n_d,\mathbb{Q}_p)} |\det\lambda|_p^{s+\frac{n_d}{2}} \chi_p(\det\lambda)\mathfrak{w}_d(\lambda)\mathcal{F}_{\epsilon_d^P\chi^{-1}}(\lambda)\,d\lambda = \prod_{j=n'+1}^{n} G(\epsilon_d^P\chi^{-1})\left(\phi(p)^{-1}\alpha_j^{-1}p^{s-\frac{1}{2}}\right)^c$$
$$= \prod_{j=n'+1}^{n} \gamma_p\left(\frac{1}{2}-s, \phi_p^{-1}\chi_p^{-1}\theta_j^{-1}\right) = \gamma_p\left(\frac{1}{2}-s, \widetilde{\sigma}_d \otimes \phi_p^{-1}\chi_p^{-1}\epsilon_{d,p}^P\right).$$

Lastly, we consider the integral in the second line of (2.8.6) with $\epsilon^P\chi^{-1} = \text{triv}$. The formula for the Hecke action on spherical representations of $\mathrm{GL}(\mathbb{Q}_p)$ gives

$$\int_{\mathrm{GL}(n_d,\mathbb{Z}_p)\begin{pmatrix} p^{-1}I_j & 0 \\ 0 & I_{n_d-j}\end{pmatrix}\mathrm{GL}(n_d,\mathbb{Z}_p)} \mathfrak{m}_d(\lambda g)\,dg = p^{\frac{j(n_d-j)}{2}} \sum_{\substack{\iota:\{1,\dots,j\}\hookrightarrow\{1,\dots,n_d\} \\ \iota(1)<\cdots<\iota(j)}} \alpha_{n'+\iota(1)}^{-1}\dots\alpha_{n'+\iota(j)}^{-1}\mathfrak{m}_d(\lambda).$$

Combining this with (2.8.9), we get

$$\begin{aligned}
&\int_{\mathrm{GL}(n_d,\mathbb{Q}_p)} |\det\lambda|_p^{s+\frac{n_d}{2}} \chi_p(\det\lambda)\mathfrak{w}_d(\lambda)\mathcal{F}_{\mathrm{triv}}(\lambda)\,d\lambda \\
&= \prod_{j=n'+1}^{n} (1-\phi_p(p)^{-1}\alpha_j^{-1}p^{s-\frac{1}{2}}) \int_{\mathrm{GL}(n_d,\mathbb{Q}_p)} |\det\lambda|_p^{s+\frac{n_d}{2}} \phi_p\epsilon_{d,p}^P(\det\lambda)\mathfrak{w}_d(\lambda)\mathbb{1}_{M_{n_d}(\mathbb{Z}_p)}(\lambda)\,d\lambda \qquad (2.8.12) \\
&= \prod_{j=n'+1}^{n} (1-\phi_p(p)^{-1}\alpha_j^{-1}p^{s-\frac{1}{2}}) \cdot L_p\left(s+\frac{1}{2}, \sigma\otimes\phi_p\right) = \gamma_p\left(\frac{1}{2}-s, \widetilde{\sigma}_d\otimes\phi_p^{-1}\right).
\end{aligned}$$

The identities (2.8.10), (2.8.11), and (2.8.12) allows us to compute (2.8.4) and (2.8.5) by induction, and this finishes the proof. ☐

3. The derivative of the p-adic standard L-function

In this section we explicit the trivial zeros of the p-adic L-function $mu\mathscr{C}_P, \phi, \beta_1, \beta_2$ of Theorem 2.6.2 and interpret them from the point of view of the (conjectural) associated p-adic Galois representation. This will allow us to interpret a factor appearing in the derivative of the p-adic L-function as Greenberg's ℓ-invariant in the case when the trivial zero is semi-stable (or of type M as called in [Gre94]). At the end, we shall prove the main theorem of the paper which relates the derivative of $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}$ at the semi-stable trivial zero to the ℓ-invariant and the complex special value.

3.1. **Greenberg–Benois conjecture and ℓ-invariants.** That idea at the base of the conjecture by Greenberg and Benois is that when a p-adic L-function has trivial zeros one should be able to recover the value of the complex L-function from a suitable derivative of the p-adic L-function. The first studied case is the one of an elliptic curve with split multiplicative reduction at p [MTT86]. They conjectured that the order of the p-adic L-function is 1 plus the order of the complex L-function and that the leading coefficient of the p-adic L-function is the same as the one of the complex L-function, up to an error term equal to $\log_p(q_E)/\mathrm{ord}_p(q_E)$ (for q_E the Tate uniformizer of E) that they call the ℓ-invariant.

Later Greenberg interpreted this number in Galois theoretical terms and proposed a similar conjecture for a great number of ordinary motives [Gre94]; succesively Benois generalised this conjecture to most semistable representations [Ben11]. Let us be more precise. Let V the p-adic Galois representation associated with the motive, and suppose that it is absolutely irreducible and satisfies Pantchishkine condition, *i.e.* that there is a sub space V' of V stable under the action of $G_{\mathbb{Q}_p}$ and containing all positive Hodge–Tate weights. Assume moreover that the Frobenius acts semisimply on the semi-stable module (in the sense of Fontaine) associated with V.

Greenberg defines two subspaces $V^{11} \subset V^{+} \subset V^{00}$ such that V^{+}/V^{11} contains all the eigenvalues p and V^{00}/V^{+} contains all the eigenvalues 1. Then we can decompose V^{00}/V^{11} as $\mathbb{Q}_p^{t_0} \oplus M \oplus \mathbb{Q}_p(1)^{t_1}$, where M is a non split extension of $\mathbb{Q}_p^{t}$ by $\mathbb{Q}_p(1)^{t}$. According to Greenberg's conjecture, the number of trivial zeros of the p-adic L-function is $e = t_0 + t + t_1$. Assume furthermore that $t_1 = 0$ (certain motivic conjectures imply $t_1 t_0 = 0$, so this hypothesis is not really restrictive). Then by picking a subspace $\widetilde{\mathbf{T}} \subset H^1(G_{\mathbb{Q}}, V)$ of dimension e, Greenberg defines an e-dimensional subspace $\widetilde{\mathbf{T}}_p$ inside $H^1(G_{\mathbb{Q}_p}, V)/H^1_{\mathrm{f}}(G_{\mathbb{Q}_p}, V)$ which injects into

$$H^1(G_{\mathbb{Q}_p}, V^1/V') \cong \bigoplus_{i=1}^{t+e_0} \mathbb{Q}_p \cdot \mathrm{ord}_p \oplus \mathbb{Q}_p \cdot \log_p .$$

If we denote by p_u (resp. p_c) the projection of $\widetilde{\mathbf{T}}_p$ to $\bigoplus_{i=1}^{t+e_0} \mathbb{Q}_p \cdot \mathrm{ord}_p$ (resp. $\bigoplus_{i=1}^{t+e_0} \mathbb{Q}_p \cdot \log_p$), then Greenberg shows that p_c is an invertible map and defines $\ell(V) := \det(p_u \circ p_c^{-1})$. Note that in general $\widetilde{\mathbf{T}}_p$ depends on V as $G_{\mathbb{Q}}$-representation, but if $e_0 = 0$ too then it depends only on the restriction to $G_{\mathbb{Q}_p}$ [Gre94, p. 169][Ros15, p. 1239]. We can finally state:

Conjecture 3.1.1. *[Greenberg–Benois conjecture] Let r be the order of the complex L-function $L(s, V)$ at $s = 0$, and suppose that $\mathcal{L}_p(S, V)$ has $e_0 + t$ trivial zeros, then S^{e_0+t+r} divides $\mathcal{L}_p(S, V)$ exactly and*

$$\mathcal{L}_p(S, V)/S^{e_0+t+r} \equiv \ell(V^*(1))(E_p(0, V)L_p(0, V)^{-1})^* \frac{L^{\mathrm{alg},(\mathrm{r})}(0, V)}{\log_p(1+p)^{e_0+t+r}(e_0+t+r)!} \bmod S,$$

where $E_p(s, V)$ is defined as in [Coa91, §6], $V^*(1)$ is the dual representation twisted by 1, $L_p(s, V)$ is the Euler factor of the motivic L-function for V and $(E_p(s, V)L_p(s, V)^{-1})^*$ is obtained from $E_p(s, V)L_p(s, V)^{-1}$ by eliminating all the Euler factors vanishing at $s = 0$. The function $L^{\mathrm{alg}}(s, V)$ is the L-function for V divided by the period.

Note that the conjecture implies the non-vanishing of the ℓ-invariant $\ell(V)$.

This conjecture has been first shown in the case of an elliptic curve with bad multiplicative reduction by Greenberg–Stevens [GS93]. Their method has been generalised many times to different contexts [Mok09, Ros16b, Ros16a, Rosss] and is at the base of our current approach. It is worth to point out that many new strategies have recently arisen [DDP11, Das16, Spi14, Dep16].

3.2. **The trivial zero of the p-adic standard L-function.** Let $\mathcal{L}_{\mathscr{C}_P,\phi\omega^j,\beta_1,\beta_2} = \mathcal{L}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(S, x) \in \mathbb{I}_{\mathscr{C}_P}[[S]] \otimes_{\mathbb{I}_{\mathscr{C}_P}} F_{\mathscr{C}_P}$ to be the Mellin transform of the component corresponding to the character ω^j on $(\mathbb{Z}/p)^\times$ of the measure $\mu_{\mathscr{C}_P,\phi,\beta_1,\beta_2}$ constructed in Theorem 2.6.2.

Suppose that $x_0 : \mathbb{I}_{\mathscr{C}_P} \to F$ is a point as in Theorem 2.6.2 and it is classical. Denote by π_{x_0} a cuspidal irreducible automorphic representation of $\mathrm{Sp}(2n, \mathbb{A})$ generated by a P-ordinary Siegel modular $\mathbb{T}^{0,N}_{P\text{-ord}}$-eigenform of weight $\iota(\underline{t}^P)$ whose eigenvalues are parametrized by x. The isomorphism class of $\pi_{0,v}$ is determined by x_0 for $v \nmid N$ (the isomorphism class of $\pi_{x_0,p}$ can be read off from the eigenvalues for all $\mathbb{U}_p^P$-operators by the discussion in §2.3.1. We are interested in possible trivial zeros of $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}$ at $S = (1+p)^{n+1} - 1$, where the corresponding critical L-value is the near-central value $L(0, \pi_x \times \phi)$. There are two types of trivial zeroes that can show up there.

3.2.1. *Crystalline trivial zero.* We say that $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}$ has a crystalline trivial zero at $((1+p)^{n+1} - 1, x_0)$ if $\phi(p) = 1$ and the local L-factor $L_p(s, \pi_{x_0} \times \phi)$ contains the factor $(1 - \phi_p(p)p^{-s})$. In particular, if $\pi_{x_0,p}$ is unramified, then a crystalline trivial zero shows up at $S = (1+p)^{n+1} - 1$.

One can also think of types of trivial zeroes in terms of the corresponding Galois representation restricted to $G_{\mathbb{Q}_p}$ (if admitting the local-global comptibility for ρ_{x_0} attached to π_{x_0}). Let $\rho_{x_0} : G_{\mathbb{Q}} \to \mathrm{GL}(2n+1, \overline{\mathbb{Q}}_p)$ be the Galois representation attached to π_{x_0} [Art13, CH13]. Then conjecturally $\rho_{x_0}|_{G_{\mathbb{Q}_p}}$ admits the following description. The Hodge–Tate weights are $0, \pm(t_1^P - 1), \ldots, \pm(t_1^P - N_1), \pm(t_2^P - N_1 - 1), \ldots, \pm(t_2 - N_2), \ldots, \pm(t_d^P - N_{d-1} - 1), \ldots, \pm(t_d^P - n)$, and the eigenvalues of the Frobenius are $1, \alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}$. Hence the Hodge polygon and the Newton polygon meet at the points with horizontal coordinates $0, N_1, N_2, \ldots, N_d, 2n+1-N_d, \ldots, 2n+1-N_2, 2n+1-N_1, 2n+1$. Therefore, $\rho_{x_0} : G_{\mathbb{Q}}$ admits a decreaing filtration Fil^j, $-d \leq j \leq d$, such that $\mathrm{Fil}^j/\mathrm{Fil}^{j+1}$ has Hodge–Tate weights $t_{d-j+1}^P - N_{d-j} - 1, \ldots, t_{d-j+1}^P - N_{d-j+1}$ (resp. $t_{-j}^P - N_{-j+1} - 1, \ldots, t_{-j}^P - N_{-j}$) if $1 \leq j \leq d$ (resp. $-d \leq j \leq -1$), and $\mathrm{Fil}^0/\mathrm{Fil}^1$ is one dimensional with trivial $G_{\mathbb{Q}_p}$-action. If $\mathrm{Fil}^0/\mathrm{Fil}^2$ is a trivial extension of $\overline{\mathbb{Q}}_p$ by $\mathrm{Fil}^1/\mathrm{Fil}^2$, then $\mathcal{L}_{\mathscr{C}_P, \phi\omega^{n+1}, \beta_1, \beta_2}$ has a crystalline trivial zero at $((1+p)^{n+1} - 1, x_0)$ if $\phi(p) = 1$.

3.2.2. *Semi-stable trivial zero.* The other case is when the local L-factor $L_p(s, \pi_{x_0} \times \phi)$ does not contain the factor $(1 - \phi_p(p)p^{-s})$, but the factor $(1 - \phi_p(p)^{-1}\alpha_n^{-1}p^{s-1})$ in $E_p(s, \pi_{x_0} \times \phi)$ contributes a trivial zero at $((1+p)^{n+1} - 1, x_0)$ when $\phi(p) = 1$ and $\alpha_n = p^{-1}$. We call this type of trivial zero semi-stable. Since Newton polygon lies above the Hodge polygon and for $\rho_{x_0}|_{G_{\mathbb{Q}_p}}$ they coincide along the slope 0 segment, we have $v_p(\alpha_n) \leq -t_d^P + n \leq -1$. If $\alpha_n = p^{-1}$, then $t_d^P = n+1$ and the Newton and Hodge polygons coincide along the the segments of slope $-1, 0, 1$. Therefore, if $\mathcal{L}_{\mathscr{C}_P, \phi\omega^{n+1}, \beta_1, \beta_2}(S, x_0) \in \mathcal{O}_F[[S]] \otimes_{\mathcal{O}_F} F$ has a semi-stable trivial zero at $S = (1+p)^{n+1} - 1$, then there exists a partition $n = n_1 + \cdots + n_d$ with $n_d = 1$ such that π_{x_0} is P-ordinary for $P \subset \mathrm{GL}(n)$ the standard parabolic subgroup attached to the partition.

The above discussion shows that in order to use p-adic deformation of π_{x_0} to study the trivial zero at $S = (1+p)^{n+1} - 1$ of the p-adic L-function attached to π_{x_0}, one can always choose P such that $n_d = 1$ and consider the P-ordinary families passing through x_0. In the following, we assume that $n_d = 1$.

In terms of the local Galois representation, a semi-stable appears if $\mathrm{Fil}^0/\mathrm{Fil}^2$ is a non-trivial extension of $\overline{\mathbb{Q}}_p$ by $\overline{\mathbb{Q}}_p(1)$.

3.3. **A formula for the derivative.** In this section we prove the main theorem of the paper applying the strategy of Greenberg–Stevens. Recall that $\mathfrak{a}_i$, $1 \leq \mathfrak{a}_i \leq n$, denotes the eigenvalue of the operator $U_{p,i}^P$ (see Proposition 2.3.2). They are invertible elements inside $\mathbb{I}_{\mathscr{C}_P}$. In order to state [Ben10, Theorem 2] in our setting, we need to fix a coordinate for a rigid analytic open neighborhood of x_0 in $\mathscr{C}_P$. Since we have assumed that the weight projection map is étale at x_0, we can take the coordinate to be the natural coordinate $T_1^P, \ldots, T_d^P$ of the weight space. Then it follows from [Ben10, Theorem 2] and [Ben11, Proposition 2.2.24] that

Theorem 3.3.1. *Let $x_0 \in \mathscr{C}_P(F)$ be a classical point where the weight projection map is étale and has image $\underline{\tau}_{x_0}^P \in \mathrm{Hom}_{\mathrm{cont}}(T^P(\mathbb{Z}_p), F^\times)$. Suppose that $S = (1+p)^{n+1} - 1$ is a semi-stable trivial zero for $\mathcal{L}_{\mathscr{C}_P, \phi\omega^{n+1}, \beta_1, \beta_2}(S, x_0)$ and the local-global compatibility is satisfied by the p-adic Galois representation ρ_{x_0}. Then*

$$\ell(\rho_{x_0}) = \ell(\rho_{x_0}^*(1)) = -\left.\frac{\partial \log_p \left(\mathfrak{a}_n(T_1^P, \ldots T_d^P)/\mathfrak{a}_{n-1}(T_1^P, \ldots, T_d^P)\right)}{\partial T_d^P}\right|_{(T_1^P, \ldots, T_d^P) = \underline{\tau}_{x_0}^P(1+p)}.$$

(For the proof see [Ros15, Theorem 1.3], where the theorem is stated for parallel weight, but the proof is the same.)

Remark 3.3.2. When $n = 2$ one can calculate the ℓ-invariant also for crystalline trivial zero using the method of [Hid06].

Remark 3.3.3. In our case, there is a trival zero for the whole p-adic L-function $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}$ at $S=(1+p)^{n+1}-1$. It is possible to define an algebraic ℓ-invariant $\ell(\mathscr{C}_P)$, which is a meromorphic function on $\mathscr{C}_P$, using the definition of [Ben11] in the context of [Pot13].

Define $j(\mathscr{C}_P)\in\mathbb{Z}/(p-1)$ by $\omega^{j(\mathscr{C}_P)}=\tau^P_d|_{(\mathbb{Z}/p)^\times}$ for a $\underline{\tau}^P$ inside the image of the projection of $\mathscr{C}_P$ to the weight space. The relation of the improved p-adic L-function $\mathcal{L}^{P\text{-imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}$ and the restriction of $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{j(\mathscr{C}_P)},\beta_1,\beta_2}$ to $\kappa=\tau^P_{d,x}$ is given as:

Proposition 3.3.4. *Suppose that $n_d=1$. As elements in $F_{\mathscr{C}_P}$,*

$$\mathcal{L}_{\mathscr{C}_P,\phi\omega^{j(\mathscr{C}_P)},\beta_1,\beta_2}(\tau^P_{d,x},x)=\left(1-\phi_p(p)^{-1}\mathfrak{a}_{n-1}(x)/\mathfrak{a}_n(x)\right)\mathcal{L}^{P\text{-imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(x).$$

Proof. This follows straightforwardly from the two interpolation formulae of Theorem 2.6.2 by noticing that when $n_d=1$, $\mathcal{A}^P(\pi\times\xi)=(1-\phi_p(p)^{-1}\alpha_n^{-1}p^{n-t^P_d})$, so

$$E_p(n+1-t^P_{d,x},\pi_x\times\phi\epsilon^P_d)=\left(1-\phi_p(p)^{-1}\mathfrak{a}_{n-1}(x)/\mathfrak{a}_n(x)\right)E^{P\text{-imp}}_p(n+1-t^P_{d,x},\pi_x\times\phi\epsilon^P_d)$$

(cf. (2.3.13)). □

Now we are ready to prove the main theorem.

Theorem 3.3.5. *Let x_0 be an F-point of $\mathscr{C}_P$ where the weight projection map $\Lambda_P\to\mathbb{T}^{1,N}_{P\text{-ord}}$ is étale and maps x_0 to $\underline{\tau}^P_0$. Suppose that the p-adic L-function $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}\in\mathbb{I}_{\mathscr{C}_P}[[S]]\otimes_{\mathbb{I}_{\mathscr{C}_P}}F_{\mathscr{C}_P}$ has a semi-stable trivial zero at $((1+p)^{n+1}-1,x_0)$ (so $j(\mathscr{C}_P)=n+1$) and the local-global compatibility is satisfied by the p-adic Galois representation ρ_{x_0}. Then we have*

$$\left.\frac{\mathrm{d}\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}(S,x_0)}{\mathrm{d}S}\right|_{S=(1+p)^{n+1}-1}=-\,\ell(\rho_{x_0})\cdot C_{\underline{t}^P_0}\cdot\sum_{\varphi\in\mathfrak{s}_{x_0}}\frac{\mathfrak{c}(\varphi,\beta_1)\mathfrak{c}(e_P\mathcal{W}(\varphi),\beta_2)}{\langle\varphi,\overline{\varphi}\rangle}$$
$$\times E^{P\text{-imp}}_p(0,\pi_{x_0}\times\phi)\cdot L^{Np\infty}(0,\pi_{x_0}\times\phi),$$

Proof. Again, we use the natural coordinate $T^P_1,\dots,T^P_d$ of the weight space to parametrize a rigid analytic open neighborhood of x_0 in $\mathscr{C}_P$. Note that by Remark 3.3.3, we know that $\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}((1+p)^{n+1}-1,x)$ is identically vanishing, so

$$\left.\frac{\partial\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}(S,T^P_1,\dots,T^P_d)}{\partial T^P_d}\right|_{S=(1+p)^{n+1}-1}=0.$$

It follows that

$$\left.\frac{\mathrm{d}\mathcal{L}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(S,x_0)}{\mathrm{d}S}\right|_{S=(1+p)^{n+1}-1}$$
$$=\left.\left(\frac{\partial\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}(S,T^P_1,\dots,T^P_d)}{\partial S}+\frac{\partial\mathcal{L}_{\mathscr{C}_P,\phi\omega^{n+1},\beta_1,\beta_2}(S,T^P_1,\dots,T^P_d)}{\partial T^P_d}\right)\right|_{S=(1+p)^{n+1}-1,(T^P_1,\dots,T^P_d)=\underline{\tau}^P_0(1+p)}$$
$$=\frac{\partial}{\partial T^P_d}\left.\left(\left(1-\phi_p(p)^{-1}\mathfrak{a}_{n-1}(T^P_1,\dots,T^P_d)/\mathfrak{a}_n(T^P_1,\dots,T^P_d)\right)\mathcal{L}^{\text{imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(T^P_1,\dots,T^P_d)\right)\right|_{(T^P_1,\dots,T^P_d)=\underline{\tau}^P_0(1+p)}$$
$$=\left.\frac{\partial\left(\mathfrak{a}_n(T^P_1,\dots T^P_d)/\mathfrak{a}_{n-1}(T^P_1,\dots,T^P_d)\right)}{\partial T^P_d}\right|_{(T^P_1,\dots,T^P_d)=\underline{\tau}^P_{x_0}(1+p)}\cdot\mathcal{L}^{\text{imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}(x_0).$$

Then the theorem follows from Theorem 3.3.1 and the interpolation property of $\mathcal{L}^{\text{imp}}_{\mathscr{C}_P,\phi,\beta_1,\beta_2}$ in Theorem 2.6.2. □

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