Probability

We want to address the following questions?

(1) What do you mean

"The probability of an event is 0.2".

$$P(A) = 0.2$$

(2) How probability is determined.

(3) What mathematical rules that probability must obey?

Basic Set Theory

Set : A collection of objects

set of odd numbers:

$\{1, 3, 5, 7, \dots\}$

set of all possible outcomes if a coin is flipped twice. H - head T - tail

$\{HH, HT, TH, TT\}$

Empty Set ϕ, or $\{\ \}$.

Subset

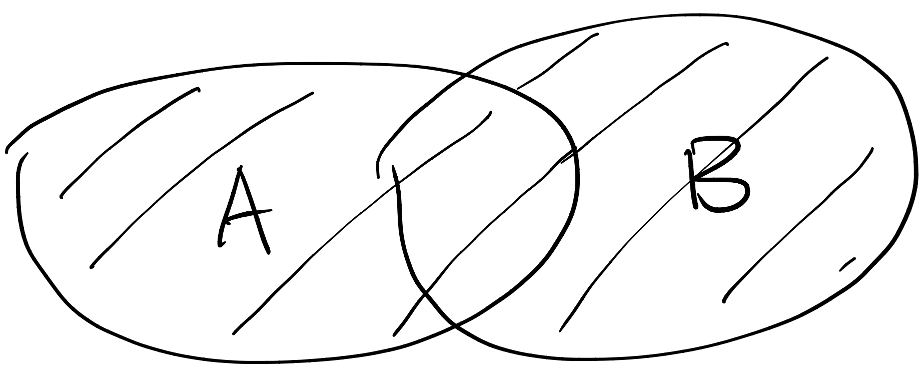
If A is a subset of B, then every element of A is also an element of B.

$$A \subseteq B$$

If $A \subseteq B$, $B \subseteq A$, then $A = B$

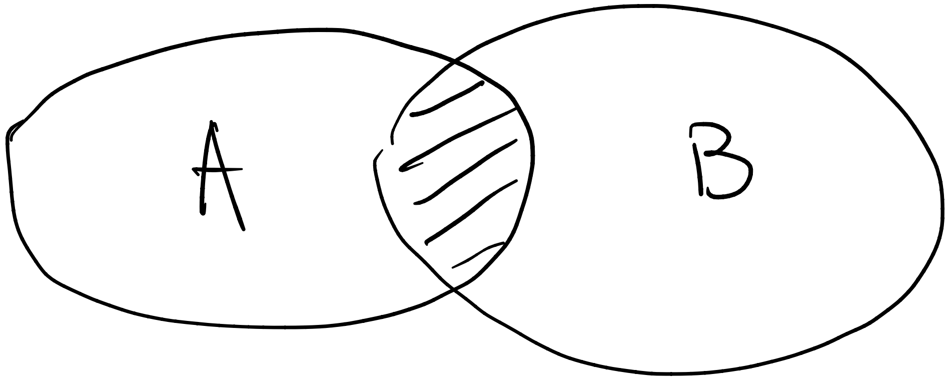
Union : the set of objects that are in A or B (or both)

$$A \cup B$$



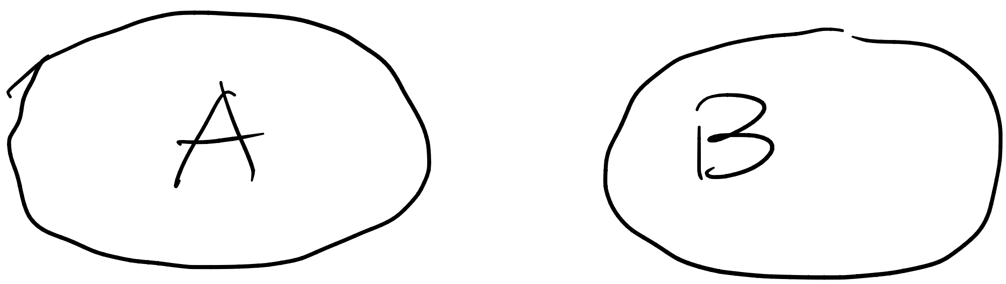
Intersection : set of all objects that are in both A and B.

$$A \cap B$$



Ex: $A \cap B \subseteq A \cup B$

Disjoint: A and B are disjoint if $A \cap B = \phi$



Ex

$A_1 \cup A_2 \cup \cdots \cup A_n$

is the set of all objects that are in at least one of the A_j's.

$$A_1 \cap A_2 \cap \cdots \cap A_n$$

is the set of all objects that are in all of the A_j's

<u>Full set</u> : the set that contains all possible objects.

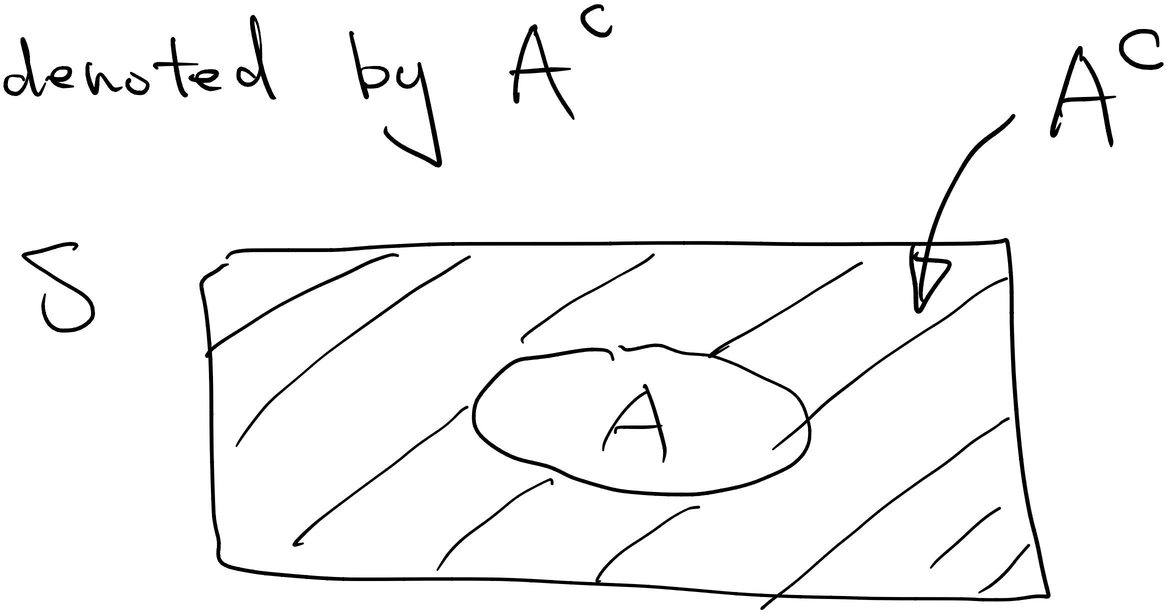
denoted by S

$$A \cap S = A$$

$$A \cup S = S$$

Complement of A : the set of all objects in S that are not in A.

denoted by A^c



Properties of set operations

Commutative laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative laws

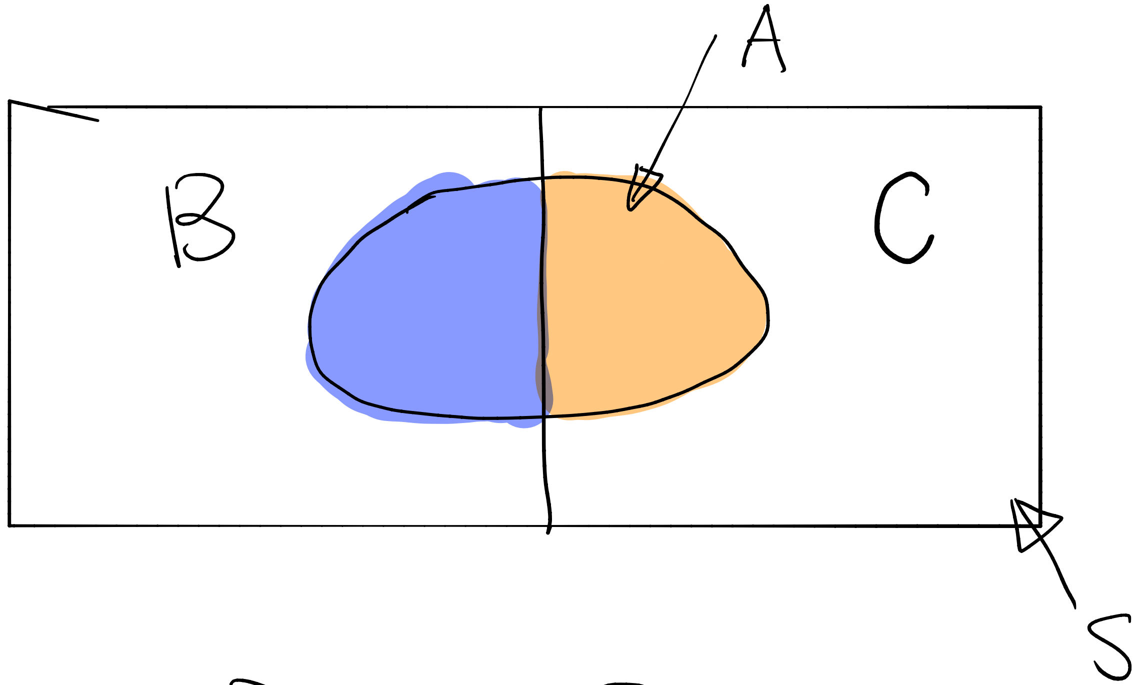
$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



since $B \cup C = S$.

then $A \cap (B \cup C) = A$.

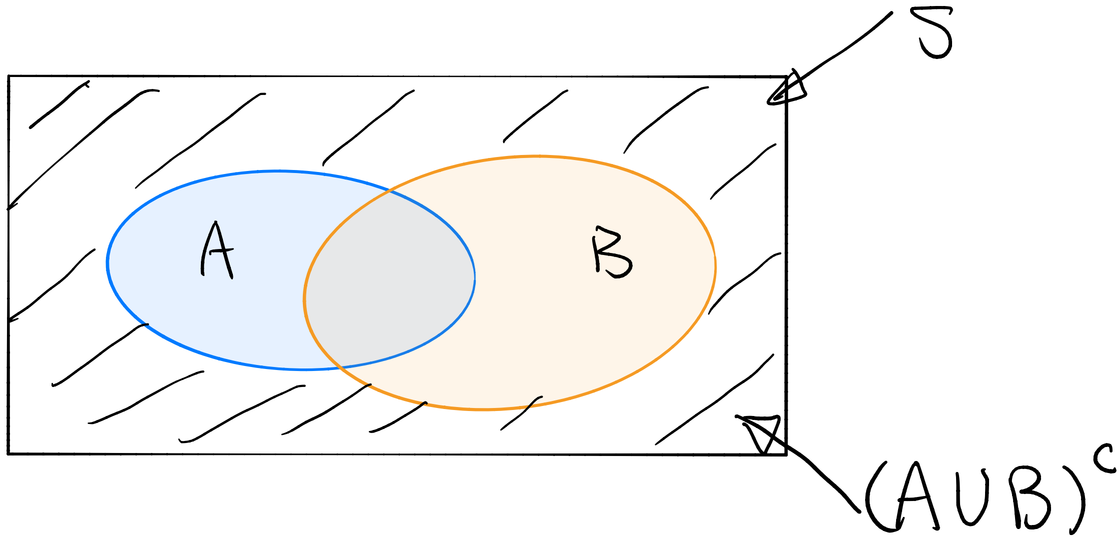
on the other hand

$$(A \cap B) \cup (A \cap C) = A.$$

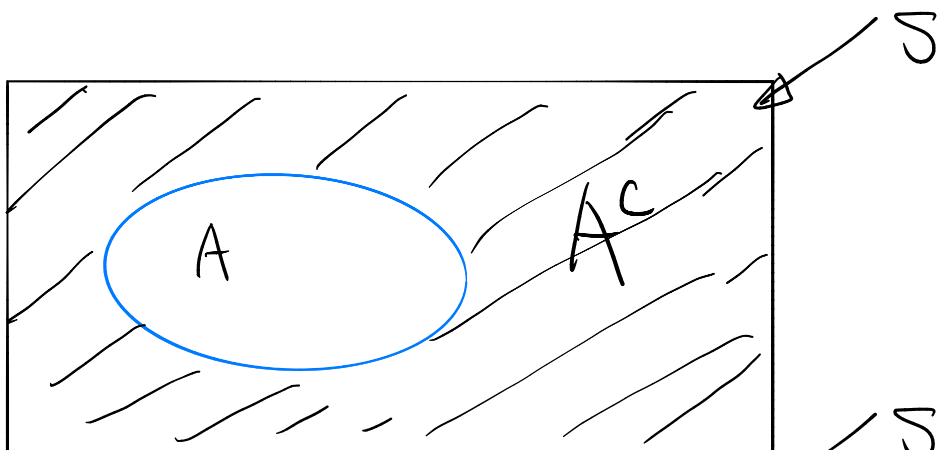
De Morgan's laws.

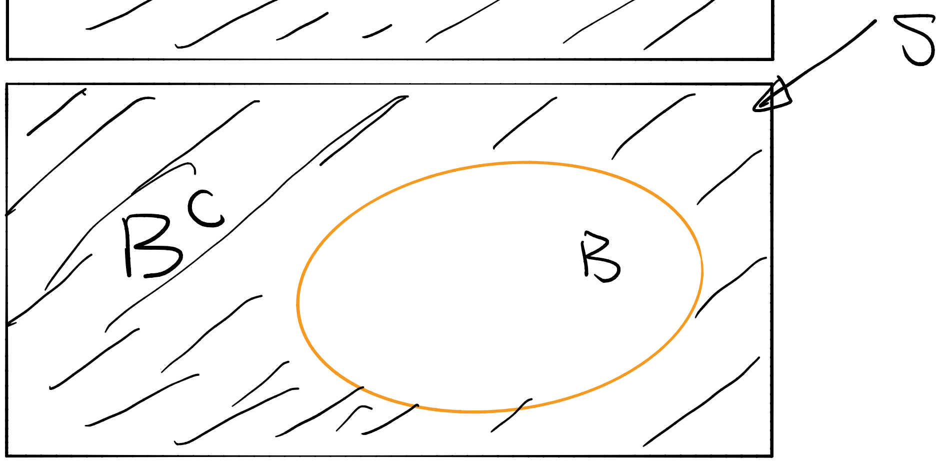
(1) $(A \cup B)^c = A^c \cap B^c$

(2) $(A \cap B)^c = A^c \cup B^c$



The shaded area is $(A \cup B)^c$





so $A^c \cap B^c = (A \cup B)^c$ is true

Probability

Experiment: a process of observations that leads to a single outcome that can not be predicted with certainty.

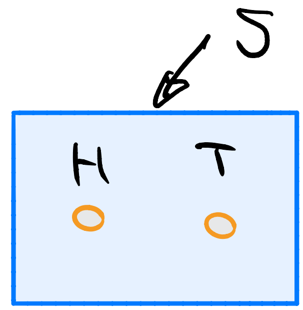
Ex: A Coin is tossed once, the up face is observed.

Experiment: tossing a coin once.

Observation: face.

Sample point(s) is the most basic outcome of an experiment.

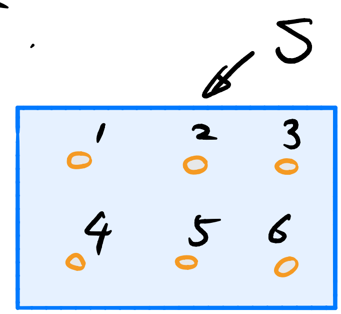
Sample space : (S) is the set of all possible outcomes of the experiment.



Experiment 1. Toss a coin once

Sample space : $S = \{s_1, s_2\} = \{H, T\}$.

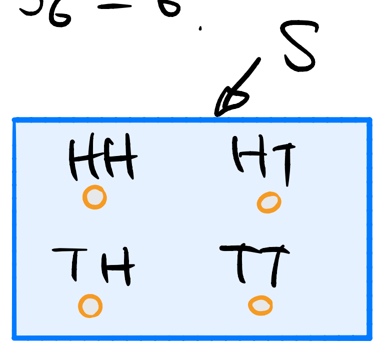
Sample points : $s_1 = H$, $s_2 = T$.



Experiment 2 . Toss a die once

Sample space . $S = \{1, 2, 3, 4, 5, 6\}$

sample points . $s_1 = 1$, $s_2 = 2$, ..., $s_6 = 6$.



Experiment 3 Toss a coin twice

$$S = \{HH, HT, TH, TT\}$$

$s_1 = HH$ $s_2 = HT$ $s_3 = TH$. $s_4 = TT$

Sometimes a collection of sample points are of interest.

Event : is a subset of sample points of the sample space S

event A. $s \in A$ $A \subseteq S$

Ex : Die tossing

$S = \{1, 2, 3, 4, 5, 6\}$

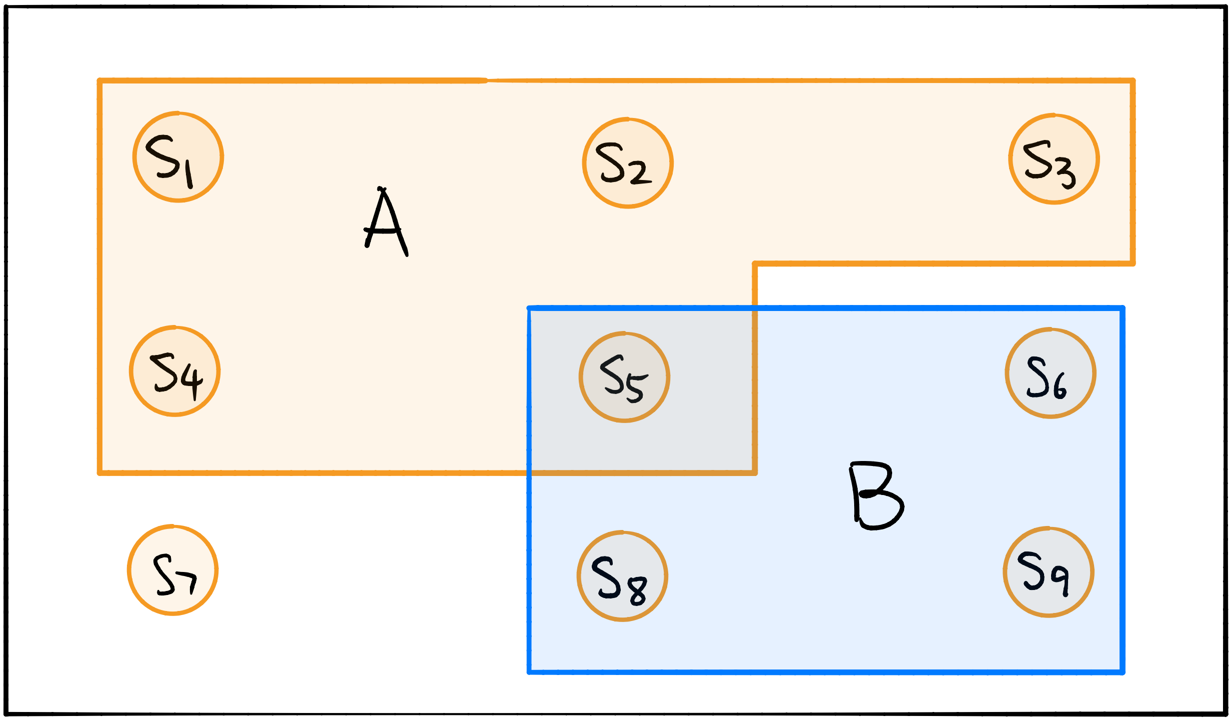
$A = \{\text{odd observed}\} = \{1, 3, 5\}$

$B = \{\text{even observed}\} = \{2, 4, 6\}$

> We say that an event A occured if the actual outcome is in A.

Ex : In die tossing, if the actual outcome is 1, 3, or 5, we say that the event A happened.

Pebble World = S



Performing the experiment amounts to randomly select one pebble, if all the pebbles are of the same mass, all the pebble are equally likely to be chosen.

(A general case that allows pebble to differ in mass will be discussed later)

Events can be described in English or set notation. English description is easier to interpret, but set notation is easier to manipulate.

English	Sets
Events and Occurrences	
Sample space	S
s is a possible outcome	$s \in S$
A is an event	$A \subseteq S$
A occurred	$s_{actual} \in A$
something must happen	$s_{actual} \in S$

Set operations (Unions, intersections, complements) can be used to build new events upon already-defined events.

Let $A, B \subseteq S$ be events

$A \cup B$: the event that occurs iff (if and only if) at least one of A and B occurs

$A \cap B$: the event that occurs iff both A and B occur.

A^c : the event that occurs iff A does not occur.

EX.

Roll a die twice.

S

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)				
(1, 4)	(2, 4)				
(1, 5)	(2, 5)				
(1, 6)	(2, 6)				

A $\quad$ A_2 $\quad$ A_1

$A_1 = \{\text{the first is } 2\}$

$A_2 = \{\text{the second is } 2\}$.

$$A = \{\text{both tosses are } 2\}$$

$$= \{\underbrace{\text{the first is } 2}_{A_1} \text{ and } \underbrace{\text{the second is } 2}_{A_2}\}$$

$$= A_1 \cap A_2$$

$$B = \{\text{at least one toss is } 2\}$$

$$= \{\underbrace{\text{the first is } 2}_{A_1} \text{ or } \underbrace{\text{the second is } 2}_{A_2}\}$$

$$= A_1 \cup A_2$$

$$C = \{\text{none of the dice is } 2\}$$

$$= \{\underbrace{\text{the first is not } 2}_{A_1^c} \text{ and } \underbrace{\text{the second is not } 2}_{A_2^c}\}$$

$$= A_1^c \cap A_2^c$$

<u>De Morgan's</u> $(A_1 \cup A_2)^c$

$$= \{\overbrace{\text{It is not}}^{\text{complement of}} \text{ the case that } \underbrace{\text{at least one of two tosses is } 2}_{A_1 \cup A_2}\}$$

$D = \{$ the first is 2 or the second is 2

but not both are 2 $\}$.

$D_1 = \{ \underbrace{\text{the first is 2}}_{A_1} \underset{\cap}{\text{and}} \underbrace{\text{the second is not 2}}_{A_2^c} \}$

$= A_1 \cap A_2^c$

$D_2 = \{ \underbrace{\text{the second is 2}}_{A_2} \underset{\cap}{\text{and}} \underbrace{\text{the first is not 2}}_{A_1^c} \}$

$= A_2 \cap A_1^c$

$D = D_1 \underset{\text{or}}{\cup} D_2$

Summary

English	Sets
A or B	$A \cup B$
A and B	$A \cap B$
not A	A^c
A but not B	$A \cap B^c$
A or B, but not both	$(A \cap B^c) \cup (A^c \cap B)$
at least one of $A_1, \ldots, A_n$	$A_1 \cup A_2 \cdots \cup A_n$
all of $A_1, \ldots, A_n$	$A_1 \cap \cdots \cap A_n$

Ex

Flip a coin 10 times, the outcome is a sequence $s \in S$.

Sample point

$$\text{HHHTTTH} \cdots \text{T} \qquad \begin{array}{cc} H & 1 \\ T & 0 \end{array}$$

$$s = (1\ 1\ 1\ 0\ 0\ 1 \cdots 0)$$

Sample space

the set of all possible sequences.

$$S = \{(c_1, c_2 \cdots\cdots, c_{10})\} \text{ where } c_j \in \{0, 1\}.$$

1. Let A_1 be the event that the first is head.

$$A_1 = \{(1, c_2, c_3, \ldots, c_{10})\}$$

$$c_2 \cdots c_{10} \in \{0, 1\}$$

$$A_j = \{(c_1 \ldots \underset{\substack{\uparrow \\ c_j}}{1} \ldots c_{10})\}$$

A_j is the event that the j-th is 1

$$j = 1, \ldots, 10$$

2. Let B the event that at least one flip is 1.

$$B = \bigcup_{j=1}^{10} A_j = A_1 \cup A_2 \cup \cdots \cup A_{10}$$

3 Let C be the event that all tosses are 1

$$C = \bigcap_{j=1}^{10} A_j = A_1 \cap A_2 \cdots \cap A_{10}$$

4. Let D be the event that there were at least two consecutive 1's

$$D = \bigcup_{j=1}^{9} (A_j \cap A_{j+1})$$

$$= (A_1 \cap A_2) \cup (A_2 \cap A_3) \cup \cdots$$

$$\cdots \cup (A_9 \cap A_{10})$$

Cardinality

If A is a finite set (event)

$$|A| = \text{the number of outcomes in } A$$

which is called the cardinality or size of A

Ex

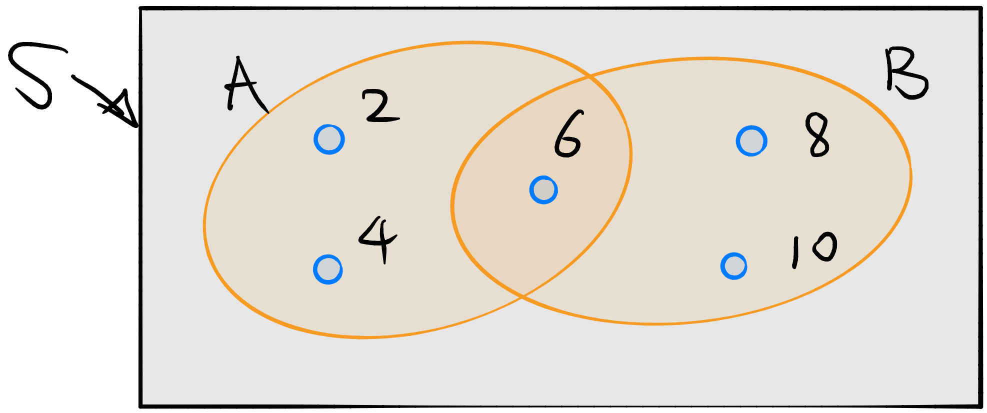
$$|\{2, 4, 6, 8, 10\}| = 5$$

Rule 1 :

If A and B are finite sets

$$A = \{2, 4, 6\}$$

$$B = \{6, 8, 10\}$$



$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$5 = 3 + 3 - 1$$

Rule 2 :

If $A \cap B = \phi$, then $|A \cap B| = 0$

Rule 3 :

If $A \cap B = \phi$, A and B are disjoint. (mutually exclusive)

$$|A \cup B| = |A| + |B|$$

since $|A \cup B| = |A| + |B| - \overbrace{|A \cap B|}^{=0}$

Rule 4:

A and A^c are disjoint $A \cap A^c = \emptyset$

$$|S| = |A \cup A^c| = |A| + |A^c|$$

$$|A^c| = |S| - |A|$$

Rule 5:

If $A_1, \ldots, A_n$ are a partition of S

$A_1 \cup A_2 \cdots \cup A_n = S$, $A_i \cap A_j = \emptyset$ for $i \neq j$

then

$$|S| = |A_1| + |A_2| + \cdots + |A_n|$$

Naive definition of probability

Definition: (Naive version of probability)

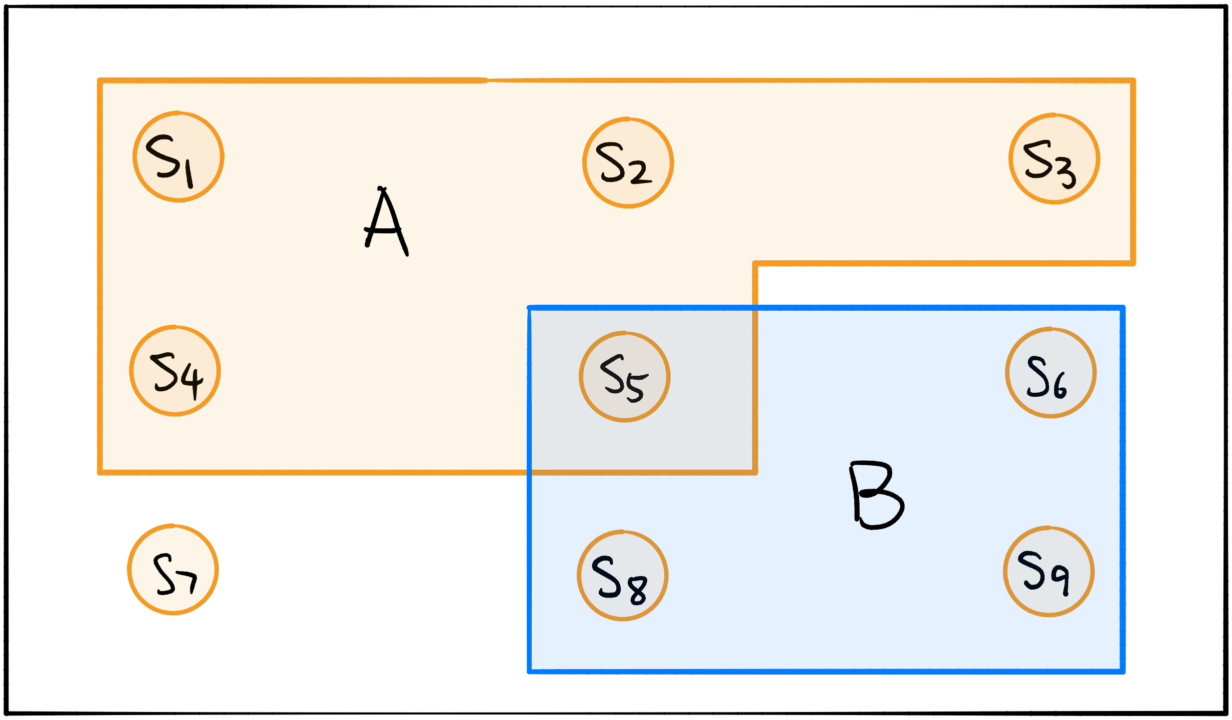
Let A be an event for an experiment with finite sample S. $A \subseteq S$

The naive probability of A is

$$P_{nv}(A) = \frac{|A|}{|S|} = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

In Pebble World, the probability of A is the fraction of pepples in A.

Pebble World $= S$



$$P_{nv}(A) = \frac{5}{9} \; ; \; P_{nv}(B) = \frac{4}{9}$$

$$P_{nv}(A \cup B) = \frac{8}{9} \; ; \; P_{nv}(A \cap B) = \frac{1}{9}$$

$$P_{nv}(A^c) = \frac{4}{9} \; ; \; P_{nv}(B^c) = \frac{5}{9}$$

$$P_{nv}((A \cup B)^c) = \frac{1}{9}$$

$$P_{nv}\left((A \cap B)^c\right) = \frac{8}{9}$$

Note:

(1) Naive definition of probability requires

- S has finite outcomes.
- Equal $\underbrace{\text{mass}}_{\text{probability}}$ for each pebble.

$$S = \{ s_1, s_2, s_3, s_4, s_5 \}$$

$$P(\{s_1\}) = \frac{|\{s_1\}|}{|S|} = \frac{1}{5}$$

$$P(\{s_2\}) = \quad '' \quad = \frac{1}{5}$$

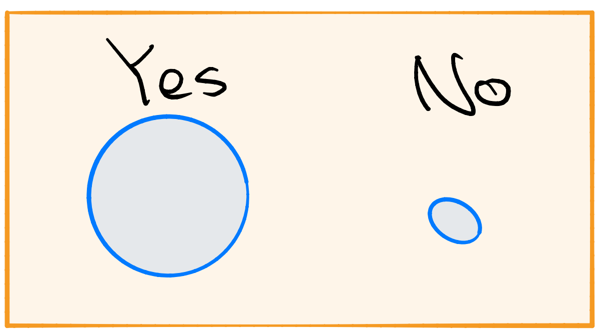
''   ''   ''

$$P(\{s_5\}) = \quad " \quad = \frac{1}{5}$$

Sometimes those conditions might not be satisfied.

Ex :

The probability of life on Mars



$$P(\{Yes\}) \neq P(\{no\})$$

Types of problems where the naive definition is appliable :

(1) problem is symmetric

coin tossing, well shuffled decks of cards,

(2) outcomes are equally likely by design

Conducting a survey of n people in a population of N people each person is equally likely to be selected.

Rules

(1) $P(\phi) = \frac{|\phi|}{|S|} = \frac{0}{|S|} = 0$

(2) $P(S) = \frac{|S|}{|S|} = 1$

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

since $P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= P(A) + P(B) - P(A \cap B)$$

(4) If $A \cap B = \phi$, $P(A \cap B) = 0$

(5) If $A \cap B = \phi$. (mutually exclusive)

$$P(A \cup B) = P(A) + P(B)$$

(6) Since $A \cap A^c = \phi$

$$P(A^c) = 1 - P(A)$$

Since

$$1 = P(\underbrace{A \cup A^c}_{S}) = P(A) + P(A^c)$$

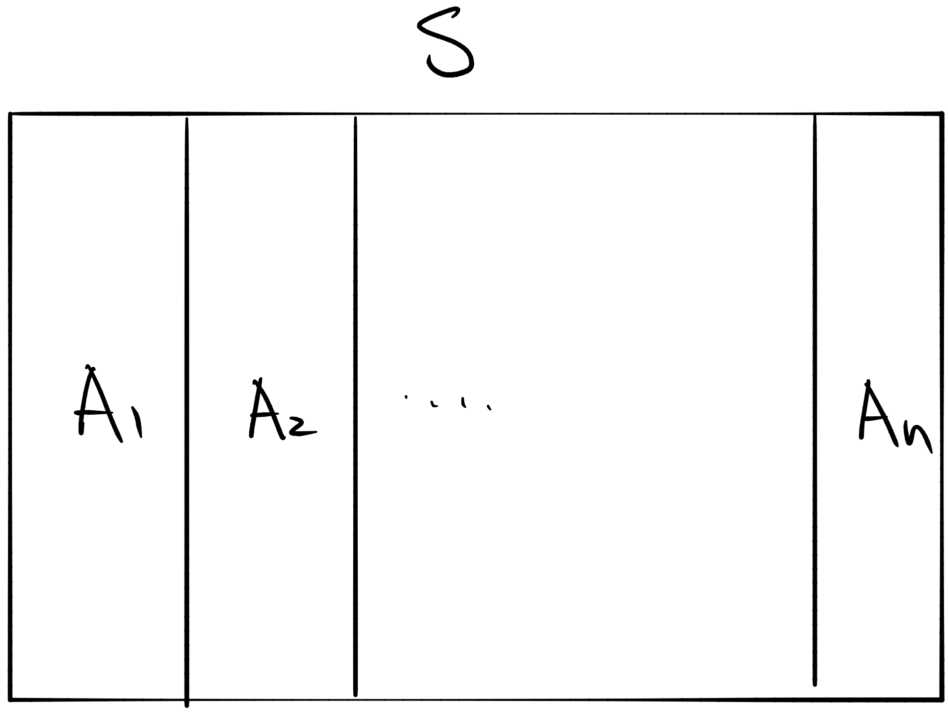
(7) If $A_1, \dots, A_n$ are a partition of S

then $A_1 \cup A_2 \cdots \cup A_n = S$, $A_i \cap A_j = \phi \ (i \neq j)$

$$P(A_1) + P(A_2) + \cdots + P(A_n) = 1$$

Since

$$1 = P(S) = \frac{|S|}{|S|} = \frac{|A_1| + |A_2| + \cdots + |A_n|}{|S|} = P(A_1) + \cdots + P(A_n)$$


S
A1
A2
....
An

EX

Roll a die twice.

S

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)				
(1,4)	(2,4)				
(1,5)	(2,5)				
(1,6)	(2,6)				

A $\quad$ A_2 $\quad$ A_1

$A_1 = \{\text{the first is } 2\}$ $\quad$ $A_2 = \{\text{the second toss is } 2\}$

$A = \{\text{both are } 2\} = A_1 \cap A_2$

$$P(A) = \frac{|A_1 \cap A_2|}{|S|} = \frac{1}{36}$$

$B = \{\text{at least one is } 2\} = A_1 \cup A_2$

$$P(B) = \frac{|A_1 \cup A_2|}{|S|} = \frac{11}{36}$$

$C = \{$none of the dice is 2$\}$

$$= A_1^c \cap A_2^c$$

$$P(C) = P(A_1^c \cap A_2^c)$$

$$= P((A_1 \cup A_2)^c)$$

$$= 1 - P(A_1 \cup A_2)$$

$$= 1 - P(B)$$

$$= 1 - \frac{11}{36}$$

$$= \frac{25}{36}$$

$D = \{$the first is 2 or the second is 2 but not both$\}$

$$= (A_1 \cap A_2^c) \cup (A_1^c \cap A_2)$$

$$P(D) = P((A_1 \cap A_2^c) \cup (A_1^c \cap A_2))$$

$$= P(A_1 \cap A_2^c) + P(A_1^c \cap A_2)$$

$$- P(\overbrace{(A_1 \cap A_2^c) \cap (A_1^c \cap A_2)}^{=\phi}) = 0$$

since $(A_1 \cap A_2^c) \cap (A_1^c \cap A_2)$

$$= A_1 \cap A_2^c \cap A_1^c \cap A_2$$

$$= \phi \qquad (\text{by } \phi \cap A = \phi)$$

$$= P(A_1 \cap A_2^c) + P(A_1^c \cap A_2)$$

$$= \frac{|A_1 \cap A_2^c|}{|S|} + \frac{|A_1^c \cap A_2|}{|S|}$$

$$= \frac{5}{36} + \frac{5}{36}$$

$$= \frac{10}{36}$$

How to count

Since $P_{nv}(A) = \frac{|A|}{|S|}$, we need to count the number of pebbles in event A and the number of pebbles in sample space S. We introduce several methods for counting.

Multiplicative rule

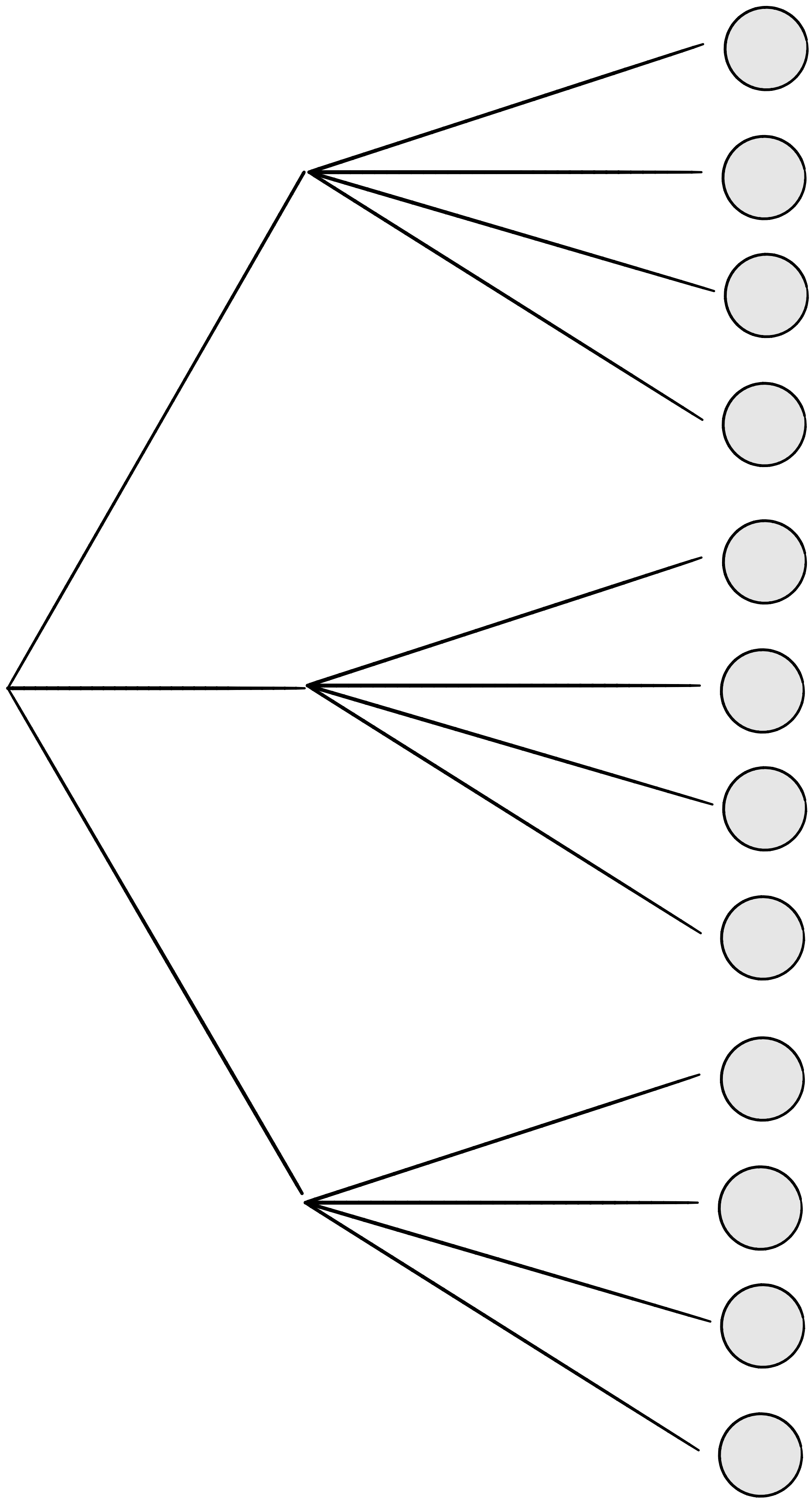
Multiplicative rule
- Sampling without replacement (SWOR)
- sampling with replacement (SWR)

Definition:

Consider a compound experiment consisting of two sub-experiments. Experiment A and

Experiment B, suppose A has a possible outcomes, and B has b possible outcomes. Then the compound experiment has $a \times b$ possible outcomes.

EX: If Experiment A has 3 possible outcomes and Experiment B has 4 possible outcomes then overall there are $3 \times 4 = 12$ possible outcome



Note: It is often easier to think about the experiments as being in chronological order, but there is no requirement in in the multiplicative rule that Experiment A has to be performed before Experiment B.

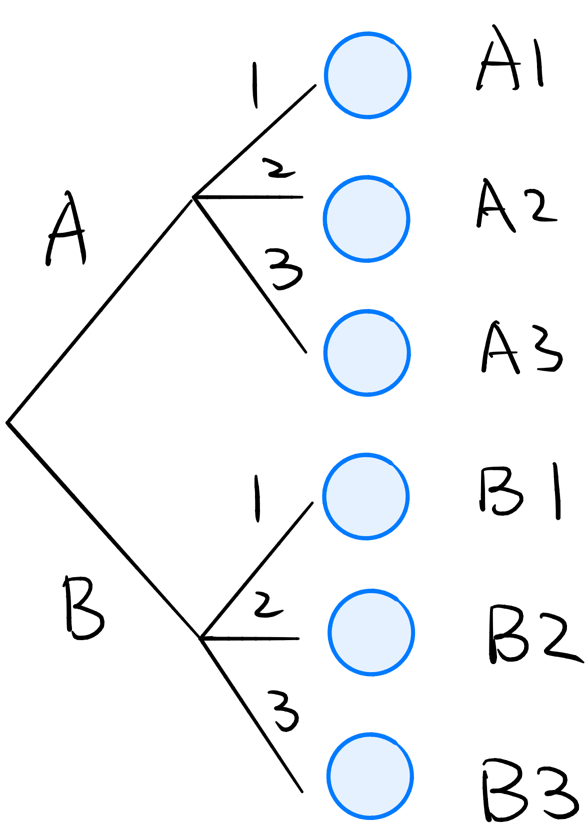
EX

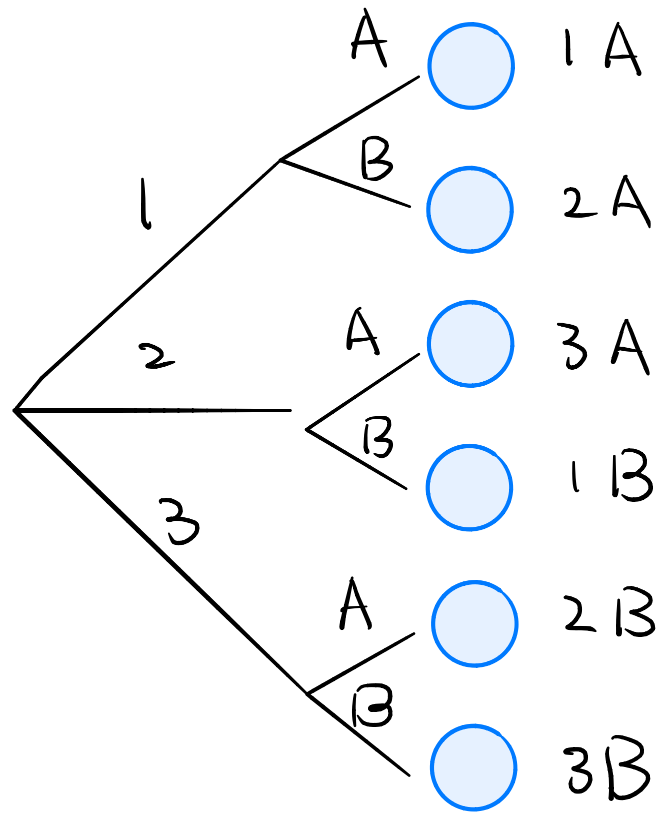
Buying an ice cream cone. You can choose the cone type $\{A, B\}$

cake cone   waffle cone

and flavor $\{1, 2, 3\}$

orange, vanilla, strawberry





By multiplicative rule

$2 \times 3 = 6$ possibilities

Note: Doesn't matter whether choose the type of cones first or flavor first.

$2 \times 3 = 3 \times 2 = 6$

Now suppose you buy two ice cream cones on a certain day, one in afternoon one in evening. for example

$\{A1, B2\}$

By multiplicative rule,

$$6 \times 6 = 36 \text{ possibilities}$$

EX

A set with n elements has 2^n subsets which include ϕ and the set itself.

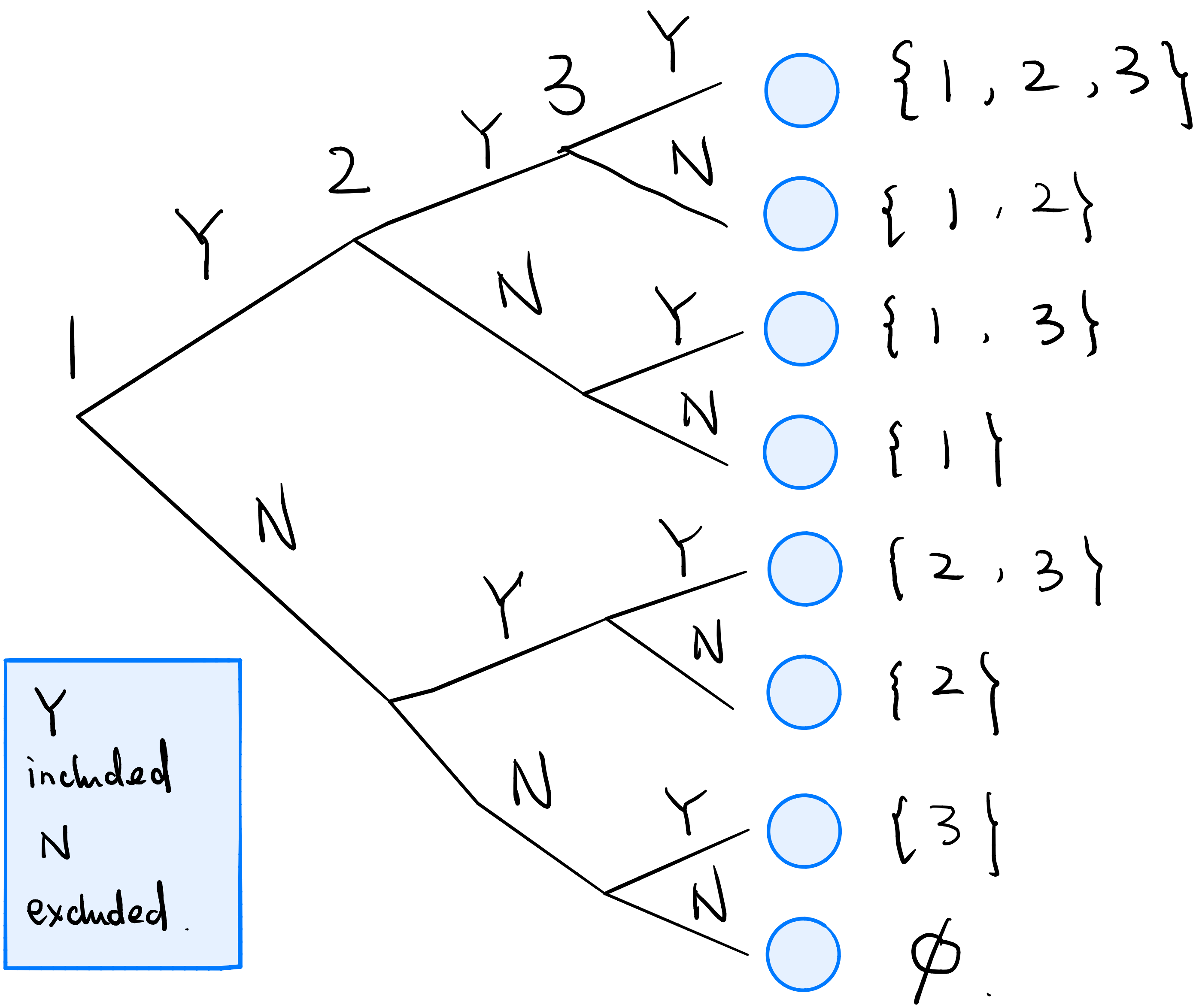
e.g. $S = \{1, 2, 3\}$. $n=3$

has $2^3 = 8$ subsets

$\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}$ $\{2, 3\}$ $\{1, 3\}$

$\{1, 2, 3\}$.

S contains 1, 2, 3. three elements, each element can be included or excluded in a certain subset.



$2 \times 2 \times 2 = 8$ subsets

Sampling with replacement (SWR)

Definition: Consider n objects and making k choices from them, one at a time with replacement (i.e. choosing a certain object doesn't preclude it from being chosen again)

Then there are

$$\underbrace{n \times n \times \cdots \times n}_{k} = n^k \text{ possible outcomes}$$

EX Toss a die twice, $k=2$, $n=6$

6×6 possible outcomes

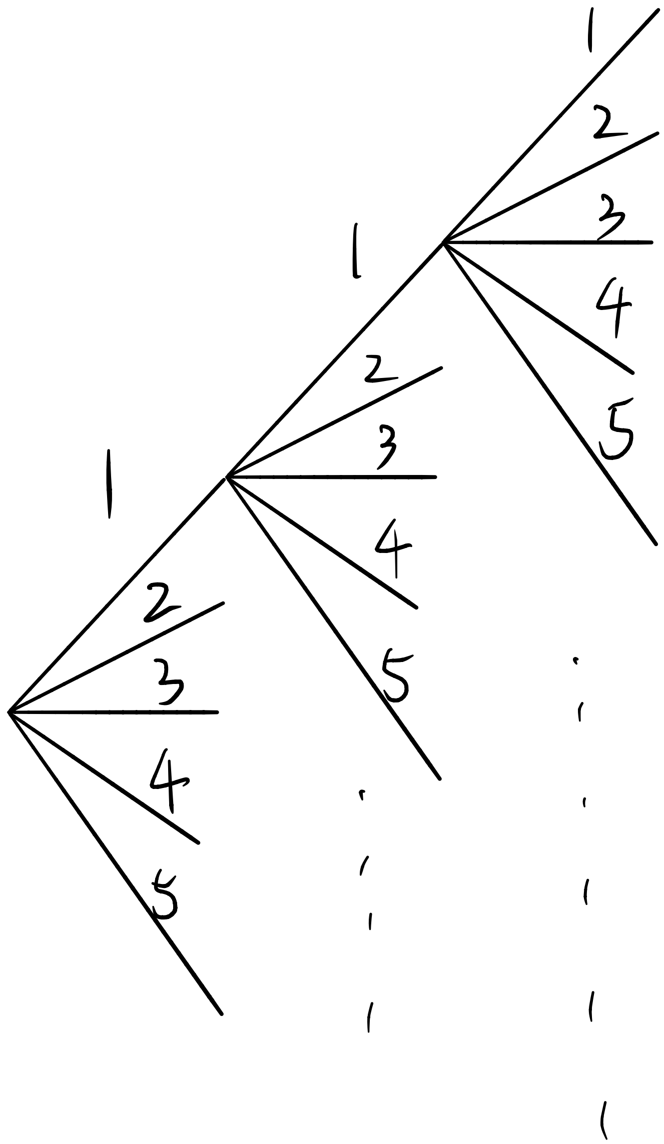
$$\underbrace{\{1,6\} \; \{2,6\} \; \cdots\cdots \; \{6,6\}}_{36 \text{ possibilities}}$$

<u>Ex</u> A jar with $n = 5$ balls, labled from 1 to 5. Sample $k=3$ balls one at a time with replacment.

(meaning that each time a ball is chosen it is returned to the jar)

Each sampling of the ball is a sub-experiment with $n=5$ possible outcomes, and there are $k=3$ sub-experiments. By SWR rule.

$(n=5)$ $\underbrace{5 \times 5 \times 5}_{k=3} = 125$ possible outcomes



$5 \times 5 \times 5 = 125$

(permutation rule)

Sampling without Replacement (SWOR)

Consider n objects and making k choices from them $(k \le n)$, one at a time without replacement (i.e. choosing a certain object precludes it from being chosen again)

Then there are

$$\underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}_{k \text{ sub-experiments.}} \quad (*)$$

possible outcomes.

Alternative formula for SWOR

$$(*) = \frac{n!}{(n-k)!}$$

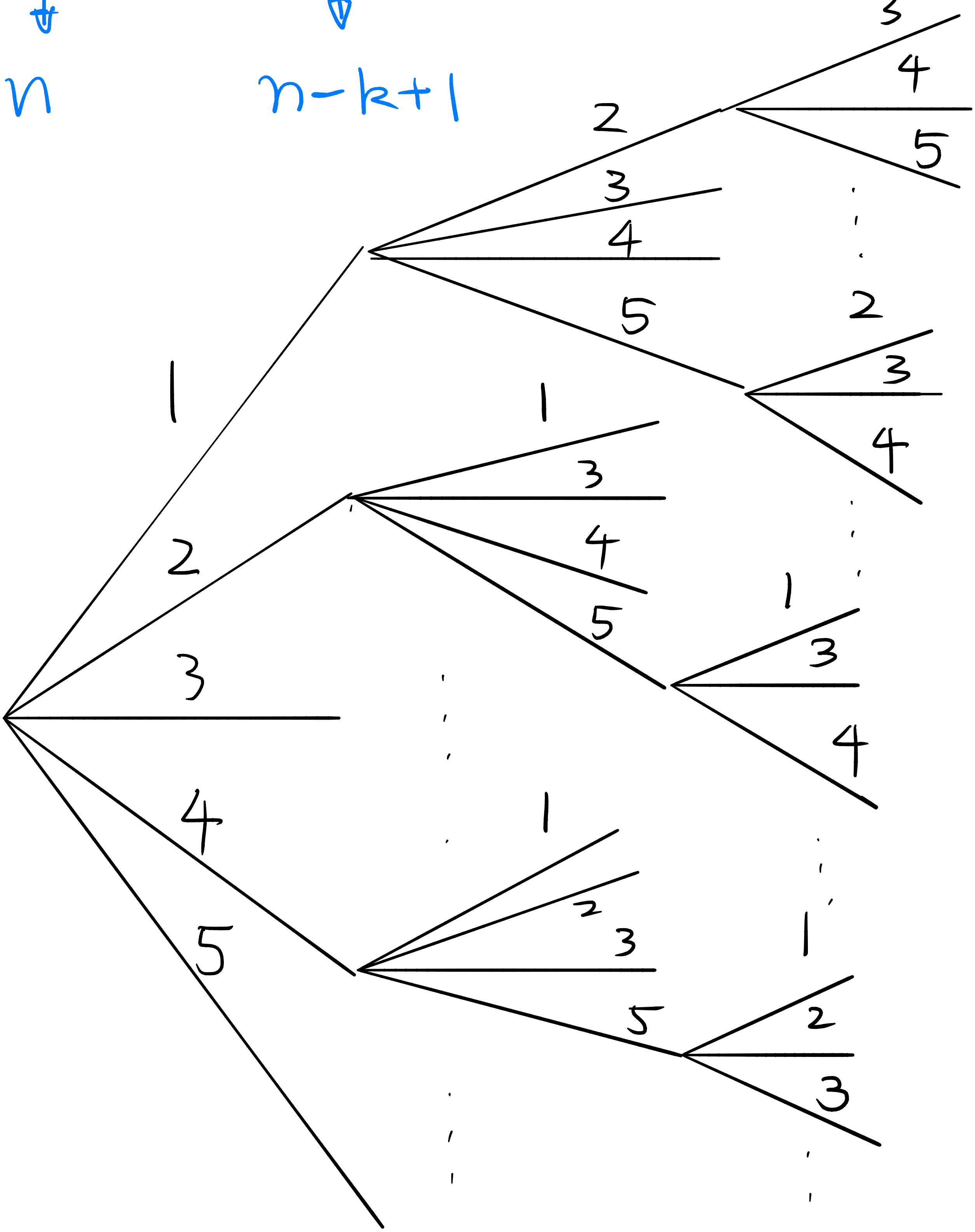
$$= \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) \cancel{\times (n-k) \times \cdots \times 2 \times 1}}{\cancel{(n-k) \times (n-k-1) \times \cdots \times 2 \times 1}}$$

$$= n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

Ex. There are $n=5$ balls in a jar labled from 1 to 5. we sample $k=3$ balls from the jar one at a time without replacement (each ball can be only chosen once)

Then there are

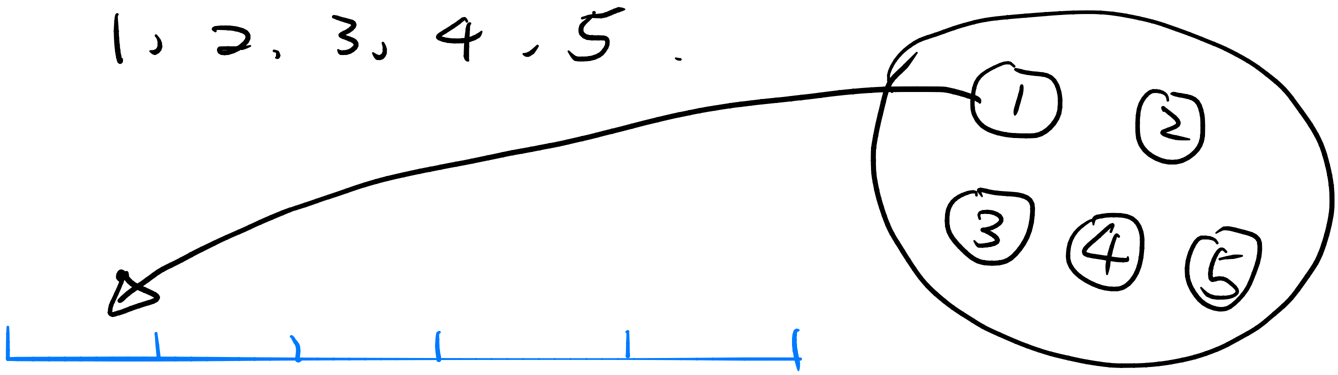
$$\underset{\downarrow\atop n}{5} \times 4 \times \underset{\downarrow\atop n-k+1}{3} = 60 \text{ possible outcomes}$$



EX (permutation)

A permutation of $1, 2, \dots n$ is an arrangment of them in some order.

e.g. $3, 5, 1, 2, 4$ is a permutation of $1, 2, 3, 4, 5$.



$$5 \times 4 \times 3 \times 2 \times 1 = 5!$$

$n = 5, \ k = 5$

$$n \times (n-1) \times (n-2) \cdots \times (n-k+1)$$

$$= 5 \times 4 \times \cdots \times 1$$

$$= 5!$$

Remember alternative formula.

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

$$= \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!$$

(Define $0! = 1$)

Ex (birthday problem)

There are $k \overset{=60}{}$ people in the room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude Feb 29). And their birthday is independent (no twins).

What is the probability that two or more people in the group have the same birthday?

2020 Calendar

January

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

February

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29

March

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

April

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30		

May

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

June

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

July

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

August

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

September

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

October

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

November

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

December

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Solution:

Lable the 365 days of a year using the numbers from 1 to 365. A possible outcome (sample point) of our experiment is a vector of length 60.

$$s = \underbrace{(300, 120, 11, 365, \dots, 5)}_{\text{length} = 60}$$

where each element of the vector represents the birthday of one of the 60 people.

The sample space S is the set that contain all possible values of s.

$$S = \{s_1, s_2, \dots\}$$

where each s_j is a vector of length 60.

Denote

$$A = \{\text{at least 1 birthday match in a group of } k = 60 \text{ people}\}$$

$P(A)$ is difficult to compute directly.

Denote the event

$$A^c = \{\text{no birthday match}\}$$

We can compute $P(A^c)$, then compute $P(A)$ by

$$P(A) = 1 - P(A^c)$$

$$P(A^c) = \frac{|A^c|}{|S|} = \frac{\left(\text{The \# of ways to assigne } n=365 \text{ birthdays to } k=60 \text{ people such that no two people share a birthday}\right)}{\left(\text{total \# of ways to assign } n=365 \text{ birthdays to } k=60 \text{ people}\right)}$$

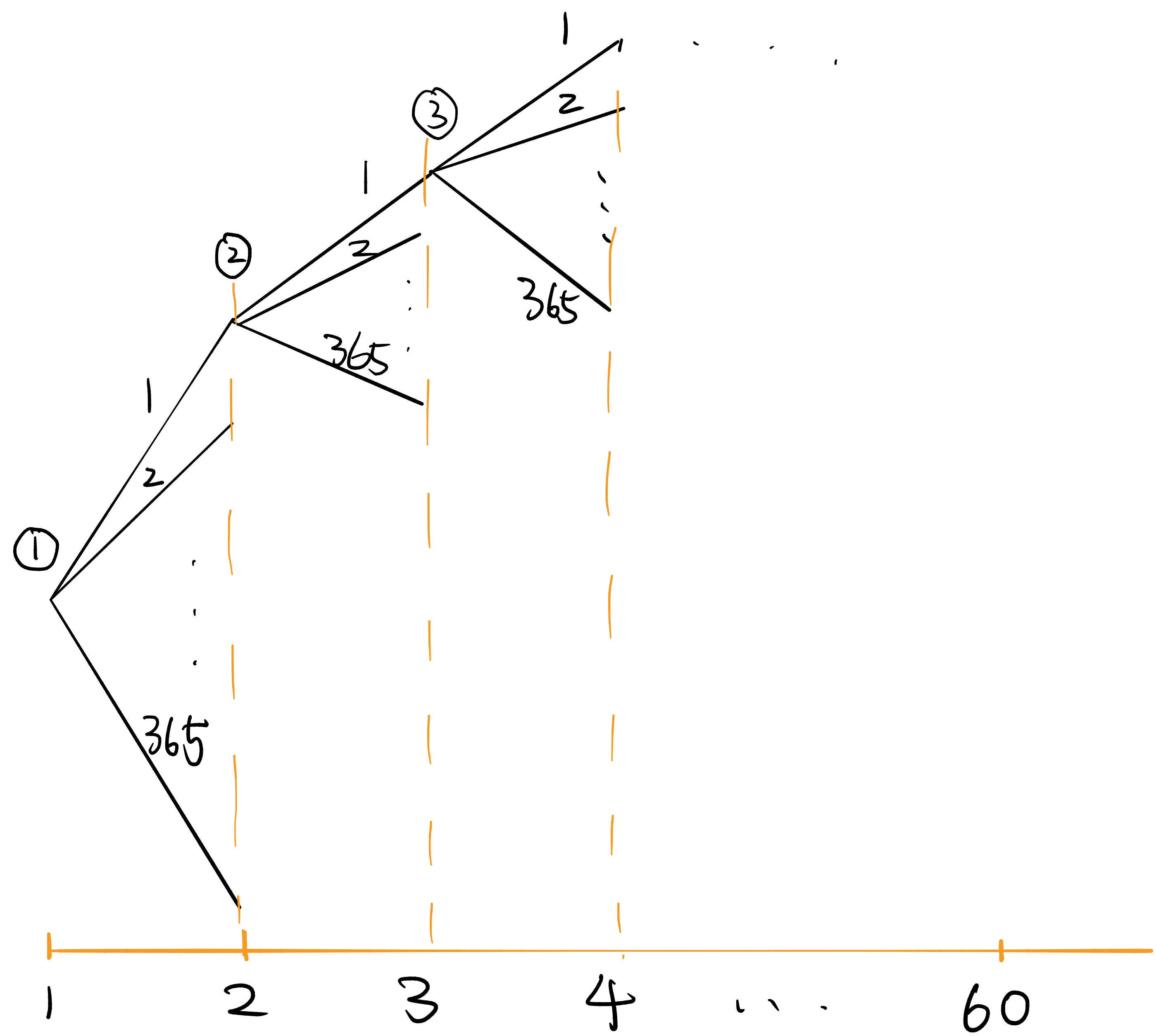
$|S|$ can be counted by using SWR rule

Assigning birthdays to 60 people can be viewed as performing 60 sub-experiments. in each of which a number between 1-365 is chosen and then assigned to a person.

This is equivalent to SWR, make $k=60$ choices from $n=365$ objects, one at a time with replacement.

$$|S| = 365 \times 365 \times \cdots \times 365 = 365^k$$

Diagram for counting $|S|$



$$365 \times 365 \times 365 \times 365 \times \cdots \times 365 = 365^k$$

$|A^c|$ can be counted by using SWOR rule

amounts to making $k=60$ choice from

$n=365$ objects, one at a time, without

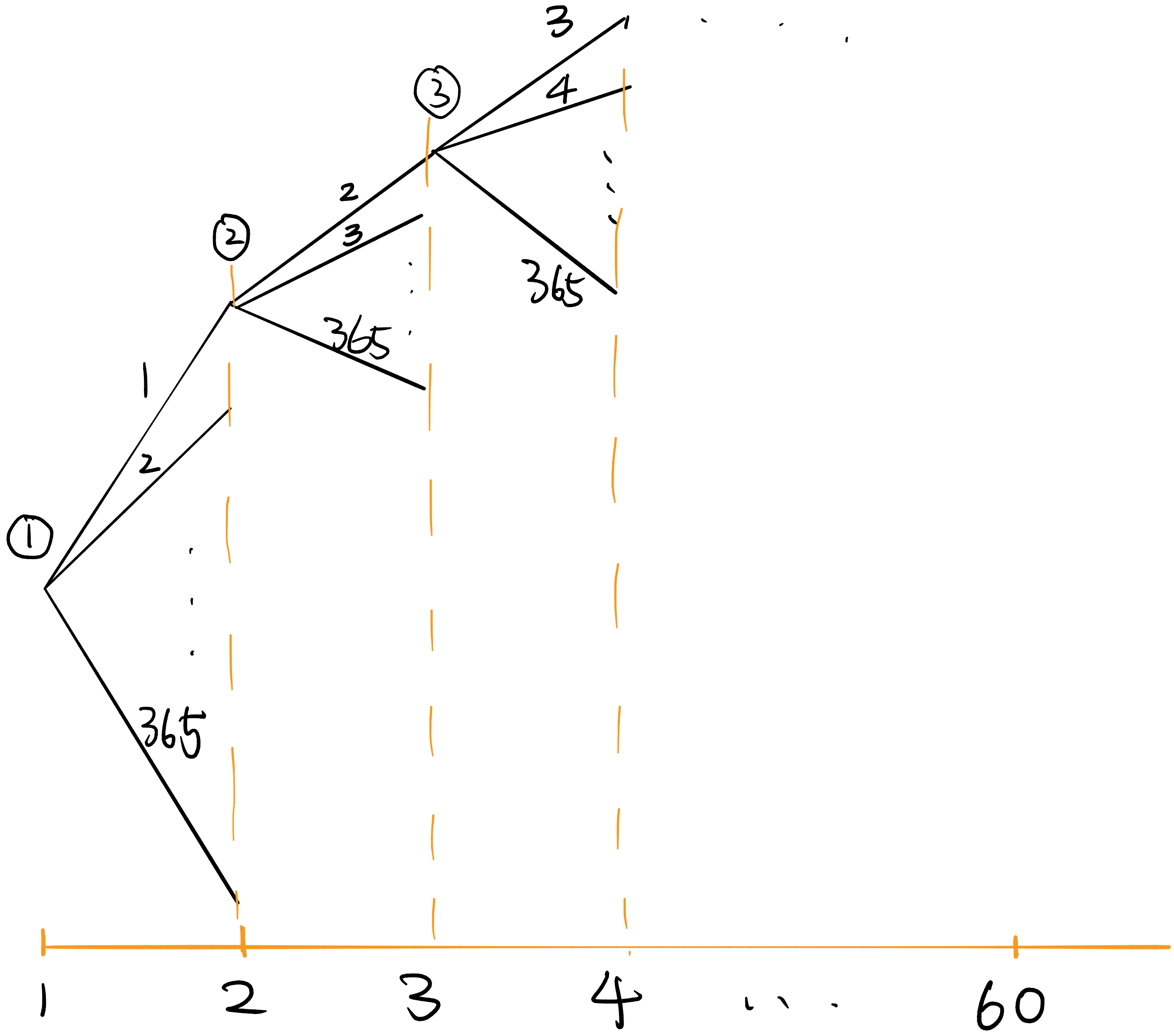
replacement (any date can be only chosen

once since there is no birthday match)

$$|A^c| = 365 \times 364 \times \cdots \times (365-k+1)$$

where $k=60$.

Diagram for counting $|A^c|$



$$365 \times 364 \times 363 \times 362 \times \cdots \times (365-60+1)$$

$$= \frac{365!}{(365-60)!}$$

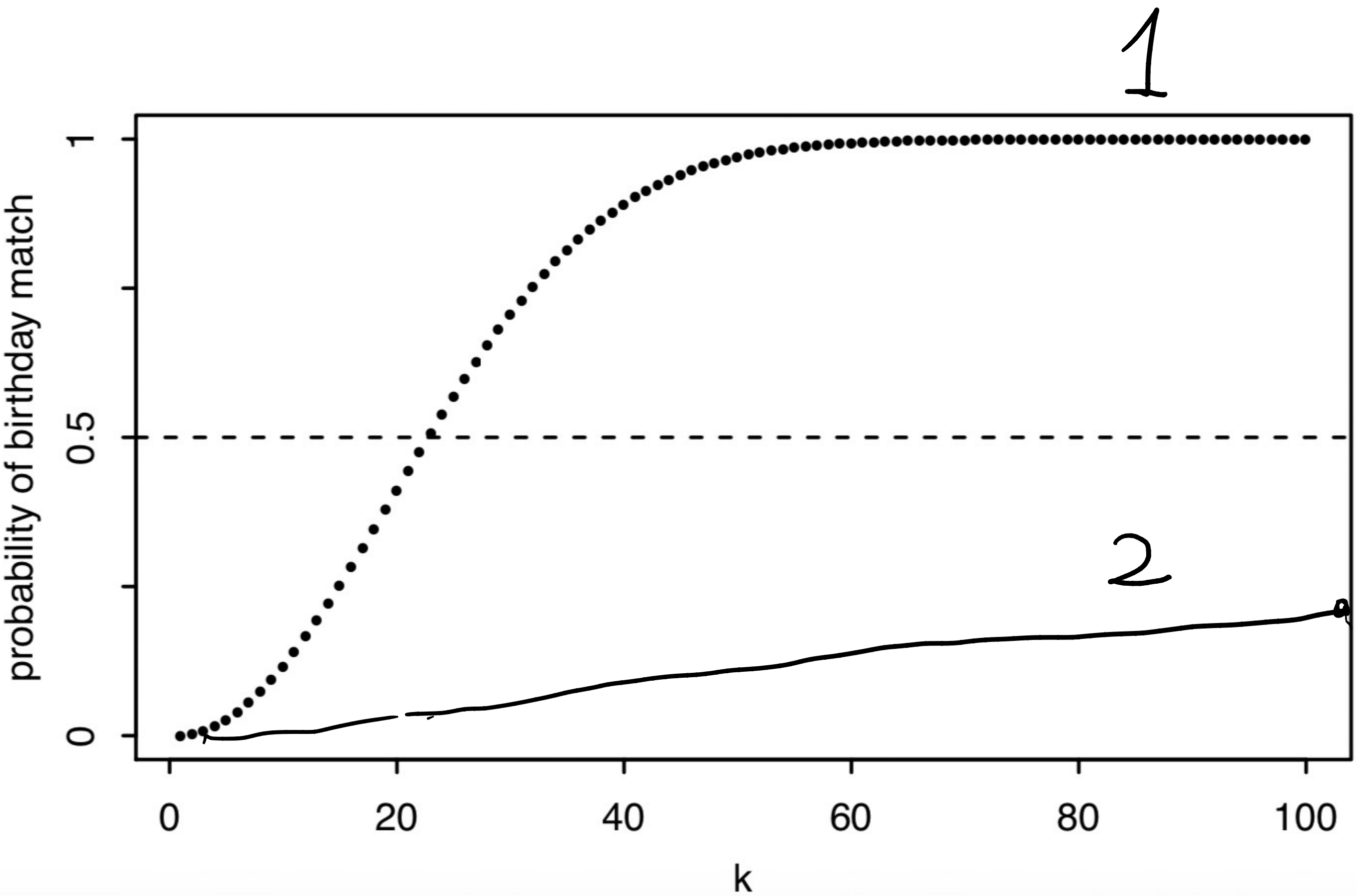
Therefore

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|S|}$$

$$= 1 - \frac{365 \times 364 \times \cdots \times (365-k+1)}{365^k}$$

$(k = 60)$

P(A) is a function of k.



When k = 23

P(A) > 0.5

When k = 60

P(A) > 0.99 !!!

Another birthday problem

There are $k=60$ people in the class. What is the probability that at least one of them has the same birthday as Yi's.

$A = \{\text{at least one birthday matches with Yi}\}$

$A^c = \{\text{no birthday match with Yi}\}$

$|S|$ is still 365^k

$|A^c|$ is 364^k since each person's birthday can be any day of the year except that it can't be the same as Yi's, each person has 364 options. SWR with $n=364$, $k=60$

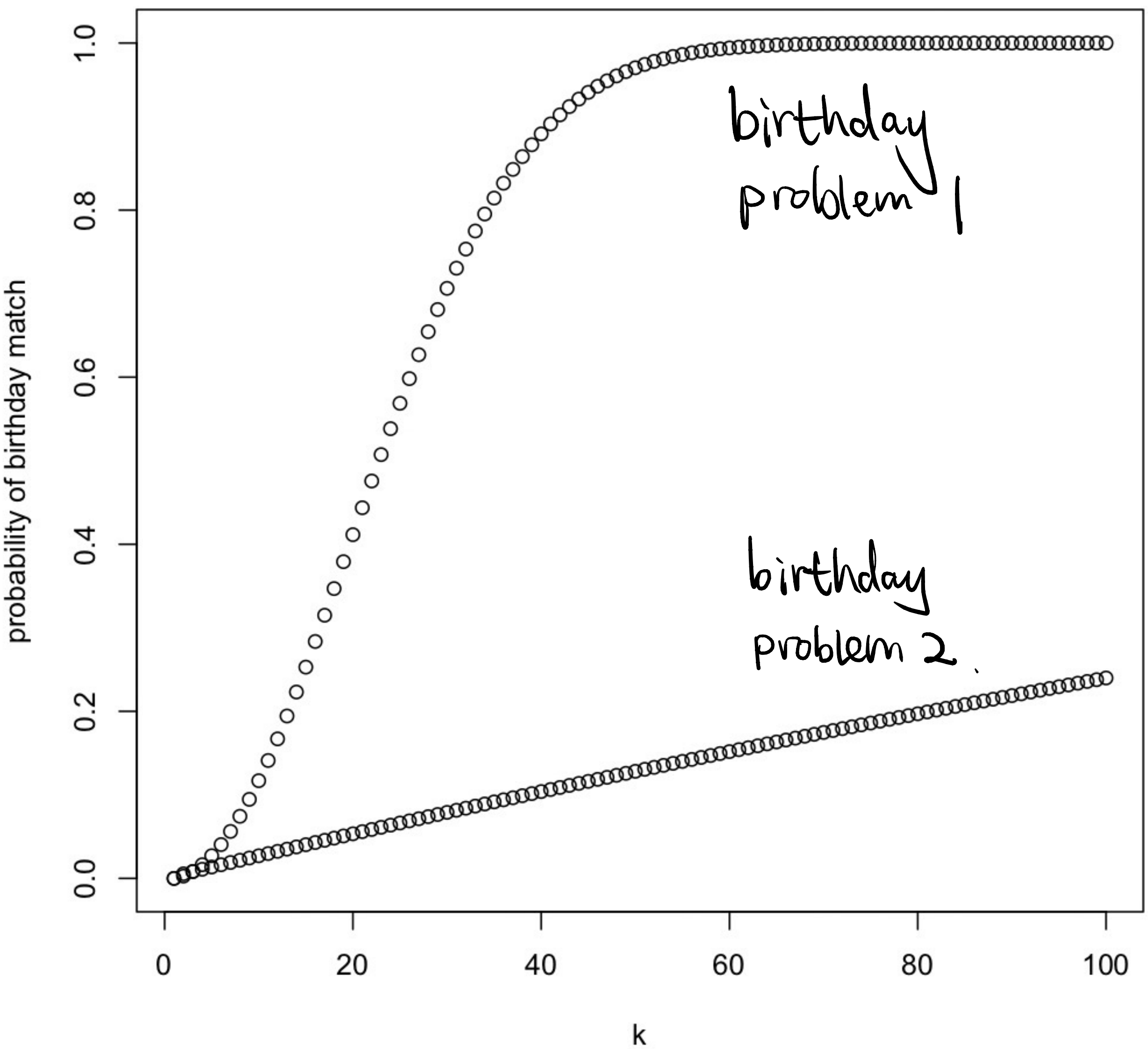
$$P(A) = 1 - P(A^c)$$

$$= 1 - \frac{|A^c|}{|S|} = 1 - \frac{364^k}{365^k}$$

$$= 1 - \left(\frac{364}{365}\right)^k$$

$k = 60$ $\quad$ $P(A) = 0.15$

$k = 365$ $\quad$ $P(A) = 0.63$


birthday problem 1
birthday problem 2
probability of birthday match
k
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100

Adjusting for overcounting (combination rule)

In many counting problem, it is not easy to directly count each outcome once and only once. If however. we are able to count each possibility exactly C time, then we can adjust by deviding by C

Ex (Committees and teams)

Consider a group of four people

$1, 2, 3, 4$

(a) How many ways are there to choose a two-person committee?

Method 1:

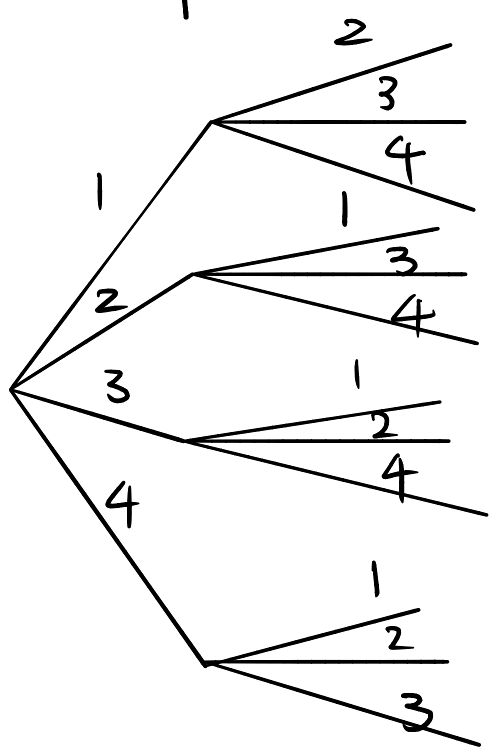
List them out

12, 13, 14, 23, 24, 34

$\underbrace{12,\ 13,\ 14,\ 23,\ 24,\ 34}_{6 \text{ ways}}$

Method 2:

Use multiplicative rule (SWOR)



(1, 2)
(1, 3)
(1, 4)
(2, 1)
(2, 3)
(2, 4)
(3, 1)
(3, 2)
(3, 4)
(4, 1)
(4, 2)
(4, 3)

$4 \times 3 = 12$

Since we choose 2 persons from 4, one at a time, without replacement, in order

the number of possible outcomes is $4 \times 3 = 12$ (SWOR), but $(1,2) = (2,1)$. Since we overcounted the problem by factor 2. thus the actual number of possibilitie are

$$\frac{12}{2} = 6 \text{ ways.}$$

(b) How many ways are there to break 4 people into 2 teams of 2.

Method 1: list them out

12	34	13	24	14	23

Method 2: use SWOR, then adjust

12 | 34

13 | 24

14 | 23

23 | 14

24 | 13

34 | 12

There are 6 ways to pick one team

But 12 | 34 = 34 | 12

So we adjust the overconnting by a factor of 2.

$$\frac{6}{2} = 3 \text{ ways.}$$

Binomial Coefficients

We count the number of ways to choose k objects out of n objects without replacement and without distinguishing between different orders in which they could be chosen. (SWORO), i.e. we count the number of subsets with size k for a set with size n.

Definition: (Binomial coefficient
Combination
SWORO)

For any nonnegative k and n binomial coefficient $\binom{n}{k}$, read "n choose k" is the number of subsets of size k for a set of size n

$$\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

For $k > n$, we set $\binom{n}{k} = 0$

Ex Consider a group of four people

1 , 2 , 3 , 4

the number of ways to choose a two-person committee is

$$\binom{n}{k} = \binom{4}{2} = \frac{4 \times 3}{2 \times 1} = 6$$

proof: (binomial coefficient)

SWOR

$$\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!}$$

overcounted adjust it by $k!$

$\binom{n}{k}$: # of subsets of size k, of a group of n people.

There are $n(n-1)\cdots(n-k+1)$ ways to make an ordered choice of k people without replace.
If the order doesn't matter, we overcounted each subset by a factor of $k!$
(There are $k!$ ways to order k objects)

$$\binom{n}{k} = \frac{\text{SWOR}}{k!}$$

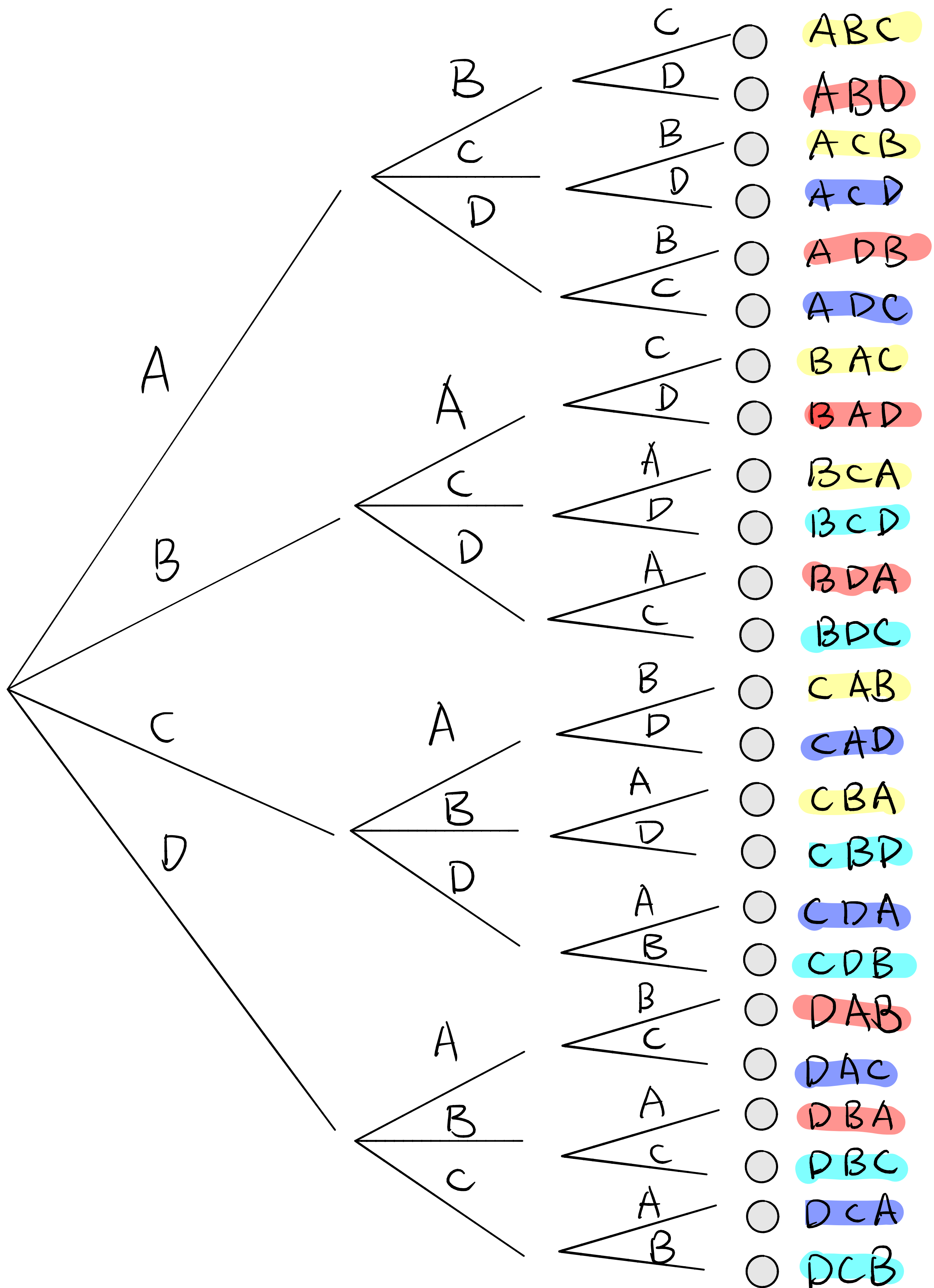
Ex (letter)

Draw a subset of size 3 from the set $\{A, B, C, D\}$, (without replacement
order doesn't matter)

The # of all possible subsets of size 3.

$n = 4 \quad k = 3$

$$\binom{n}{k} = \binom{4}{3} = \frac{4 \times 3 \times 2}{\underbrace{3 \times 2 \times 1}_{3!}} = 4$$



$$\frac{4 \times 3 \times 2}{3!} = 4$$

$3! = 6$	$3! = 6$	$3! = 6$	$3! = 6$
ABC	BCD	ABD	ACD
ACB	BDC	ADB	ADC
BAC	CBD	BAD	CAD
BCA	CDB	BDA	CDA
CAB	DBC	DAB	DAC
CBA	DCB	DBA	DCA

4 combinations.

Note:

For the computation of $\binom{n}{k}$, use

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{instead of} \quad \frac{n!}{(n-k)!\,k!}$$

$$\binom{100}{2} = \frac{100 \times 99}{2} = \cancel{\frac{100!}{98!\,2!}}$$

Ex Lotto

Pick-6 lottery, select 6 #'s from a set of numbers ranging from 1 - 53

There is only one winning combination.

What is the probability of winning the lotto?

$$P(\text{winning}) = \frac{\# \text{ of winng combination}}{\text{Total } \# \text{ of possible combination}}$$

$$= \frac{1}{\binom{53}{6}}$$

$$= \frac{1}{23 \text{ million}}.$$

Ex (Full house in poker)

Draw 5 cards out of 52 cards

Full house : ex. 3 7's . 2 10's

$$P(\text{full house}) = \frac{A}{\binom{52}{5}}$$

To compute A, we use the multiplicative rule.

$$= \frac{13\binom{4}{3}12\binom{4}{2}}{\binom{52}{5}}$$

$$= \frac{3744}{2598960} \simeq 0.00144.$$

Step 1	Step 2	Step 3	Step 4
Choose a # for 3-card set	choose 3 cards out of 4	choose a # for 2-card set	choose 2 cards out of 4
$13 \times$	$\binom{4}{3} \times$	$12 \times$	$\binom{4}{2}$

Multiplicative rule:

order doesn't matter

Step 1	Step 2	Step 3	Step 4
Choose a # for 2-card set	choose 2 cards out of 4	choose a # for 3 card set	choose 3 cards out of 4
$13 \times$	$\binom{4}{2} \times$	$12 \times$	$\binom{4}{3}$

Ex (permutation of a word)

How many ways to permute the letters in the word LALALAAA?

Just to choose where the 5 A's to go

$$\binom{8}{5}$$

or equivalently to decide where 3L to go

$$\binom{8}{3} = \binom{8}{5} = \frac{8 \times 7 \times 6}{3!} = 56$$

$$\binom{n}{k} = \binom{n}{n-k}$$

How about the word STATISTICS

3 S 3 T 2 I 1 C 1 A.

Method 1 :

(1) $$\underset{S}{\binom{10}{3}}\underset{T}{\binom{7}{3}}\underset{I}{\binom{4}{2}}\underset{C}{\binom{2}{1}}\underset{A}{\binom{1}{1}}$$

$$\underset{I}{\binom{10}{2}}\underset{S}{\binom{8}{3}}\underset{T}{\binom{5}{3}}\underset{A}{\binom{2}{1}}\underset{C}{\binom{1}{1}}$$

Method 2 :

STATISTICS

1 2 3 4 5 6 7 8 9 10

$$\frac{10!}{\underset{S}{3!}\,\underset{T}{3!}\,\underset{I}{2!}}$$

Ex Newton - Pepy problem.

A. at least one 6 appears when 6 fair dice are rolled.

B. at least two 6 appears 12 dice

C. at least three 6 appears 18 dice

Solution:

(1) $A = \{\text{at least one } 6\}$

$A^c = \{\text{getting no } 6\}$

$$P(A) = 1 - P(A^c) = 1 - \frac{5^6}{6^6}$$

$$= 0.67$$

(2) $B = \{\text{at least two } 6,\ 12 \text{ dice}\}$

$B^c = \{\text{get no } 6\text{'s or get exactly one } 6\}$

$$P(B) = 1 - P(B^c) = 1 - \frac{5^{12} + \binom{12}{1} 5^{11}}{6^{12}}$$

$= 0.62$

(3) $C = \{\text{at least three } 6\text{'s},\ 18 \text{ dice}\}$

$C^c = \{\text{get zero, one or two } 6\text{'s}$

$\text{in } 18 \text{ dice}\}$

$$P(C) = 1 - P(C^c) = 1 - \frac{5^{18} + \binom{18}{1} 5^{17} + \binom{18}{2} 5^{16}}{6^{18}}$$

$= 0.60$

Summary:

Sampling Table:

Choose k objects out of n.

	order matters	order doesn't.
replace	n^k $(k \geq 1)$ k could be greater n	 $\binom{n+k-1}{k}$
Don't replace	$n \times (n-1) \times \cdots \times (n-k+1)$ $(k \leq n)$	$\binom{n}{k}$ $(k \leq n)$

Non-naive definition of probability

Definition

A probability space consists of S, P.

S is the sample space

P is the probability function

input : events $A \subseteq S$

ouput : $P(A) \in [0, 1]$

The function P must satisfy the following

2 axioms:

(1) $0 \le P(A) \le 1$ for any $A \subseteq S$

special cases :

$$P(\phi) = 0 \quad , \quad P(S) = 1$$

(2) If $A_1, A_2 \ldots$ are disjoint events

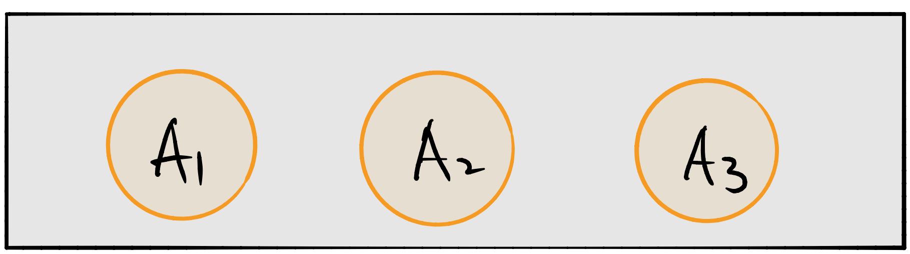
then

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

special case:

$$P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} P(A_j)$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) \qquad (n=2)$$



Probability rules

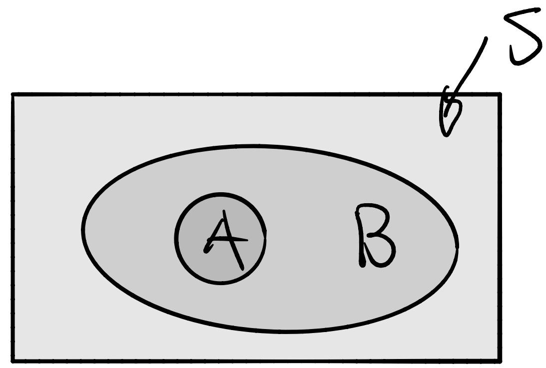
(1) $P(A^c) = 1 - P(A)$

proof: A and A^c are disjoint.

$$1 \overset{(1)}{=} P(S) = P(A \cup A^c) \overset{(2)}{=} P(A) + P(A^c)$$

(2) If $A \subseteq B$

$$P(A) \leq P(B)$$



proof:

$$B = \underbrace{A \cup (B \cap A^c)}_{\text{disjoint}}$$

$$P(B) = P(A \cup (B \cap A^c)) = P(A) + \overbrace{P(B \cap A^c)}^{\geq 0}$$

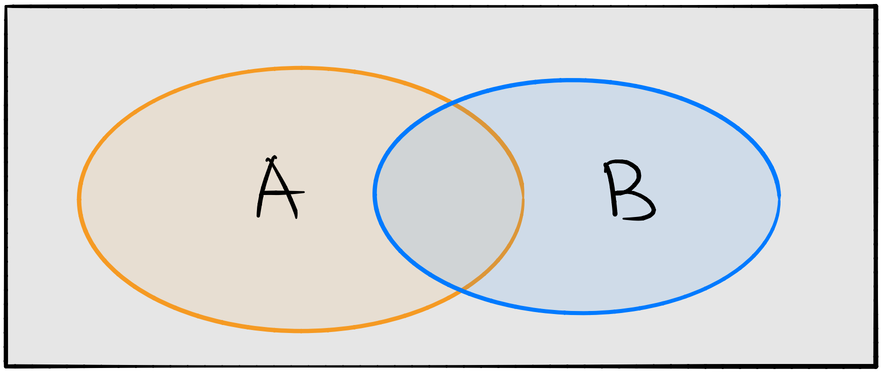
Then $P(B) \geq P(A)$

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

proof:

disjoint

$$P(A \cup B) = P(A \cup (B \cap A^c))$$



Axiom 2

$$= P(A) + P(B \cap A^c)$$

$$\stackrel{?}{=} P(A) + P(B) - P(A \cap B)$$

It is suffice to show that .

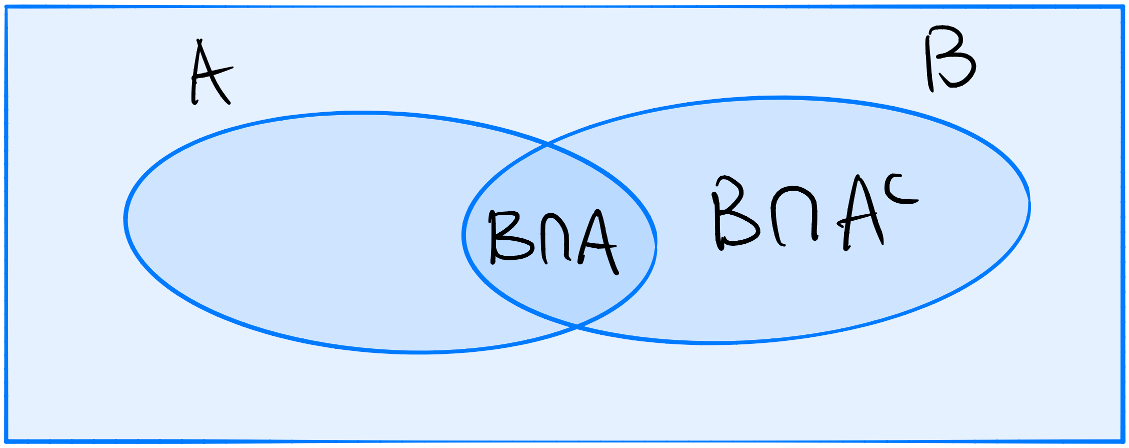
$$P(B \cap A^c) = P(B) - P(A \cap B)$$

$$\Leftrightarrow P(B) = P(B \cap A^c) + P(B \cap A)$$

$$\Leftrightarrow P(\overbrace{(B \cap A) \cup (B \cap A^c)}^{= B}) = P(B \cap A^c) + P(B \cap A)$$

Since $B = (B \cap A) \cup (B \cap A^c)$

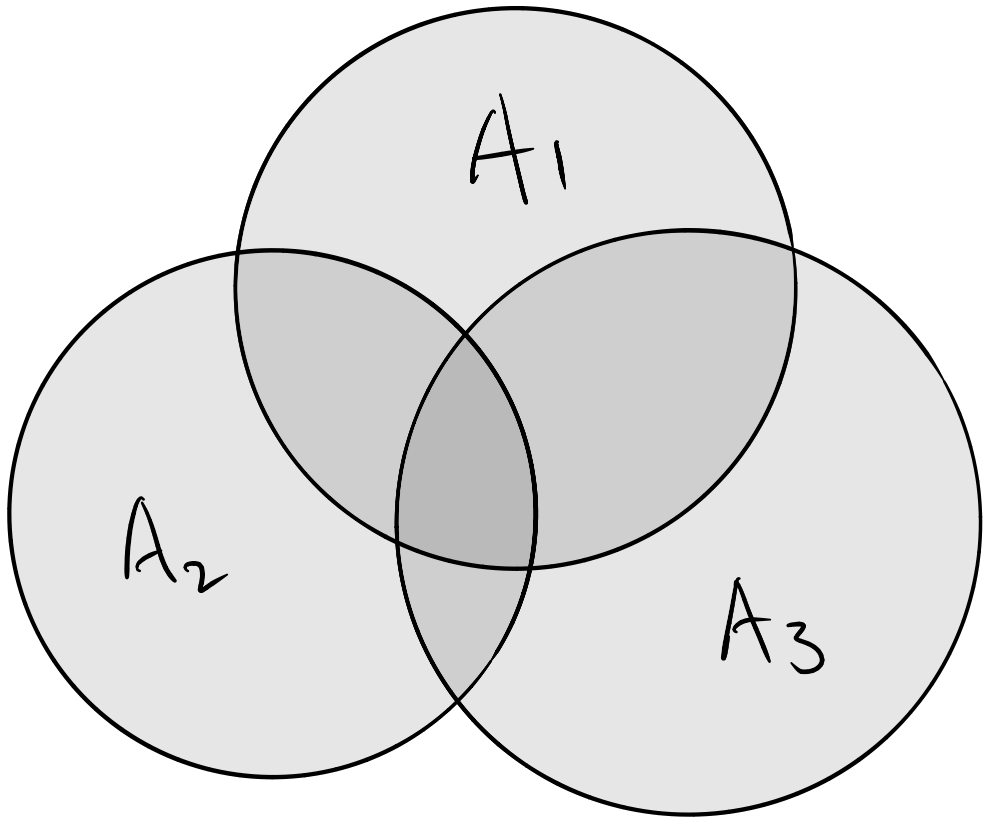
disjoint .



(4) Inclusion-exclusion rule

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$



$$P(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) -$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

EX (de Montmort's match problem)

1, 2, ..., n cards.

Define A_i be the event that the ith card has the number i written on it (match).

$$P(\text{Winning}) = P(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

$\ldots$

$$P(A_1 \cap A_2 \cap \cdots A_n) = \frac{1}{n!}$$

There are

$\binom{n}{1}$ terms of $P(A_i)$

$\binom{n}{2}$ terms of $P(A_i \cap A_j)$

$\binom{n}{3}$ terms of $P(A_i \cap A_j \cap A_k)$

$\vdots$

$\binom{n}{n}$ term of $P(A_1 \cap A_2 \ldots \cap A_n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \binom{n}{1} \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{1}{n(n-1)}$$

$$+ \binom{n}{3} \frac{1}{n(n-1)(n-2)} - \cdots + (-1)^{n+1} \cdot 1 \cdot \frac{1}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \cdot \frac{1}{n!}$$

$$\xrightarrow{n \to \infty} 1 - e^{-1}$$

Independence

Definition:

Events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Note: Completely different from disjointness

Ex1 : Independent events.

$A = \{$Tom'll call me today$\}$

$B = \{$Yi'll call me today$\}$

$$P(A \cap B) = P(A) \cdot P(B).$$

If $P(A) \cdot P(B) \neq 0$. $P(A \cap B) \neq 0$.

Thus A and B are not disjoint

unless one of them is $\emptyset$

Ex 2

Disjoint events

$A = \{$ Tom 'll call me exactly once Today $\}$

$B = \{$ Tom 'll call me more than once today $\}$

$$A \cap B = \phi.$$

But $P(A \cap B) = 0 \neq P(A) \cdot P(B)$

Thus A and B are not indgpendent unless one of them is ϕ.

Disjoint: if A occurs then B cannot possibly occur

Independent: If A occurs, it tells us nothing what so ever whether B occurs.

To generalize the notion of independence to 3 events

A B C are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap C) = P(A) \cdot P(C)$$

$$P(B \cap C) = P(B) \cdot P(C)$$

$$P(A \cap B \cap C) = P(A) P(B) P(C)$$

Independence : A and B are independent.

$$P(A \cap B) = P(A)P(B)$$

Remember

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B)$$

$$P(A) + P(B)(1 - P(A))$$

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

when A and B are independent.

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{=0}$$

when A are B are disjoint.

Conditional Probability

Ex: $P(\text{date the next person})$

$P(\text{date} \mid A, B, c)$

How should you update prob / beliefs / uncertainty based on new evidence.

A - billionaire

B - lost all money recently

c - to save your life.

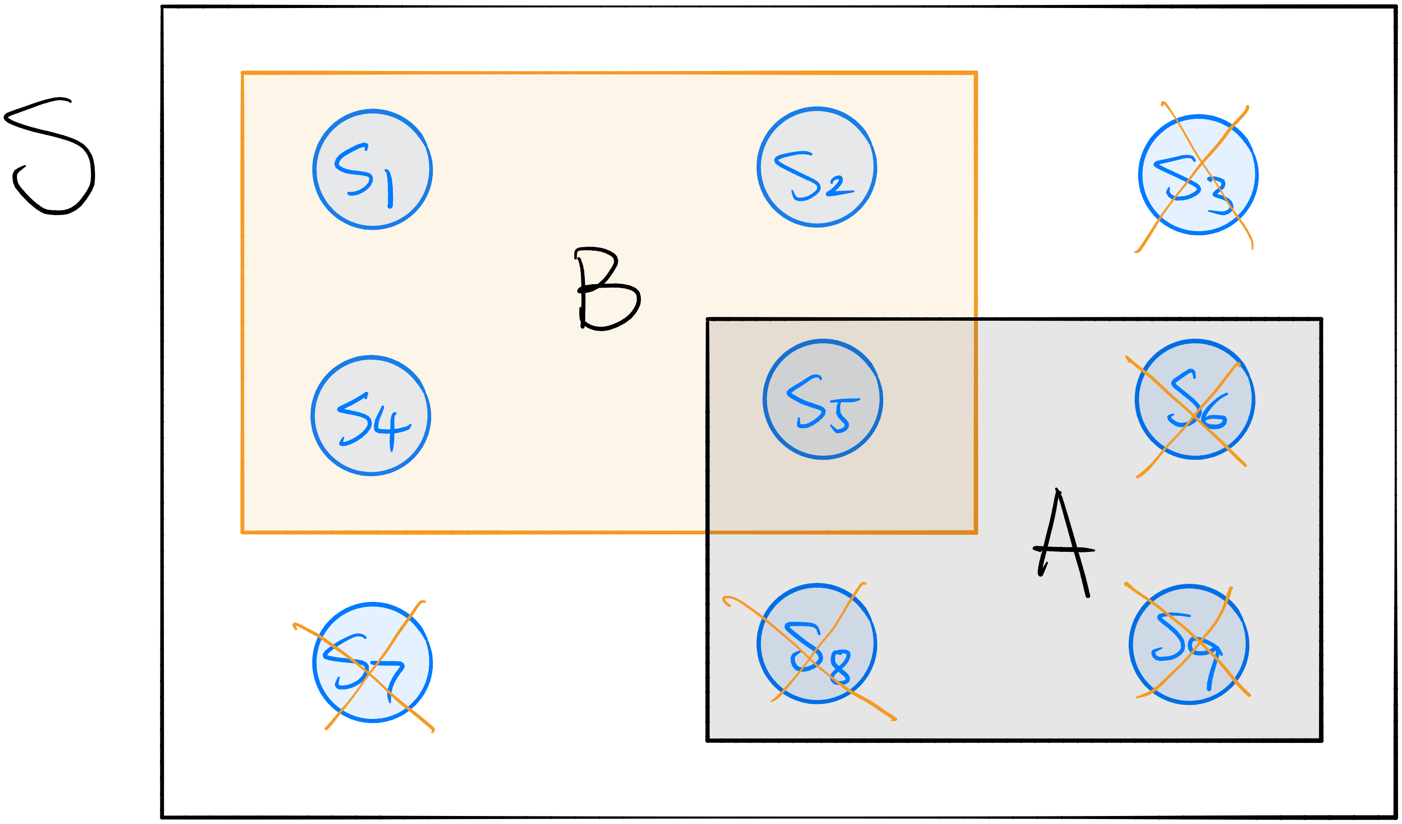
Definition:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

If $P(B) = 0$ $\quad P(A \mid B) = 0$.

Intuition: Pebble world

Originally $P(A) = \frac{4}{9}$



$$P(s_1) = P(s_2) = \cdots = P(s_9) = \frac{1}{9}$$

The total mass is 1.

Why $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{9}}{\frac{4}{9}}$

$P(A \mid B)$ is the prob that A occurs given that B occurred.

conditional on

(1) Conditioned on B means get rid of

pebbles in B^c

(2) Universe now restricted to B

Pebbles in this new universe don't have total mass 1

$$P(S_1) + P(S_2) + P(S_4) + P(S_5) = \frac{4}{9}$$

(3) Divide the prob. of any event in the new universe B by $P(B)$ to make the total mass 1 again.

(renormalization)

e.g. $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{9}}{\frac{4}{9}} = \frac{1}{4}$$

$$= P(S_5|B)$$

After normalization, in the new universe B

$$P(S_1|B) + P(S_2|B) + P(S_4|B) + P(S_5|B) = 1$$

Total mass is 1 again.

For example if $A = B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(A)} = 1$$

Thm 1

By definition $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B) \cdot P(B)$$

$$P(A \cap B) = P(B|A)\, P(A)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

If A and B are independent.

$$P(A \cap B) = P(A|B) \cdot \cancel{P(B)} = P(A) \cdot \cancel{P(B)}$$

$$P(A|B) = P(A)$$

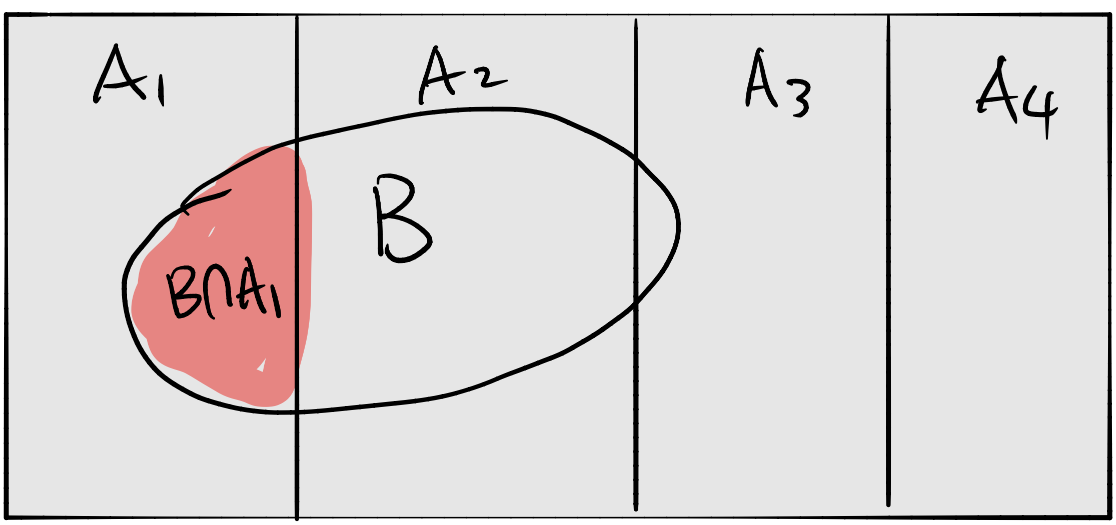
Thm 2 (Bayes' rule)

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$$P(A|B)\cdot P(B) = P(A \cap B) = P(B|A)\cdot P(A)$$

divide by $P(B)$ on both sides.

Thm 3 (Law of total probability)



Given $A_1, A_2 \cdots A_n$ a partation of S

$$A_1 \cap A_2 \cdots \cap A_n = \phi$$

$$A_1 \cup A_2 \cdots \cup A_n = S.$$

$$P(B) = P(B \cap A_1) + P(B \cap A_2) \cdots + P(B \cap A_n)$$

$$= P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)$$

Thm 4

$$P(A_1, \ldots, A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2)$$

$$\cdots P(A_n|A_1, \ldots, A_{n-1})$$

Ex Roll 6 dice and what is the prob. of getting 6 1's.

$A_j = \{ \text{the } j\text{th die is } 1 \}$

$$P(A_1 \cap A_2 \cap \cdots A_6) = P(A_1) P(A_2) \cdots P(A_2)$$

$$= \left(\frac{1}{6}\right)^6$$

or use naive definition of prob.

$$P(A_1 \cap A_2 \cap \cdots \cap A_6) = \frac{|A_1 \cap A_2 \cap \cdots \cap A_6|}{|S|}$$

$$= \frac{1}{6^6}$$

Ex Given a family with four children

bbbb, bgbg or gggg.

In Canada 105 vs 100

0.51 vs 0.49

boy girl

$$P(bbbb) = (0.51)^4$$

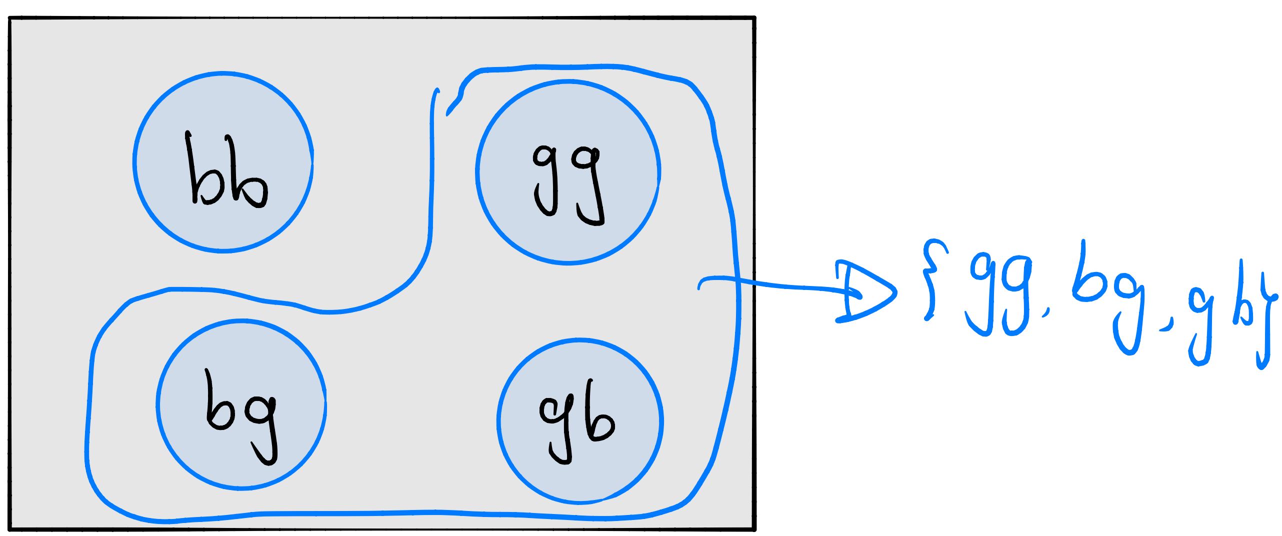
$$P(bgbg) = (0.51)^2(0.49)^2$$

$$P(gggg) = (0.49)^4$$

Ex A family has two children

it is known that at least one of the two is a girl, then what is the prob that both girls? What if it is known that the young child is a girl?

Solution: $P(\text{girl}) = P(\text{boy}) = \frac{1}{2}$



$P(gg \mid \text{at least one girl})$

$= P(gg \mid \{gg, bg, gb\})$

$$= \frac{P(\{gg\} \cap \{gg, bg, gb\})}{P(\{gg, bg, gb\})}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}$$

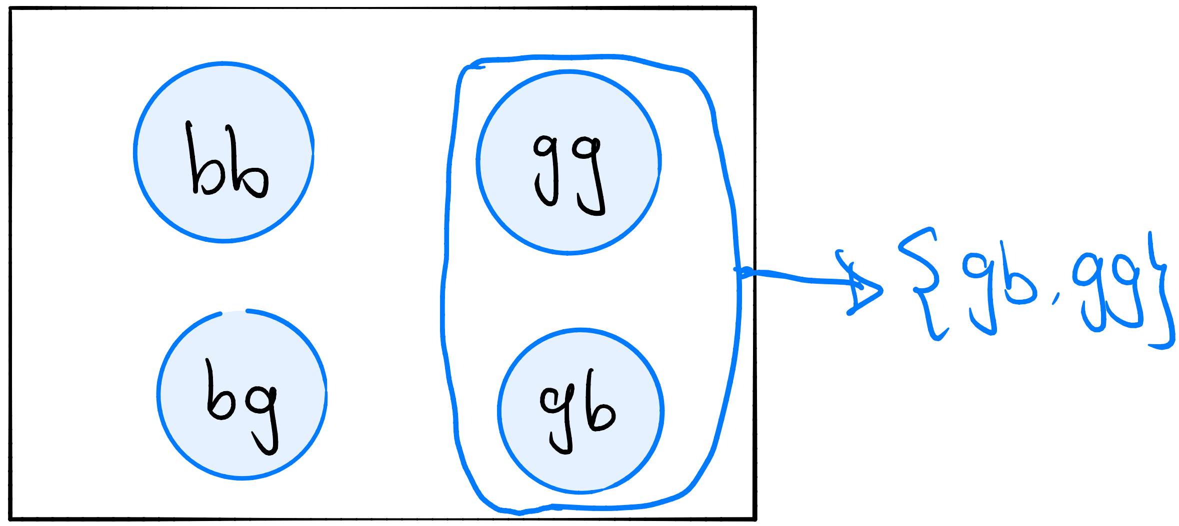
on the other hand

$$P(gg \mid \text{the younger child is a girl})$$

$$= P(gg \mid \{gb, gg\})$$

$$= \frac{P(\{gg\} \cap \{gb, gg\})}{P(\{gb, gg\})} = \frac{P(\{gg\})}{P(\{gb, gg\})}$$

$$= \frac{1/4}{1/2} = \frac{1}{2}$$



Ex

It is known that at least one of two is a girl born in winter

$P(\text{both girls} \mid \text{at least one winter girl})$

Assume $P(\text{boy}) = P(\text{girl}) = \frac{1}{2}$

$$P(Sp) = P(Su) = P(Au) = P(Wi) = \frac{1}{4}$$

$P(\text{both girls} \mid \text{at least one winter girl})$

$$= \frac{P(\{\text{both girl}\} \cap \{\text{at least one winter girl}\})}{P(\{\underbrace{\text{at least one winter girl}}_{A}\})} \quad (*)$$

$$P(A) = 1 - P(\text{no winter girls})$$

$$= 1 - \left(\frac{7}{8}\right)^2$$

Sp G	Su G	Sp B	Su B
Au G	Wi G	Au B	Wi B

$$P(\underbrace{\{\text{both girl}\} \cap \{\text{at least one winter girl}\}}_{A \cap B})$$

$$= P(\{\text{both girl}\} \cap \{\text{at least one winter child}\})$$

$$= P(\text{both girl})\, P(\text{at least one winter child})$$

$$= \left(\frac{1}{2}\right)^2 \times \left(1 - P(\text{both are non-winter})\right)$$

$$= \left(\frac{1}{2}\right)^2 \times \left(1 - \left(\frac{3}{4}\right)^2\right)$$

$$(*) = \frac{P(A \cap B)}{P(B)} = \frac{7}{15} > \frac{1}{3}$$

Ex Patient gets tested for diesease, which afflicts 1% of population, tests positive

Suppose test as advertised as "95% accurate"

suppose this means

D : patient has diesease.

T : patient tests positive.

$$P(T|D) = 0.95 = P(T^c|D^c)$$

Patient interested in $P(D|T)$

use Bayes' rule :

$$P(D|T) = \frac{\overset{0.95}{P(T|D)} \cdot \overset{0.01}{P(D)}}{P(T)}$$

$$= 0.16$$

Since by the Law of total probability.

$$P(T) = P(T \cap D) + P(T \cap D^c)$$

$$= P(T|D)P(D) + P(T|D^c)P(D^c)$$

$$= 0.95 \times 0.01 + 0.05 \times 0.99$$

$$= 0.059$$

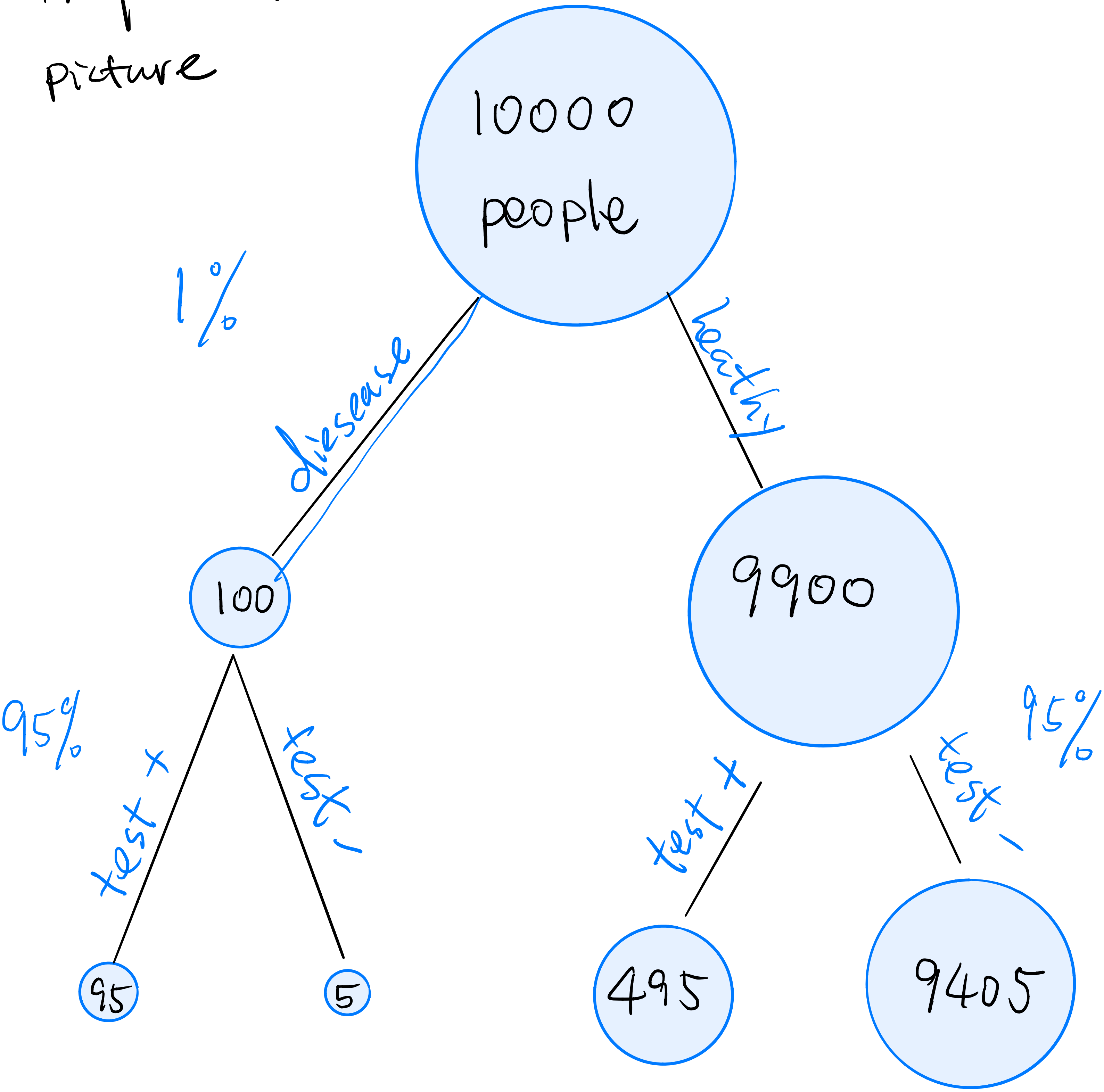
where

$$P(T|D^c) = 1 - P(T^c|D^c)$$

$$= 1 - 0.95$$

$$= 0.05$$

Frequentist picture



$$P(D|T) = \frac{P(D \cap T)}{P(T)}$$

$$= \frac{95/10000}{(95+495)/10000}$$

$$= \frac{95}{95+495} = 0.16$$

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Children

Sally Clark, mother wrongly convicted of killing her sons, found dead at home

- Family says she never recovered from court case
- Cause of death to be determined by coroner

Thair Shaikh
Sat 17 Mar 2007 09.22 GMT

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Sudden Infant Death Syndrome.

(SIDS) $\frac{1}{8500}$ chance.

Timeline

December 1996 Sally Clark's son Christopher, aged 11 weeks, is found dead while her husband is out

January 1998 Her second son, Harry, dies, aged eight weeks

February 1998 Mrs Clark is arrested

October 1999 Mrs Clark's trial begins at Chester crown court. Professor Roy Meadow appears as a witness, telling the jury there is a "one in 73m" chance of two children dying from cot deaths in an affluent family

November 1999 Mrs Clark is found guilty and given two life sentences

October 2000 First appeal fails

January 2003 Mrs Clark's conviction quashed by the court of appeal

March 2007 Sally Clark dies

Prosecutor's fallacy

people often confuse P(A|B) with P(B|A)

A : innocence

B : evidence

Sally Clark case

Witness : Sudden Infant Death Syndrom (SIDS)

$$P(\underbrace{2 \text{ SIDS death}}_{\text{evidence}} \mid \text{innocence}) = \left(\frac{1}{8500}\right)^2$$

$$= \frac{1}{73m}$$

Witness claimed the probability of Clark's innocence was 1 in 73 million.

Fallacy 1 :

$$\frac{1}{8500} \cdot \frac{1}{8500}$$

that assume independence.

Since we have

$$P(A \cap B) = P(A) \cdot P(B)$$

only if A and B are independent.

Fallacy 2 :

Witness confused $P(\text{evidence} \mid \text{innocence})$ with

$P(\text{innocence} \mid \text{evidence})$

witness calculated:

$$P(2 \text{ SIDS deaths} \mid \text{innocence})$$

$$= P(\text{evidence} \mid \text{innocence})$$

But our concern is actually:

$$P(\text{innocence} \mid \text{evidence})$$

$$= \frac{P(\text{evidence} \mid \text{innocence})\, P(\text{innocence})}{P(\text{evidence})}$$