Duality

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- **Primal problem:**

$$\begin{aligned} &\min_{x} \ f(x) \\ &\text{subject to } h_i(x) \leq 0, \ i = 1, \ldots, m \\ &\qquad\qquad\quad \ell_j(x) = 0, \ j = 1, \ldots, r, \end{aligned}$$

  so we have m inequality constraints and r equality constraints. In unconstrained problems, we have $m = r = 0$. The above says that we have a properly defined optimization problem to solve.

- **Primal feasible:** x is primal feasible if it satisfies

$$\begin{aligned} h_i(x) &\leq 0, \ 1 \leq i \leq m \\ \ell_j(x) &= 0, \ 1 \leq j \leq r. \end{aligned}$$

  Let C denotes the set of x's that are primal feasible, then C is called a **primal feasible set.**

- **Primal optimal:** define primal optimal $x^\star$ as

$$x^\star = \arg\min_{x \in C} f(x)$$

  Denote

$$f^\star = f(x^\star).$$

- **Lagrangian:**

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x). \qquad u \in \mathbb{R}^m,\ u \geq 0$$

  Important property: for each feasible x

$$f(x) \geq L(x, u, v).$$

- **Lagrangian dual function:**

$$g(u, v) = \min_x L(x, u, v).$$

Proposition 1. *$g(u, v)$ is concave.*

Proof. For any (u_1, v_1), (u_2, v_2) and $0 \leq \alpha \leq 1$, let $u^\alpha = \alpha u_1 + (1-\alpha)u_2$, $v^\alpha = \alpha v_1 + (1-\alpha)v_2$ and

$$g(u^\alpha, v^\alpha) = \min_x L(x, u^\alpha, v^\alpha).$$

Note that

$$\begin{aligned} L(x, u^\alpha, v^\alpha) &= f(x) + \sum_{i=1}^{m} u_i^\alpha h_i(x) + \sum_{j=1}^{r} v_j^\alpha \ell_j(x) \\ &= \alpha L(x, u_1, v_1) + (1-\alpha) L(x, u_2, v_2), \end{aligned}$$

which implies that

$$\begin{aligned} L(x, u^\alpha, v^\alpha) &\geq \alpha \min_x L(x, u_1, v_1) + (1-\alpha) \min_x L(x, u_2, v_2) \\ &= \alpha g(u_1, v_1) + (1-\alpha) g(u_2, v_2). \end{aligned}$$

It follows that

$$g(u^\alpha, v^\alpha) = \min_x L(x, u^\alpha, v^\alpha) \geq \alpha g(u_1, v_1) + (1-\alpha) g(u_2, v_2).$$

□

Proposition 2. *Let C denote primal feasible set. If $u_i \geq 0$, then Lagrange dual function is always a lower bound of $f^\star$. i.e.*

$$g(u, v) = \min_x L(x, u, v) \leq \min_{x \in C} L(x, u, v) \leq f^\star \leq f(x).$$

- **Dual problem:**

$$\max_{u,v} \; g(u,v)$$
$$\text{subject to } u \geq 0$$

Dual is a concave maximization problem $\Leftrightarrow \min_{u\geq 0} -g(u,v)$ is a convex minimization problem.

- **Dual feasible:** u is dual feasible if $u \geq 0$.

- **Dual optimal:** define the optimal solution $(u^\star, v^\star)$ of dual problem as

$$(u^\star, v^\star) = \arg\max_{u\geq 0, v} g(u,v),$$

Denote

$$g^\star = g(u^\star, v^\star).$$

- **Duality gap:** given primal feasible x and dual feasible u, v, the quantity $f(x) - g(u,v)$ is called the duality gap between x and u, v. Note that since $f^\star \geq g(u,v)$

$$f(x) - f^\star \leq f(x) - g(u,v)$$

Proposition 3. *If the duality gap $f(x_0) - g(u_0, v_0) = 0$, then x_0 is primal optimal (and similarly, u_0, v_0 are dual optimal).*

Proof. Since

$$f(x_0) - f^\star \leq f(x_0) - g(u_0, v_0) = 0$$

So

$$f(x_0) = f^\star$$

So x_0 is primal optimal. Similarly by

$$g^\star - g(u_0, v_0) \leq f(x_0) - g(u_0, v_0) = 0$$

we know that u_0 and v_0 are dual optimal. □

- **Weak duality:** weak duality $g^\star \leq f^\star$ is always true by Proposition 2.

- **Slater's condition:** there exists at least one strictly feasible $x_0 \in \mathbb{R}^n$, in other words

  $$h_1(x_0) < 0, \ldots, h_m(x_0) < 0 \text{ and } \ell_1(x_0) = 0, \ldots, \ell_r(x_0) = 0.$$

  Slater's condition states that the feasible region must have **an interior point**

- **Strong duality:** $g^\star = f^\star$ is not always true. It requires two conditions:

Proposition 4. *Strong duality holds if **(1)** the primal is a convex problem (i.e. f and $h_1, \ldots, h_m$ are convex, $\ell_1, \ldots, \ell_r$ are affine) and **(2)** Slater's condition is satisfied.*