Convex Sets

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Definition 1. Convex set: $C \subset \mathbb{R}^n$ such that if $x, y \in C$ then $tx + (1-t)y \in C$ for all $0 \leq t \leq 1$.

Definition 2. Convex combination of $x_1, ..., x_k \in \mathbb{R}^n$ is any linear combination

$$\theta_1 x_1 + ... + \theta_k x_k$$

with $\theta_i \geq 0 \;\; i = 0, ..., k$, and $\sum_{i=1}^{k} \theta_i = 1$.

Definition 3. Convex hull of a set C, denoted $\text{conv}(C)$, is the set of all the convex combinations of elements. It is the smallest convex set that contains set C (note C is not necessarily convex).

Examples of convex sets:

- Norm ball: $\{x : ||x|| \leq r\}$. It is a convex set by using the triangular inequality of the norm

$$\begin{aligned} \|tx + (1-t)y\| &\leq t\|x\| + (1-t)\|y\| \\ &\leq r \end{aligned}$$

- Affine space:

$$Cx = d \Longleftrightarrow Cx \leq d \;\; -Cx \leq -d$$

- Polyhedron:

$$\{x : Ax \leq b\} \Longleftrightarrow \{x : \; a_i^T x \leq b_i \; i = 1, \ldots, m\}$$

  a_i is the i-th row of A (note: this is an intersection of m halfspaces);

  - This definition generalizes to $\{x : Ax \leq b, Cx = d\}$ since we can rewrite the $Cx = d$ constraint using inequality constraints.

- Simplex: special case of polyhedra, given by

$$\text{conv}\{x_0, ..., x_k\}$$

  where these points are affinely independent. The canonical example is the probability simplex

$$w^T e = \text{conv}\{e_1, ..., e_n\} = \{w : w \geq 0, 1^T w = 1\}$$

  where

$$e = [e_1, e_2, \ldots, e_n]$$

  where

$$e_1 = (1, 0, \ldots, 0)^T$$

  where $e_1, ..., e_n$ are the standard basis vectors in $\mathbb{R}^n$ and $w \in \mathbb{R}^n$.

- **Note:** $x_0, \ldots, x_k$ are affine independent $\Longleftrightarrow x_1 - x_0, ..., x_k - x_0$ are linearly independent.

Definition 4. Cone $C \subset \mathbb{R}^n$ such that $x \in C \implies tx \in C$ for all $t \geq 0$.

- **Note:** 0 must be lies in the cone.

Definition 5. Normal cone: $N_c(x)$ is a normal cone to set C at the point $x \in C$ which satisfies

$$N_c(x) = \{g : \langle g, y - x\rangle \leq 0 \text{ for all } y \in C\}$$

Proposition 1. *Normal cone is convex cone.*

Proof. We show $N_c(x)$ is a cone and convex □

1. To show $N_c(x)$ is a cone
   (a) Fix any $g \in N_c(x)$ and $t \geq 0$
   (b) By definition
   $$\langle g, y - x\rangle \leq 0 \text{ for all } y \in C$$
   then
   $$\langle tg, y - x\rangle = t\,\langle g, y - x\rangle \leq 0 \text{ for all } y \in C$$
   Thus $tg \in N_c(x)$. Therefore $N_c(x)$ is a cone
2. To show $N_c(x)$ is a convex set $g_1, g_2 \in N_c(x)$, we want to show $tg_1 + (1-t)g_2 \in N_c(x)$
   (a) Fix $g_1, g_2 \in N_c(x)$ and $t \in [0, 1]$
   $$\langle g_1, y - x\rangle \leq 0 \text{ for all } y \in C$$
   $$\langle g_2, y - x\rangle \leq 0 \text{ for all } y \in C$$

Thus

$$\langle tg_1 + (1-t)g_2, y - x\rangle = t\langle g_1, y - x\rangle + (1-t)\langle g_2, y - x\rangle$$
$$\leq 0 \text{ for all } y \in C$$

Therefore

$$tg_1 + (1-t)g_2 \in N_c(x)$$

Thus $N_c(x)$ is a convex set.

Note:

- For $N_c(x)$, set C can be any set, not necessarily convex
- $N_c(x) = \{0\}$ for any x inside C

Basic linear algebra facts:

- $X \in \mathbb{S}^n \implies \lambda(X) \in \mathbb{R}^n$
- $X \in \mathbb{S}^n_+ \implies \lambda(X) \in \mathbb{R}^n_+$
- $X \in \mathbb{S}^n_{++} \implies \lambda(X) \in \mathbb{R}^n_{++}$

We can define an inner product over $\mathbb{S}^n$: given $X, Y \in \mathbb{S}^n$,

$$\langle X, Y\rangle = \mathrm{tr}(XY)$$

We can define a partial ordering over $\mathbb{S}^n$: given $X, Y \in \mathbb{S}^n$,

$$X \succeq Y \iff X - Y \in \mathbb{S}^n_+$$

Proposition 2. *The set of all positive semidefinite matrices $x \in \mathbb{S}_+^n$ is a convex cone*

1. If $x \in \mathbb{S}_+^n$ then $tx \in \mathbb{S}_+^n$ for $t \geq 0$
2. $x, y \in \mathbb{S}_+^n$ then $tx + (1-t)y \in \mathbb{S}_+^n$ for $t \geq 0$

Proposition 3. *The set of all the points x that satisfies*

$$\{x : \ x_1 A_1 + \cdots x_k A_k \preceq B\}$$

is a convex set.

Proof. Method 1: Let $C = \{x : \ x_1 A_1 + \cdots x_k A_k \preceq B\}$.
Assume $\forall x, \ y \in C$ and let

$$z = tx + (1-t)y \ , t \in [0,1]$$

We want to show $z \in C$

$$\begin{aligned}
& B - (z_1 A_1 + \cdots + z_k A_k) \\
=& B - ((tx_1 + (1-t)y_1))A_1 + \cdots + (tx_k + (1-t)y_k)A_k) \\
=& t(B - (x_1 A_1 + \cdots + x_k A_k)) + (1-t)(B - (y_1 A_1 + \cdots + y_k A_k)) \\
\succeq& 0
\end{aligned}$$

Method 2:

$$\{x : \ x_1 A_1 + \cdots x_k A_k \preceq B\} \Longleftrightarrow \{x : \ f(x) \succeq 0\}, \ \mathbb{S}_+^n \text{ is convex so } f^{-1}(\mathbb{S}_+^n) \text{ is convex}$$

□

Question: We have two disjoint convex sets. Do we always have a hyperplane which strictly separate two sets.

***Answer**:* No

Proof. For example let $C = \{x : a^T x \leq b\}$, $D = \{x : a^T x > b\}$. C and D can not be strictly separated by a hyperplane. One of the set is open. $\square$