β_E^*	$\theta_1^{*\top}$	$\theta_2^{*\top}$	$\cdots$	$\theta_p^{*\top}$	γ_{1E}^*	γ_{2E}^*	$\cdots$	γ_{pE}^*
$\boldsymbol{\phi}_1^{*\top}$	$\boldsymbol{\phi}_2^{*\top}$	$\boldsymbol{\phi}_3^{*\top}$	$\cdots$	$\boldsymbol{\phi}_{p+1}^{*\top}$	$\boldsymbol{\phi}_{p+2}^{*\top}$	$\boldsymbol{\phi}_{p+3}^{*\top}$	$\cdots$	$\boldsymbol{\phi}_{2p+1}^{*\top}$
$\lambda(1-\alpha)w_E$	$\lambda(1-\alpha)w_2$	$\lambda(1-\alpha)w_3$	$\cdots$	$\lambda(1-\alpha)w_{p+1}$	$\lambda\alpha w_{p+2,E}$	$\lambda\alpha w_{p+3,E}$	$\cdots$	$\lambda\alpha w_{2p+1,E}$
λ_1	λ_2	λ_3	$\cdots$	λ_{p+1}	λ_{p+2}	λ_{p+3}	$\cdots$	λ_{2p+1}

Table 4
Correspondence between parameters used to simplify the notation in the proofs. The first row shows the actual parameters used in the loss function. The second row shows the corresponding parameters in the simplified notation. The third row shows the actual tuning parameters used in the penalty function. The fourth row shows the corresponding tuning parameters in the simplified notation. This correspondence greatly simplifies the notation used in the proofs.

A. Proofs

As shown in the main text, we simplified the notation to make the proofs easier to follow. We summarize the original notation and the corresponding simplified notation in Table 4. This notation then allows us to write down the `sail` estimates as

$$\widehat{\boldsymbol{\Phi}}_n = \arg\min_{\boldsymbol{\Phi}} Q_n(\boldsymbol{\Phi}) = -L_n(\boldsymbol{\Phi}) + n\lambda_m \sum_{m=1}^{2p+1} \|\boldsymbol{\phi}_m\|_2 \,. \tag{17}$$

A.1. Regularity Conditions

(C1) The observation $\{\mathbf{V}_i \,:\, i = 1,\ldots,n\}$ are independent and identically distributed with a probability density $f(\mathbf{V},\boldsymbol{\Phi})$, which has a common support. We assume the density f satisfies the following equations:

$$E_{\boldsymbol{\Phi}}\left[\nabla_{\boldsymbol{\phi}_j} \log f\left(\boldsymbol{V},\boldsymbol{\Phi}\right)\right] = \mathbf{0} \quad \text{for } j = 1,\ldots,2p+1,$$

and

$$\begin{aligned}\mathbf{I}_{j_1k_1j_2k_2}(\boldsymbol{\Phi}) &= E_{\boldsymbol{\Phi}}\left[\frac{\partial}{\partial\phi_{j_1k_1}}\log f(V,\boldsymbol{\Phi})\cdot\frac{\partial}{\partial\phi_{j_2k_2}}\log f(V,\boldsymbol{\Phi})\right]\\ &= E_{\boldsymbol{\Phi}}\left[-\frac{\partial^2}{\partial\phi_{j_1k_1}\phi_{j_2k_2}}\log f(V,\boldsymbol{\Phi})\right],\end{aligned}$$

for any $j_1, j_2 = 1,\ldots,2p+1$, and $k_1 = 1,\ldots,p_{j1}$, $k_2 = 1,\ldots,p_{j2}$, where j_1, j_2 are the index of group, k_1, k_2 be the index of elements within the corresponding group, p_{j_1}, p_{j_2} are the group size of j_1, j_2 respectively.

(C2) The Fisher information matrix

$$\mathbf{I}(\boldsymbol{\Phi}) = E\left[\left(\frac{\partial}{\partial\boldsymbol{\Phi}}\log f(V,\boldsymbol{\Phi})\right)\left(\frac{\partial}{\partial\boldsymbol{\Phi}}\log f(V,\boldsymbol{\Phi})\right)^{\top}\right],$$

is finite and positive definite at $\boldsymbol{\Phi} = \boldsymbol{\Phi}^*$.

(C3) There exists an open set ω of Ω that contains the true parameter point $\boldsymbol{\Phi}^*$ such that for almost all $\mathbf{V}$ the density $f(\mathbf{V},\boldsymbol{\Phi})$ admits all third derivatives $\frac{\partial^3 f(\mathbf{V},\boldsymbol{\Phi})}{\partial\phi_{j_1k_1}\partial\phi_{j_2k_2}\partial\phi_{j_3k_3}}$ for all $\boldsymbol{\Phi}$ in ω and any $j_1, j_2, j_3 = 1,\ldots,2p+1$, and $k_1 = 1,\ldots,p_{j1}$, $k_2 = 1,\ldots,p_{j2}$ and $k_3 = 1,\ldots,p_{j3}$. Furthermore, there exist functions $M_{j_1k_1j_2k_2j_3k_3}$ such that

$$\left|\frac{\partial^3}{\partial\phi_{j_1k_1}\partial\phi_{j_2k_2}\partial\phi_{j_3k_3}}\log f(\mathbf{V},\boldsymbol{\Phi})\right| \le M_{j_1k_1j_2k_2j_3k_3}(\mathbf{V}) \quad \text{for all } \boldsymbol{\Phi}\in\omega,$$

and $m_{j_1k_1j_2k_2j_3k_3} = E_{\boldsymbol{\Phi}^*}[M_{j_1k_1j_2k_2j_3k_3}(\mathbf{V})] < \infty$.

A.2. Lemma 1 proof

Let $\eta_n = \frac{1}{\sqrt{n}} + a_n$ and $\{\boldsymbol{\Phi}^* + \eta_n \boldsymbol{\delta} : \|\boldsymbol{\delta}\|_2 \leq C\}$ be the ball around $\boldsymbol{\Phi}^*$ for $\boldsymbol{\delta} \in \mathbb{R}^d$, where d is the dimension of the design matrix and C is some constant. Under the regularity assumptions, we show that there exists a local minimizer $\widehat{\boldsymbol{\Phi}}_n$ of $Q_n(\boldsymbol{\Phi})$ such that $\|\widehat{\boldsymbol{\Phi}}_n - \boldsymbol{\Phi}^*\|_2 = O_p(\frac{1}{\sqrt{n}})$. For this proof, we adopt the approaches outlined in (Fan and Li, 2001; Choi et al., 2010; Nardi et al., 2008; Wang et al., 2007) and extend it to our situation. Let $\eta_n = \frac{1}{\sqrt{n}} + a_n$ and $\{\boldsymbol{\Phi}^* + \eta_n \boldsymbol{\delta} : \|\boldsymbol{\delta}\|_2 \leq C\}$ be the ball around $\boldsymbol{\Phi}^*$ for $\boldsymbol{\delta} = (\mathbf{u}_1^\top, \mathbf{u}_2^\top, , \ldots, \mathbf{u}_{p+1}^\top, \mathbf{u}_{p+2}^\top, \ldots, \mathbf{u}_{2p+1}^\top)^\top \in \mathbb{R}^d$, where d is the dimension of the design matrix and C is some constant. The objective function is given by

$$Q_n(\boldsymbol{\Phi}) = -L_n(\boldsymbol{\Phi}) + n\lambda_m \sum_{m=1}^{2p+1} \|\boldsymbol{\phi}_m\|_2 .$$

Define

$$D_n(\boldsymbol{\delta}) \equiv Q_n(\boldsymbol{\Phi}^* + \eta_n \boldsymbol{\delta}) - Q_n(\boldsymbol{\Phi}^*).$$

Then for $\boldsymbol{\delta}$ that satisfies $\|\boldsymbol{\delta}\|_2 = C$, we have

$$\begin{aligned}
D_n(\boldsymbol{\delta}) &= -L_n(\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta}) + L_n(\boldsymbol{\Phi}^*) + n \sum_{m=1}^{2p+1} \lambda_m(\|\boldsymbol{\theta}_m^* + \eta_n \mathbf{u}_m\|_2 - \|\boldsymbol{\theta}_m^*\|_2) \\
&\overset{(a)}{\geq} -L_n(\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta}) + L_n(\boldsymbol{\Phi}^*) + n \sum_{m\in\mathcal{A}_1} \lambda_m(\|\boldsymbol{\theta}_m^* + \eta_n \mathbf{u}_m\|_2 - \|\boldsymbol{\theta}_m^*\|_2) \\
&\quad + n \sum_{m\in\mathcal{A}_2} \lambda_m(\|\boldsymbol{\theta}_m^* + \eta_n \mathbf{u}_m\|_2 - \|\boldsymbol{\theta}_m^*\|_2) \\
&\overset{(b)}{\geq} -L_n(\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta}) + L_n(\boldsymbol{\Phi}^*) - n\eta_n \sum_{m\in\mathcal{A}_1} \lambda_m \|\mathbf{u}_m\|_2 - n\eta_n \sum_{m\in\mathcal{A}_2} \lambda_m \|\mathbf{u}_m\|_2 \\
&\overset{(c)}{\geq} -L_n(\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta}) + L_n(\boldsymbol{\Phi}^*) - n\eta_n^2 \sum_{m\in\mathcal{A}_1} \|\mathbf{u}_m\|_2 - n\eta_n^2 \sum_{m\in\mathcal{A}_2} \|\mathbf{u}_m\|_2 \\
&\geq -L_n(\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta}) + L_n(\boldsymbol{\Phi}^*) - n\eta_n^2(|\mathcal{A}_1| + |\mathcal{A}_2|)C \\
&\overset{(d)}{=} -[\nabla L_n(\boldsymbol{\Phi}^*)]^\top(\eta_n\boldsymbol{\delta}) - \frac{1}{2}(\eta_n\boldsymbol{\delta})^\top[\nabla^2 L_n(\boldsymbol{\Phi}^*)](\eta_n\boldsymbol{\delta})(1 + o(1)) \\
&\quad - n\eta_n^2(|\mathcal{A}_1| + |\mathcal{A}_2|)C.
\end{aligned} \tag{18}$$

Inequality (a) is by the fact that $\sum_{m\notin\mathcal{A}_1} \|\boldsymbol{\phi}_m^*\|_2 = 0$ and $\sum_{m\notin\mathcal{A}_2} \|\boldsymbol{\phi}_m^*\|_2 = 0$. Inequality (b) is due to the reverse triangle inequality $\|a\|_2 - \|b\|_2 \geq -\|a - b\|_2$. Inequality (c) is by $\lambda_m \leq a_n \leq \eta_n$ for $m \in \mathcal{A}_1$ and $m \in \mathcal{A}_2$. Equality (d) is by the standard argument on the Taylor expansion of the loss function:

$$\begin{aligned}
L_n(\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta}) &= L_n(\boldsymbol{\Phi}^* + \eta_n \cdot \mathbf{0}) + \eta_n \nabla L_n(\boldsymbol{\Phi}^* + \eta_n \cdot \mathbf{0})^\top(\boldsymbol{\delta} - \mathbf{0}) \\
&\qquad + \frac{1}{2}(\boldsymbol{\delta} - \mathbf{0})^\top \nabla^2 L_n(\boldsymbol{\Phi}^* + \eta_n \cdot \mathbf{0})(\boldsymbol{\delta} - \mathbf{0})\{1 + o(1)\} \\
&= L_n(\boldsymbol{\Phi}^*) + \eta_n \nabla L_n(\boldsymbol{\Phi}^*)^\top \boldsymbol{\delta} + \frac{1}{2}\boldsymbol{\delta}^\top \nabla^2 L_n(\boldsymbol{\Phi}^*)\boldsymbol{\delta}\eta_n^2\{1 + o(1)\}.
\end{aligned}$$

We split (18) into three parts:

$$\begin{aligned}
D_1 &= -\left[\nabla L_n\left(\boldsymbol{\Phi}^*\right)\right]^\top \left(\eta_n\boldsymbol{\delta}\right) \\
D_2 &= -\frac{1}{2}\left(\eta_n\boldsymbol{\delta}\right)^\top \left[\nabla^2 L_n\left(\boldsymbol{\Phi}^*\right)\right]\left(\eta_n\boldsymbol{\delta}\right)(1 + o(1)) \\
D_3 &= -n\eta_n^2(|\mathcal{A}_1| + |\mathcal{A}_2|)C.
\end{aligned}$$

Then

$$
\begin{aligned}
D_1 &= -\eta_n \left[\nabla L_n\left(\boldsymbol{\Phi}^*\right)\right]^\top \boldsymbol{\delta} \\
&= -\sqrt{n}\eta_n \left(\frac{1}{\sqrt{n}}\nabla L_n\left(\boldsymbol{\Phi}^*\right)\right)^\top \boldsymbol{\delta} \\
&= -\sqrt{n}\eta_n \left(\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} \nabla \log f\left(\boldsymbol{V}_i, \boldsymbol{\Phi}\right)\Big|_{\boldsymbol{\Phi}=\boldsymbol{\Phi}^*}\right)^\top \boldsymbol{\delta} \\
&= -\sqrt{n}\eta_n \left(\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} \nabla \log f\left(\boldsymbol{V}_i, \boldsymbol{\Phi}\right)\Big|_{\boldsymbol{\Phi}=\boldsymbol{\Phi}^*} - \mathbf{0}\right]\right)^\top \boldsymbol{\delta} \\
&= -\sqrt{n}\eta_n \left(\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} \nabla \log f\left(\boldsymbol{V}_i, \boldsymbol{\Phi}\right)\Big|_{\boldsymbol{\Phi}=\boldsymbol{\Phi}^*} - E_{\boldsymbol{\Phi}^*}\nabla L\left(\boldsymbol{\Phi}^*\right)\right]\right)^\top \boldsymbol{\delta} \\
&= -\sqrt{n}\eta_n O_P(1)\,\boldsymbol{\delta} \\
&= -O_P\left(n\eta_n^2\right)\boldsymbol{\delta}.
\end{aligned} \tag{19}
$$

The last equation is by $a_n = o(\frac{1}{\sqrt{n}})$ and

$$
\begin{aligned}
O_P(n\eta_n^2) &= O_P(n(n^{-1/2} + a_n)^2) = O_P(1 + 2n^{1/2}a_n + na_n^2)) \\
&= O_P(1 + n^{1/2}a_n + (n^{1/2}a_n)^2) = O_P(1 + n^{1/2}a_n + o(1)) \\
&= O_p(n^{1/2}(n^{-1/2} + a_n)) = O_p(n^{1/2}\eta_n),
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 &= \frac{1}{2}n\eta_n^2\left\{\boldsymbol{\delta}^\top\left[-\frac{1}{n}\nabla^2 L_n\left(\boldsymbol{\Phi}^*\right)\right]\boldsymbol{\delta}\right\}\left(1 + o_p(1)\right) \\
&= \frac{1}{2}n\eta_n^2\left\{\boldsymbol{\delta}^\top\left[\mathbf{I}\left(\boldsymbol{\Phi}^*\right)\right]\boldsymbol{\delta}\right\}\left(1 + o_p(1)\right) \text{ (by the weak law of large numbers)} \\
&= O_p(n\eta_n^2\|\delta\|_2^2).
\end{aligned} \tag{20}
$$

Combining (19) and (20) with (18) gives:

$$
\begin{aligned}
D_n(\boldsymbol{\delta}) &\geq D_1 + D_2 + D_3 \\
&= -O_P\left(n\eta_n^2\right)\boldsymbol{\delta} + O_p(n\eta_n^2\|\delta\|_2^2) - n\eta_n^2(|\mathcal{A}_1| + |\mathcal{A}_2|)C.
\end{aligned}
$$

We can see that the first term D_1 is linear in $\boldsymbol{\delta}$ and the second term D_2 is quadratic in $\boldsymbol{\delta}$. We can conclude that for a large enough constant $C = \|\delta\|_2$, D_2 dominates D_1 and D_3. Note that this is a positive term since $I(\boldsymbol{\Phi})$ is positive definite at $\boldsymbol{\Phi} = \boldsymbol{\Phi}^*$ by regularity condition (C2). Therefore, for each $\varepsilon > 0$, there exists a large enough constant C such that, for large enough n

$$
P\left\{\inf_{\|\delta\|_2 = C} D_n(\boldsymbol{\delta}) > 0\right\} \geq 1 - \varepsilon.
$$

This implies with probability at least $1 - \varepsilon$ that the empirical likelihood Q_n has a local minimizer in the ball $\{\boldsymbol{\Phi}^* + \eta_n\boldsymbol{\delta} : \|\delta\|_2 \leq C\}$ (since Q_n is bounded and $\{\boldsymbol{\Phi}^* + \alpha_n\boldsymbol{\delta} : \|\delta\|_2 \leq C\}$ is closed). In other words, there exists a local solution $\widehat{\boldsymbol{\Phi}}_n$ such that $\|\widehat{\boldsymbol{\Phi}}_n - \boldsymbol{\Phi}^*\| \leq \eta_n\|\delta\|_2 \leq \eta_n C = O_P(\eta_n) = O_P(\frac{1}{\sqrt{n}} + a_n) = O_p(\frac{1}{\sqrt{n}})$, since $a_n = o(\frac{1}{\sqrt{n}})$. Hence, $\left\|\widehat{\boldsymbol{\Phi}}_n - \boldsymbol{\Phi}^*\right\|_2 = O_P\left(\frac{1}{\sqrt{n}}\right)$. □

A.3. Theorem 1 proof

We first consider consistency for the main effects $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_1^c}=\mathbf{0}\right)\to 1$. Following (Fan and Li, 2001; Choi et al., 2010), it is sufficient to show that for all $m\in\mathcal{A}_1^c$, $P\left(\widehat{\boldsymbol{\phi}}_m=\mathbf{0}\right)\to 1$, which implies that $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_1^c}=\mathbf{0}\right)\to 1$, i.e., the $\sqrt{n}$-consistent estimate $\widehat{\boldsymbol{\Phi}}$ has oracle property $\widehat{\boldsymbol{\phi}}_m=\mathbf{0}$ if $\boldsymbol{\phi}_m^*=\mathbf{0}$. Denote

$$\widehat{\boldsymbol{\phi}}_m=(\hat{\phi}_{m1},\ldots,\hat{\phi}_{mp_m}),$$

where p_m is the group size of $\widehat{\boldsymbol{\phi}}_m$. Let $\hat{\phi}_{mk}$ be the k-th entry of $\widehat{\boldsymbol{\phi}}_m$. Note that if $\widehat{\boldsymbol{\phi}}_m\neq\mathbf{0}$, then $\hat{\phi}_{mk}\neq 0$ for $k=1,\ldots,p_m$, then penalty function $\|\widehat{\boldsymbol{\phi}}_m\|_2$ becomes differentiable. Therefore ϕ_{mk} for $k=1,\ldots,p_m$ must satisfy the following normal equation

$$\begin{aligned}
\frac{\partial Q_n\left(\widehat{\boldsymbol{\Phi}}_n\right)}{\partial\phi_{mk}} &= -\frac{\partial L_n\left(\widehat{\boldsymbol{\Phi}}_n\right)}{\partial\phi_{mk}}+n\lambda_m\frac{\hat{\phi}_{mk}}{\|\widehat{\boldsymbol{\phi}}_m\|_2}\\
&= -\frac{\partial L_n\left(\boldsymbol{\Phi}^*\right)}{\partial\phi_{mk}}-\sum_{j_1=1}^{2p+1}\sum_{k_1=1}^{p_{j_1}}\frac{\partial^2 L_n\left(\boldsymbol{\Phi}^*\right)}{\partial\phi_{mk}\partial\phi_{j_1k_1}}\left(\hat{\phi}_{j_1k_1}-\phi^*_{j_1k_1}\right)\\
&\quad -\frac{1}{2}\sum_{j_1=1}^{2p+1}\sum_{k_1=1}^{p_{j_1}}\sum_{j_2=1}^{2p+1}\sum_{k_2=1}^{p_{j_2}}\frac{\partial^3 L_n(\widetilde{\boldsymbol{\Phi}})}{\partial\phi_{mk}\partial\phi_{j_1k_1}\partial\phi_{j_2k_2}}\left(\hat{\phi}_{j_1k_1}-\phi^*_{j_1k_1}\right)\left(\hat{\phi}_{j_2k_2}-\phi^*_{j_2k_2}\right)\\
&\quad +n\lambda_m\frac{\hat{\phi}_{mk}}{\|\widehat{\boldsymbol{\phi}}_m\|_2}\triangleq I_1+I_2+I_3+I_4=0,
\end{aligned}$$

where $\widetilde{\boldsymbol{\Phi}}$ lies between $\widehat{\boldsymbol{\Phi}}_n$ and $\boldsymbol{\Phi}^*$. By the regularity conditions and Lemma (1) that $\left\|\widehat{\boldsymbol{\Phi}}_n-\boldsymbol{\Phi}^*\right\|_2=O_P\left(\frac{1}{\sqrt{n}}\right)$, the first term is of the order $O_p(\sqrt{n})$

$$I_1=-\frac{\partial L_n\left(\widehat{\boldsymbol{\Phi}}_n\right)}{\partial\phi_{mk}}=-\sqrt{n}\sqrt{n}\frac{1}{n}\frac{\partial L_n\left(\widehat{\boldsymbol{\Phi}}_n\right)}{\partial\phi_{mk}}=\sqrt{n}O_p(1)=O_p(\sqrt{n}).$$

Then the second is of the order $O_P\left(\frac{1}{\sqrt{n}}\right)$ and the third term is of the order $O_P\left(\frac{1}{n}\right)$. Hence

$$\frac{\partial Q_n\left(\widehat{\boldsymbol{\Phi}}_n\right)}{\partial\boldsymbol{\Phi}_m}=\sqrt{n}\left\{O_p(1)+\sqrt{n}\lambda_m\frac{\hat{\phi}_{mk}}{\|\widehat{\boldsymbol{\phi}}_m\|_2}\right\}. \tag{21}$$

As $\sqrt{n}\lambda_m\geq\sqrt{n}b_n\to\infty$ for $m\in\mathcal{A}_1^c$ from the assumption, therefore we know that I_4 dominates I_1, I_2 and I_3 in (21) with probability tending to one. This means that (21) cannot be true as long as the sample size is sufficiently large. As a result, we can conclude that with probability tending to one, the estimate $\widehat{\boldsymbol{\phi}}_m=(\hat{\phi}_{m1},\ldots,\hat{\phi}_{mp_m})$ must be in a position where $\widehat{\boldsymbol{\phi}}_m$ is not differentiable. Hence $\widehat{\boldsymbol{\phi}}_m=\mathbf{0}$ for all $m\in\mathcal{A}_1^c$. Hence $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_1^c}=\mathbf{0}\right)\to 1$. This completes the proof.

Next, we prove that for the interactions $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_2^c}=\mathbf{0}\right)\to 1$. For $m\in\mathcal{A}_2^c$ s.t. $\boldsymbol{\phi}_m^*=\gamma_{jE}^*=0$ but $\beta_E\neq 0$ and $\boldsymbol{\theta}_j^*\neq\mathbf{0}\quad(1\leq j\leq p)$, we can prove $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_2^c}=\mathbf{0}\right)\to 1$ by a similar reasoning, which further implies that $P(\hat{\gamma}_{jE}=0)\to 0$. For $m\in\mathcal{A}_2^c$ such that $\boldsymbol{\phi}_m^*=\gamma_{jE}^*=0$ and either $\beta_E=0$ or $\boldsymbol{\theta}_j^*=\mathbf{0}\quad(1\leq j\leq p)$: without loss of generality, assume that $\boldsymbol{\theta}_j^*=\mathbf{0}$. Notice that $\hat{\boldsymbol{\theta}}_j=\mathbf{0}$ implies $\hat{\gamma}_{jE}=0$, since if $\hat{\gamma}_{jE}\neq 0$, the value of the loss function does not change but the value of the penalty function will increase. Because we already prove $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_1^c}=\mathbf{0}\right)\to 1$, therefore we get $P\left(\widehat{\boldsymbol{\Phi}}_{\mathcal{A}_2^c}=\mathbf{0}\right)\to 1$ as well for this case. □

A.4. Theorem 2 proof

By Lemma 1 and Theorem 1, there exists a $\widehat{\mathbf{\Phi}}_{\mathcal{A}}$ that is a $\sqrt{n}$-consistent local minimizer of $Q(\mathbf{\Phi}_{\mathcal{A}})$, therefore $\left\|\widehat{\mathbf{\Phi}}_{\mathcal{A}} - \mathbf{\Phi}^*_{\mathcal{A}}\right\|_2 = O_P\left(\frac{1}{\sqrt{n}}\right)$ and $P\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}^c} = \mathbf{0}\right) \to 1$. Thus satisfies (with probability tending to 1):

$$\left.\frac{\partial Q_n\left(\mathbf{\Phi}_{\mathcal{A}}\right)}{\partial \mathbf{\Phi}_m}\right|_{\mathbf{\Phi}=\begin{pmatrix}\widehat{\mathbf{\Phi}}_{\mathcal{A}}\\ 0\end{pmatrix}} = 0, \quad \forall m \in \mathcal{A}, \tag{22}$$

that is

$$\left.\frac{\partial Q_n\left(\mathbf{\Phi}_{\mathcal{A}}\right)}{\partial \mathbf{\Phi}_m}\right|_{\mathbf{\Phi}_{\mathcal{A}}=\widehat{\mathbf{\Phi}}_{\mathcal{A}}} = 0, \quad \forall m \in \mathcal{A}, \tag{23}$$

where

$$\begin{aligned}
Q_n(\mathbf{\Phi}_{\mathcal{A}}) &= -L_n(\mathbf{\Phi}_{\mathcal{A}}) + \underbrace{n \sum_{m\in\mathcal{A}_1} \lambda_m \left\|\boldsymbol{\phi}_m\right\|_2 + n \sum_{m\in\mathcal{A}_2} \lambda_m \left\|\boldsymbol{\phi}_m\right\|_2}_{\triangleq nP(\mathbf{\Phi}_{\mathcal{A}})} \\
&= -L_n(\mathbf{\Phi}_{\mathcal{A}}) + nP(\mathbf{\Phi}_{\mathcal{A}}).
\end{aligned} \tag{24}$$

From (23) and (24) we have

$$\nabla_{\mathcal{A}} Q_n\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}}\right) = -\nabla_{\mathcal{A}} L_n\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}}\right) + n\nabla_{\mathcal{A}} P\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}}\right) = \mathbf{0}, \tag{25}$$

with probability tending to 1.

Denote $\mathbf{\Sigma} = \mathrm{diag}\{o_p(1), \dots, o_p(1)\}$. We then expand $-\nabla_{\mathcal{A}} L_n\left(\mathbf{\Phi}_{\mathcal{A}}\right)$ at $\mathbf{\Phi}_{\mathcal{A}} = \mathbf{\Phi}^*_{\mathcal{A}}$ in (25):

$$\begin{aligned}
-\nabla_{\mathcal{A}} L_n\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}}\right) &= -\nabla_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) - \left[\nabla^2_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) + \mathbf{\Sigma}\right]\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}} - \mathbf{\Phi}^*_{\mathcal{A}}\right) \\
&= \sqrt{n}\left[-\frac{1}{\sqrt{n}}\nabla_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) + \left(-\frac{1}{n}\nabla^2_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) - \mathbf{\Sigma}\right)\sqrt{n}\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}} - \mathbf{\Phi}^*_{\mathcal{A}}\right)\right] \\
&= \sqrt{n}\left[-\frac{1}{\sqrt{n}}\nabla_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) + \left(\mathbf{I}\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) - \mathbf{\Sigma}\right)\sqrt{n}\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}} - \mathbf{\Phi}^*_{\mathcal{A}}\right)\right].
\end{aligned}$$

The third line follows by

$$\frac{1}{n}\nabla^2_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) = E\left\{\nabla^2_{\mathcal{A}} L\left(\mathbf{\Phi}^*_{\mathcal{A}}\right)\right\} + \mathbf{\Sigma} = -\mathbf{I}\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) + \mathbf{\Sigma}.$$

Denote

$$\mathbf{b} = (\lambda_m \mathrm{sgn}\left(\beta^*_m\right), \lambda_m \frac{\boldsymbol{\theta}^*_m}{\|\boldsymbol{\theta}^*_m\|_2}^{\top}, \lambda_m\, \mathrm{sgn}(\gamma^*_{mE}))^{\top}, \qquad m \in \mathcal{A}.$$

We also expand $n\nabla_{\mathcal{A}} P\left(\mathbf{\Phi}_{\mathcal{A}}\right)$ at $\mathbf{\Phi}_{\mathcal{A}} = \mathbf{\Phi}^*_{\mathcal{A}}$ in (25):

$$n\nabla_{\mathcal{A}} P\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}}\right) = n\left[\mathbf{b} + \mathbf{\Sigma}\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}} - \mathbf{\Phi}^*_{\mathcal{A}}\right)\right].$$

Due to the fact that $\sqrt{n}\lambda_m \leq \sqrt{n}a_n \to 0$ for $m \in \mathcal{A}$ and $\frac{\theta^*_{mk}}{\|\theta^*_m\|_2} \leq 1$ for any $1 \leq k \leq p_m$, we know that $\sqrt{n}\mathbf{b} = (o_p(1), \dots, o_p(1))^{\top}$. Thus,

$$\nabla_{\mathcal{A}} Q_n\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}}\right) = \sqrt{n}\left[-\frac{1}{\sqrt{n}}\nabla_{\mathcal{A}} L_n\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) + \left(\mathbf{I}\left(\mathbf{\Phi}^*_{\mathcal{A}}\right) + \mathbf{\Sigma}\right)\sqrt{n}\left(\widehat{\mathbf{\Phi}}_{\mathcal{A}} - \mathbf{\Phi}^*_{\mathcal{A}}\right)\right]$$

$$
\begin{aligned}
&+ \sqrt{n}\left[\sqrt{n}\mathbf{b} + \boldsymbol{\Sigma}\sqrt{n}\left(\hat{\boldsymbol{\Phi}}_{\mathcal{A}} - \boldsymbol{\Phi}^*_{\mathcal{A}}\right)\right] \\
&= \sqrt{n}\left[-\frac{1}{\sqrt{n}}\nabla_{\mathcal{A}} L_n\left(\boldsymbol{\Phi}^*_{\mathcal{A}}\right) + \sqrt{n}\mathbf{b} + \left(\mathbf{I}\left(\boldsymbol{\Phi}^*_{\mathcal{A}}\right) + \boldsymbol{\Sigma}\right)\sqrt{n}\left(\hat{\boldsymbol{\Phi}}_{\mathcal{A}} - \boldsymbol{\Phi}^*_{\mathcal{A}}\right)\right] \\
&= \mathbf{0},
\end{aligned}
$$

and

$$
\left(\mathbf{I}\left(\boldsymbol{\Phi}^*_{\mathcal{A}}\right) + \boldsymbol{\Sigma}\right)\sqrt{n}(\hat{\boldsymbol{\Phi}}_{\mathcal{A}} - \boldsymbol{\Phi}^*_{\mathcal{A}}) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\nabla_{\mathcal{A}}\log f\left(\boldsymbol{V}_i, \boldsymbol{\Phi}^*_{\mathcal{A}}\right) + o_p(1).
$$

Therefore, by the central limit theorem, we know that

$$
\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\nabla_{\mathcal{A}}\log f(V_i, \boldsymbol{\Phi}^*_{\mathcal{A}})\right] \to N(\mathbf{0}, \mathbf{I}(\boldsymbol{\Phi}^*_{\mathcal{A}})).
$$

Hence,

$$
\sqrt{n}\left(\hat{\boldsymbol{\Phi}}_{\mathcal{A}} - \boldsymbol{\Phi}^*_{\mathcal{A}}\right) \xrightarrow{d} N\left(\mathbf{0}, \mathbf{I}^{-1}\left(\boldsymbol{\Phi}^*_{\mathcal{A}}\right)\right).
$$

□

B. Algorithm Details

In this section we provide more specific details about the algorithms used to solve the `sail` objective function. We assume that Y, $\boldsymbol{\Psi}_j$, X_E and $X_E \circ \boldsymbol{\Psi}_j$ have been centered by their sample means $\overline{Y}$, $\overline{\boldsymbol{\Psi}}_j$, $\overline{X}_E$, and $\overline{X_E \circ \boldsymbol{\Psi}_j}$, respectively. Here, $\overline{\boldsymbol{\Psi}}_j \in \mathbb{R}^{m_j}$ and $\overline{X_E \circ \boldsymbol{\Psi}_j} \in \mathbb{R}^{m_j}$ represent the column means of $\boldsymbol{\Psi}_j$ and $X_E \circ \boldsymbol{\Psi}_j$, respectively. Since the intercept (β_0) is not penalized and all variables have been centered, we can omit it from the loss function and compute it once the algorithm has converged for all other parameters. The strong heredity `sail` model with least-squares loss has the form

$$
\hat{Y} = \sum_{j=1}^{p}\boldsymbol{\Psi}_j\boldsymbol{\theta}_j + \beta_E X_E + \sum_{j=1}^{p}\gamma_j\beta_E(X_E \circ \boldsymbol{\Psi}_j)\boldsymbol{\theta}_j, \tag{26}
$$

and the objective function is given by

$$
Q(\boldsymbol{\Phi}) = \frac{1}{2n}\left\|Y - \hat{Y}\right\|_2^2 + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j\left\|\boldsymbol{\theta}_j\right\|_2\right) + \lambda\alpha\sum_{j=1}^{p} w_{jE}\left|\gamma_j\right|. \tag{27}
$$

Solving (27) in a blockwise manner allows us to leverage computationally fast algorithms for ℓ_1 and ℓ_2 norm penalized regression. Denote the n-dimensional residual column vector $R = Y - \hat{Y}$. The subgradient equations are given by

$$
\frac{\partial Q}{\partial \beta_E} = -\frac{1}{n}\left(X_E + \sum_{j=1}^{p}\gamma_j(X_E \circ \boldsymbol{\Psi}_j)\boldsymbol{\theta}_j\right)^{\top} R + \lambda(1-\alpha)w_E s_1 = 0 \tag{28}
$$

$$
\frac{\partial Q}{\partial \boldsymbol{\theta}_j} = -\frac{1}{n}\left(\boldsymbol{\Psi}_j + \gamma_j\beta_E(X_E \circ \boldsymbol{\Psi}_j)\right)^{\top} R + \lambda(1-\alpha)w_j s_2 = \mathbf{0} \tag{29}
$$

and

$$
\frac{\partial Q}{\partial \gamma_j} = -\frac{1}{n}\left(\beta_E(X_E \circ \boldsymbol{\Psi}_j)\boldsymbol{\theta}_j\right)^{\top} R + \lambda\alpha w_{jE} s_3 = 0, \tag{30}
$$

where s_1 is in the subgradient of the ℓ_1 norm:

$$s_1 \in \begin{cases} \operatorname{sign}\left(\beta_E\right) & \text{if } \beta_E \neq 0 \\ [-1, 1] & \text{if } \beta_E = 0, \end{cases}$$

s_2 is in the subgradient of the ℓ_2 norm:

$$s_2 \in \begin{cases} \dfrac{\boldsymbol{\theta}_j}{\left\|\boldsymbol{\theta}_j\right\|_2} & \text{if } \boldsymbol{\theta}_j \neq \mathbf{0} \\ u \in \mathbb{R}^{m_j} : \|u\|_2 \leq 1 & \text{if } \boldsymbol{\theta}_j = \mathbf{0}, \end{cases}$$

and s_3 is in the subgradient of the ℓ_1 norm:

$$s_3 \in \begin{cases} \operatorname{sign}\left(\gamma_j\right) & \text{if } \gamma_j \neq 0 \\ [-1, 1] & \text{if } \gamma_j = 0. \end{cases}$$

Define the partial residuals, without the jth predictor for $j = 1, \dots, p$, as

$$R_{(-j)} = Y - \sum_{\ell \neq j} \boldsymbol{\Psi}_\ell \boldsymbol{\theta}_\ell - \beta_E X_E - \sum_{\ell \neq j} \gamma_\ell \beta_E (X_E \circ \boldsymbol{\Psi}_\ell) \boldsymbol{\theta}_\ell,$$

the partial residual without X_E as

$$R_{(-E)} = Y - \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j,$$

and the partial residual without the jth interaction for $j = 1, \dots, p$, as

$$R_{(-jE)} = Y - \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j - \beta_E X_E - \sum_{\ell \neq j} \gamma_\ell \beta_E (X_E \circ \boldsymbol{\Psi}_\ell) \boldsymbol{\theta}_\ell.$$

From the subgradient equations (28)–(30) we see that

$$\hat{\beta}_E = \frac{S\left(\frac{1}{n \cdot w_E}\left(X_E + \sum_{j=1}^{p} \hat{\gamma}_j (X_E \circ \boldsymbol{\Psi}_j) \hat{\boldsymbol{\theta}}_j\right)^\top R_{(-E)}, \lambda(1-\alpha)\right)}{\left(X_E + \sum_{j=1}^{p} \hat{\gamma}_j (X_E \circ \boldsymbol{\Psi}_j) \hat{\boldsymbol{\theta}}_j\right)^\top \left(X_E + \sum_{j=1}^{p} \hat{\gamma}_j (X_E \circ \boldsymbol{\Psi}_j) \hat{\boldsymbol{\theta}}_j\right)} \tag{31}$$

$$\lambda(1-\alpha) w_j \frac{\boldsymbol{\theta}_j}{\left\|\boldsymbol{\theta}_j\right\|_2} = \frac{1}{n}\left(\boldsymbol{\Psi}_j + \gamma_j \beta_E (X_E \circ \boldsymbol{\Psi}_j)\right)^\top R_{(-j)} \tag{32}$$

$$\hat{\gamma}_j = \frac{S\left(\frac{1}{n \cdot w_{jE}}\left(\beta_E (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j\right)^\top R_{(-jE)}, \lambda\alpha\right)}{\left(\beta_E (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j\right)^\top \left(\beta_E (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j\right)}, \tag{33}$$

where $S(x, t) = \operatorname{sign}(x)(|x| - t)$ is the soft-thresholding operator. Given these estimates, the intercept can be computed using the following equation:

$$\hat{\beta}_0 = \overline{Y} - \sum_{j=1}^{p} \overline{\boldsymbol{\Psi}}_j \hat{\boldsymbol{\theta}}_j - \hat{\beta}_E \overline{X}_E - \sum_{j=1}^{p} \hat{\gamma}_j \hat{\beta}_E \overline{(X_E \circ \boldsymbol{\Psi}_j)} \hat{\boldsymbol{\theta}}_j. \tag{34}$$

We see from (31) that there is a closed form solution for β_E. From (33), each γ_j also has a closed form solution and can be solved efficiently for $j = 1, \dots, p$ using a coordinate descent procedure (Friedman et al., 2010). Since there is no closed form solution for β_j, we use a quadratic majorization technique (Yang and Zou, 2015) to solve (32). Furthermore, we update each $\boldsymbol{\theta}_j$ in a coordinate wise fashion and leverage this to implement further computational speedups which are detailed in Supplemental Section B.2. From these estimates, we compute the interaction effects using the reparametrizations presented in Table 1, e.g., $\hat{\boldsymbol{\tau}}_j = \hat{\gamma}_j \hat{\beta}_E \hat{\boldsymbol{\theta}}_j$, $j = 1, \dots, p$ for the strong heredity `sail` model.

B.1. Least-Squares `sail` with Strong Heredity

A more detailed algorithm for fitting the least-squares `sail` model with strong heredity is given in Algorithm 3.

Algorithm 3 Blockwise Coordinate Descent for Least-Squares `sail` with Strong Heredity

function $\texttt{sail}(\boldsymbol{X}, Y, X_E, \texttt{basis}, \lambda, \alpha, w_j, w_E, w_{jE}, \epsilon)$ ⊳ Algorithm for solving (27)

$\boldsymbol{\Psi}_j \leftarrow \texttt{basis}(X_j)$, $\widetilde{\boldsymbol{\Psi}}_j \leftarrow X_E \circ \boldsymbol{\Psi}_j$ for $j = 1, \ldots, p$

Center all variables by their sample means

Initialize: $\beta_E^{(0)} = \boldsymbol{\theta}_j^{(0)} = \gamma_j^{(0)} \leftarrow 0$ for $j = 1, \ldots, p$.

Set iteration counter $k \leftarrow 0$

$R^* \leftarrow Y - \beta_E^{(k)} X_E - \sum_j (\boldsymbol{\Psi}_j + \gamma_j^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_j) \boldsymbol{\theta}_j^{(k)}$

repeat

- To update $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)$

  $\widetilde{X}_j \leftarrow \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_j \boldsymbol{\theta}_j^{(k)}$ for $j = 1, \ldots, p$

  $R \leftarrow R^* + \sum_{j=1}^{p} \gamma_j^{(k)} \widetilde{X}_j$

$$\boldsymbol{\gamma}^{(k)(new)} \leftarrow \arg\min_{\boldsymbol{\gamma}} \frac{1}{2n} \left\| R - \sum_j \gamma_j \widetilde{X}_j \right\|_2^2 + \lambda\alpha \sum_j w_{jE} \left| \gamma_j \right|$$

  $\Delta = \sum_j (\gamma_j^{(k)} - \gamma_j^{(k)(new)}) \widetilde{X}_j$

  $R^* \leftarrow R^* + \Delta$

- To update $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p)$

  $\widetilde{X}_j \leftarrow \boldsymbol{\Psi}_j + \gamma_j^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_j$ for $j = 1, \ldots, p$

  for $j = 1, \ldots, p$ **do**

    $R \leftarrow R^* + \widetilde{X}_j \boldsymbol{\theta}_j^{(k)}$

$$\boldsymbol{\theta}_j^{(k)(new)} \leftarrow \arg\min_{\boldsymbol{\theta}_j} \frac{1}{2n} \left\| R - \widetilde{X}_j \boldsymbol{\theta}_j \right\|_2^2 + \lambda(1-\alpha) w_j \left\| \boldsymbol{\theta}_j \right\|_2$$

    $\Delta = \widetilde{X}_j (\boldsymbol{\theta}_j^{(k)} - \boldsymbol{\theta}_j^{(k)(new)})$

    $R^* \leftarrow R^* + \Delta$

- To update β_E

  $\widetilde{X}_E \leftarrow X_E + \sum_j \gamma_j^{(k)} \widetilde{\boldsymbol{\Psi}}_j \boldsymbol{\theta}_j^{(k)}$

  $R \leftarrow R^* + \beta_E^{(k)} \widetilde{X}_E$

$$\beta_E^{(k)(new)} \leftarrow \frac{1}{\widetilde{X}_E^\top \widetilde{X}_E} S\left(\frac{1}{n \cdot w_E} \widetilde{X}_E^\top R, \lambda(1-\alpha) \right)$$

⊳ $S(x, t) = \text{sign}(x)(|x| - t)_+$

  $\Delta = (\beta_E^{(k)} - \beta_E^{(k)(new)}) \widetilde{X}_E$

  $R^* \leftarrow R^* + \Delta$

$k \leftarrow k + 1$

until convergence criterion is satisfied: $\left| Q(\boldsymbol{\Phi}^{(k-1)}) - Q(\boldsymbol{\Phi}^{(k)}) \right| / Q(\boldsymbol{\Phi}^{(k-1)}) < \epsilon$

Compute the intercept β_0

$\beta_0 \leftarrow \overline{Y} - \sum_{j=1}^{p} \overline{\boldsymbol{\Psi}}_j \hat{\boldsymbol{\theta}}_j - \hat{\beta}_E \overline{X}_E - \sum_{j=1}^{p} \hat{\gamma}_j \hat{\beta}_E \overline{(X_E \circ \boldsymbol{\Psi}_j)} \hat{\boldsymbol{\theta}}_j$

B.2. Details on Update for θ

Here we discuss a computational speedup in the updates for the $\boldsymbol{\theta}$ parameter. The partial residual (R_s) used for updating $\boldsymbol{\theta}_s$ ($s \in 1, \dots, p$) at the kth iteration is given by

$$R_s = Y - \widetilde{Y}^{(k)}_{(-s)}, \tag{35}$$

where $\widetilde{Y}^{(k)}_{(-s)}$ is the fitted value at the kth iteration excluding the contribution from $\boldsymbol{\Psi}_s$:

$$\widetilde{Y}^{(k)}_{(-s)} = \beta_E^{(k)} X_E + \sum_{\ell \neq s} \boldsymbol{\Psi}_\ell \boldsymbol{\theta}_\ell^{(k)} + \sum_{\ell \neq s} \gamma_\ell^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_\ell \boldsymbol{\theta}_\ell^{(k)}. \tag{36}$$

Using (36), (35) can be re-written as

$$\begin{aligned} R_s &= Y - \beta_E^{(k)} X_E - \sum_{j=1}^{p} (\boldsymbol{\Psi}_j + \gamma_j^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_j) \boldsymbol{\theta}_j^{(k)} + (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s) \boldsymbol{\theta}_s^{(k)} \\ &= R^* + (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s) \boldsymbol{\theta}_s^{(k)}, \end{aligned} \tag{37}$$

where

$$R^* = Y - \beta_E^{(k)} X_E - \sum_{j=1}^{p} (\boldsymbol{\Psi}_j + \gamma_j^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_j) \boldsymbol{\theta}_j^{(k)}. \tag{38}$$

Denote $\boldsymbol{\theta}_s^{(k)(\text{new})}$ the solution for predictor s at the kth iteration, given by:

$$\boldsymbol{\theta}_s^{(k)(new)} = \arg\min_{\theta_j} \frac{1}{2n} \left\| R_s - (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s) \boldsymbol{\theta}_j \right\|_2^2 + \lambda(1-\alpha) w_s \left\| \theta_j \right\|_2. \tag{39}$$

Now we want to update the parameters for the next predictor $\boldsymbol{\theta}_{s+1}$ ($s+1 \in 1, \dots, p$) at the kth iteration. The partial residual used to update $\boldsymbol{\theta}_{s+1}$ is given by

$$R_{s+1} = R^* + (\boldsymbol{\Psi}_{s+1} + \gamma_{s+1}^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_{s+1}) \boldsymbol{\theta}_{s+1}^{(k)} + (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s)(\boldsymbol{\theta}_s^{(k)} - \boldsymbol{\theta}_s^{(k)(new)}), \tag{40}$$

where R^* is given by (38), $\boldsymbol{\theta}_s^{(k)}$ is the parameter value prior to the update, and $\boldsymbol{\theta}_s^{(k)(new)}$ is the updated value given by (39). Taking the difference between (37) and (40) gives

$$\begin{aligned} \Delta &= R_t - R_s \\ &= (\boldsymbol{\Psi}_t + \gamma_t^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_t) \boldsymbol{\theta}_t^{(k)} + (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s)(\boldsymbol{\theta}_s^{(k)} - \boldsymbol{\theta}_s^{(k)(new)}) - (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s) \boldsymbol{\theta}_s^{(k)} \\ &= (\boldsymbol{\Psi}_t + \gamma_t^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_t) \boldsymbol{\theta}_t^{(k)} - (\boldsymbol{\Psi}_s + \gamma_s^{(k)} \beta_E^{(k)} \widetilde{\boldsymbol{\Psi}}_s) \boldsymbol{\theta}_s^{(k)(new)}. \end{aligned} \tag{41}$$

Therefore $R_t = R_s + \Delta$, and the partial residual for updating the next predictor can be computed by updating the previous partial residual by Δ, given by (41). This formulation can lead to computational speedups especially when $\Delta = 0$, meaning the partial residual does not need to be re-calculated.

B.3. Maximum penalty parameter (λ_{max}) for strong heredity

The subgradient equations (28)–(30) can be used to determine the largest value of λ such that all coefficients are 0. From the subgradient Equation (28), we see that $\beta_E = 0$ is a solution if

$$\frac{1}{w_E} \left| \frac{1}{n} \left(X_E + \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j \right)^\top R_{(-E)} \right| \leq \lambda(1-\alpha). \tag{42}$$

From the subgradient Equation (29), we see that $\boldsymbol{\theta}_j = \mathbf{0}$ is a solution if

$$\frac{1}{w_j} \left\| \frac{1}{n} \left(\boldsymbol{\Psi}_j + \gamma_j \beta_E (X_E \circ \boldsymbol{\Psi}_j) \right)^\top R_{(-j)} \right\|_2 \leq \lambda(1-\alpha). \tag{43}$$

From the subgradient Equation (30), we see that $\gamma_j = 0$ is a solution if

$$\frac{1}{w_{jE}} \left| \frac{1}{n} \left(\beta_E (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j\right)^\top R_{(-jE)} \right| \leq \lambda\alpha. \tag{44}$$

Due to the strong heredity property, the parameter vector $(\beta_E, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p, \gamma_1, \ldots, \gamma_p)$ will be entirely equal to $\mathbf{0}$ if $(\beta_E, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p) = \mathbf{0}$. Therefore, the smallest value of λ for which the entire parameter vector (excluding the intercept) is $\mathbf{0}$ is:

$$\lambda_{max} = \frac{1}{n(1-\alpha)} \max \left\{ \frac{1}{w_E} \left(X_E + \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j \right)^\top R_{(-E)}, \max_j \frac{1}{w_j} \left\| \left(\boldsymbol{\Psi}_j + \gamma_j \beta_E (X_E \circ \boldsymbol{\Psi}_j)\right)^\top R_{(-j)} \right\|_2 \right\}, \tag{45}$$

which reduces to

$$\lambda_{max} = \frac{1}{n(1-\alpha)} \max \left\{ \frac{1}{w_E} \left(X_E\right)^\top R_{(-E)}, \max_j \frac{1}{w_j} \left\| \left(\boldsymbol{\Psi}_j\right)^\top R_{(-j)} \right\|_2 \right\}.$$

B.4. Least-Squares `sail` with Weak Heredity

We assume the same centering constraints as in Section B.1. The least-squares `sail` model with weak heredity has the form

$$\hat{Y} = \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j + \beta_E X_E + \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j). \tag{46}$$

The objective function is given by

$$Q(\boldsymbol{\Phi}) = \frac{1}{2n} \left\| Y - \hat{Y} \right\|_2^2 + \lambda(1-\alpha) \left(w_E |\beta_E| + \sum_{j=1}^{p} w_j \left\| \boldsymbol{\theta}_j \right\|_2 \right) + \lambda\alpha \sum_{j=1}^{p} w_{jE} \left|\gamma_j\right|. \tag{47}$$

Denote the n-dimensional residual column vector $R = Y - \hat{Y}$. The subgradient equations are given by

$$\frac{\partial Q}{\partial \beta_E} = -\frac{1}{n} \left(X_E + \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \mathbf{1}_{m_j} \right)^\top R + \lambda(1-\alpha) w_E s_1 = 0 \tag{48}$$

$$\frac{\partial Q}{\partial \boldsymbol{\theta}_j} = -\frac{1}{n} \left(\boldsymbol{\Psi}_j + \gamma_j (X_E \circ \boldsymbol{\Psi}_j)\right)^\top R + \lambda(1-\alpha) w_j s_2 = \mathbf{0} \tag{49}$$

$$\frac{\partial Q}{\partial \gamma_j} = -\frac{1}{n} \left((X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) \right)^\top R + \lambda\alpha w_{jE} s_3 = 0, \tag{50}$$

where s_1 is in the subgradient of the ℓ_1 norm:

$$s_1 \in \begin{cases} \text{sign}\left(\beta_E\right) & \text{if } \beta_E \neq 0 \\ [-1, 1] & \text{if } \beta_E = 0, \end{cases}$$

s_2 is in the subgradient of the ℓ_2 norm:

$$s_2 \in \begin{cases} \dfrac{\boldsymbol{\theta}_j}{\left\|\boldsymbol{\theta}_j\right\|_2} & \text{if } \boldsymbol{\theta}_j \neq \mathbf{0} \\ u \in \mathbb{R}^{m_j} : \|u\|_2 \leq 1 & \text{if } \boldsymbol{\theta}_j = \mathbf{0}, \end{cases}$$

and s_3 is in the subgradient of the ℓ_1 norm:

$$s_3 \in \begin{cases} \text{sign}\left(\gamma_j\right) & \text{if } \gamma_j \neq 0 \\ [-1, 1] & \text{if } \gamma_j = 0. \end{cases}$$

Define the partial residuals, without the jth predictor for $j = 1, \ldots, p$, as

$$R_{(-j)} = Y - \sum_{\ell \neq j} \boldsymbol{\Psi}_\ell \boldsymbol{\theta}_\ell - \beta_E X_E - \sum_{\ell \neq j} \gamma_\ell (X_E \circ \boldsymbol{\Psi}_\ell)(\beta_E \cdot \mathbf{1}_{m_\ell} + \boldsymbol{\theta}_\ell),$$

the partial residual without X_E as

$$R_{(-E)} = Y - \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j - \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j,$$

and the partial residual without the jth interaction for $j = 1, \ldots, p$

$$R_{(-jE)} = Y - \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j - \beta_E X_E - \sum_{\ell \neq j} \gamma_\ell (X_E \circ \boldsymbol{\Psi}_\ell)(\beta_E \cdot \mathbf{1}_{m_\ell} + \boldsymbol{\theta}_\ell).$$

From the subgradient Equation (48), we see that $\beta_E = 0$ is a solution if

$$\frac{1}{w_E} \left| \frac{1}{n} \left(X_E + \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \mathbf{1}_{m_j} \right)^\top R_{(-E)} \right| \leq \lambda (1 - \alpha) \tag{51}$$

From the subgradient Equation (49), we see that $\boldsymbol{\theta}_j = \mathbf{0}$ is a solution if

$$\frac{1}{w_j} \left\| \frac{1}{n} \left(\boldsymbol{\Psi}_j + \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \right)^\top R_{(-j)} \right\|_2 \leq \lambda (1 - \alpha). \tag{52}$$

From the subgradient Equation (50), we see that $\gamma_j = 0$ is a solution if

$$\frac{1}{w_{jE}} \left| \frac{1}{n} \left((X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) \right)^\top R_{(-jE)} \right| \leq \lambda \alpha. \tag{53}$$

From the subgradient equations we see that

$$\hat{\beta}_E = \frac{S\left(\frac{1}{n \cdot w_E} \left(X_E + \sum_{j=1}^{p} \hat{\gamma}_j (X_E \circ \boldsymbol{\Psi}_j) \mathbf{1}_{m_j} \right)^\top R_{(-E)}, \lambda(1-\alpha) \right)}{\left(X_E + \sum_{j=1}^{p} \hat{\gamma}_j (X_E \circ \boldsymbol{\Psi}_j) \mathbf{1}_{m_j} \right)^\top \left(X_E + \sum_{j=1}^{p} \hat{\gamma}_j (X_E \circ \boldsymbol{\Psi}_j) \mathbf{1}_{m_j} \right)} \tag{54}$$

$$\lambda (1 - \alpha) w_j \frac{\boldsymbol{\theta}_j}{\left\| \boldsymbol{\theta}_j \right\|_2} = \frac{1}{n} \left(\boldsymbol{\Psi}_j + \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \right)^\top R_{(-j)} \tag{55}$$

$$\hat{\gamma}_j = \frac{S\left(\frac{1}{n \cdot w_{jE}} \left((X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) \right)^\top R_{(-jE)}, \lambda \alpha \right)}{\left((X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) \right)^\top \left((X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) \right)}, \tag{56}$$

where $S(x, t) = \text{sign}(x)(|x| - t)$ is the soft-thresholding operator. As was the case in the strong heredity `sail` model, there is a closed form solution for β_E, each γ_j also has a closed form solution and can be solved efficiently for $j = 1, \ldots, p$ using the coordinate descent procedure implemented in the `glmnet` package (Friedman et al., 2010), while we use the quadratic majorization technique implemented in the `gglasso` package (Yang and Zou, 2015) to solve (55). Algorithm 4 details the procedure used to fit the least-squares weak heredity `sail` model.

Algorithm 4 Coordinate descent for least-squares `sail` with weak heredity

function `sail`$(\boldsymbol{X}, Y, X_E, \texttt{basis}, \lambda, \alpha, w_j, w_E, w_{jE}, \epsilon)$ ⊳ Algorithm for solving (47)

$\Psi_j \leftarrow \texttt{basis}(X_j)$, $\widetilde{\Psi}_j \leftarrow X_E \circ \Psi_j$ for $j = 1, \ldots, p$

Center all variables by their sample means

Initialize: $\beta_E^{(0)} = \boldsymbol{\theta}_j^{(0)} = \gamma_j^{(0)} \leftarrow 0$ for $j = 1, \ldots, p$.

Set iteration counter $k \leftarrow 0$

$\boldsymbol{R}^* \leftarrow Y - \beta_E^{(k)} X_E - \sum_j \boldsymbol{\Psi}_j \boldsymbol{\theta}_j^{(k)} - \sum_j \gamma_j^{(k)} \widetilde{\boldsymbol{\Psi}}_j (\beta_E^{(k)} \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j^{(k)})$

repeat

• To update $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)$

$\widetilde{X}_j \leftarrow \widetilde{\boldsymbol{\Psi}}_j (\beta_E^{(k)} \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j^{(k)}) \qquad$ for $j = 1, \ldots, p$

$R \leftarrow R^* + \sum_{j=1}^p \gamma_j^{(k)} \widetilde{X}_j$

$$\boldsymbol{\gamma}^{(k)(new)} \leftarrow \arg\min_{\boldsymbol{\gamma}} \frac{1}{2n} \left\| R - \sum_j \gamma_j \widetilde{X}_j \right\|_2^2 + \lambda\alpha \sum_j w_{jE} \left| \gamma_j \right|$$

$\Delta = \sum_j (\gamma_j^{(k)} - \gamma_j^{(k)(new)}) \widetilde{X}_j$

$R^* \leftarrow R^* + \Delta$

• To update $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p)$

$\widetilde{X}_j \leftarrow \boldsymbol{\Psi}_j + \gamma_j^{(k)} \widetilde{\boldsymbol{\Psi}}_j$ for $j = 1, \ldots, p$

for $j = 1, \ldots, p$ **do**

$R \leftarrow R^* + \widetilde{X}_j \boldsymbol{\theta}_j^{(k)}$

$$\boldsymbol{\theta}_j^{(k)(new)} \leftarrow \arg\min_{\boldsymbol{\theta}_j} \frac{1}{2n} \left\| R - \widetilde{X}_j \boldsymbol{\theta}_j \right\|_2^2 + \lambda(1-\alpha) w_j \left\| \boldsymbol{\theta}_j \right\|_2$$

$\Delta = \widetilde{X}_j (\boldsymbol{\theta}_j^{(k)} - \boldsymbol{\theta}_j^{(k)(new)})$

$R^* \leftarrow R^* + \Delta$

• To update β_E

$\widetilde{X}_E \leftarrow X_E + \sum_j \gamma_j^{(k)} \widetilde{\boldsymbol{\Psi}}_j \mathbf{1}_{m_j}$

$R \leftarrow R^* + \beta_E^{(k)} \widetilde{X}_E$

$$\beta_E^{(k)(new)} \leftarrow \frac{1}{\widetilde{X}_E^\top \widetilde{X}_E} S\left(\frac{1}{n \cdot w_E} \widetilde{X}_E^\top R, \lambda(1-\alpha) \right)$$

⊳ $S(x, t) = \text{sign}(x)(|x| - t)_+$

$\Delta = (\beta_E^{(k)} - \beta_E^{(k)(new)}) \widetilde{X}_E$

$R^* \leftarrow R^* + \Delta$

$k \leftarrow k + 1$

until convergence criterion is satisfied: $\left| Q(\boldsymbol{\Phi}^{(k-1)}) - Q(\boldsymbol{\Phi}^{(k)}) \right| / Q(\boldsymbol{\Phi}^{(k-1)}) < \epsilon$

Compute the intercept β_0

$\beta_0 \leftarrow \overline{Y} - \sum_{j=1}^p \overline{\boldsymbol{\Psi}}_j \hat{\boldsymbol{\theta}}_j - \hat{\beta}_E \overline{X}_E - \sum_{j=1}^p \hat{\gamma}_j \hat{\beta}_E \overline{(X_E \circ \boldsymbol{\Psi}_j)} \hat{\boldsymbol{\theta}}_j$

B.4.1. Maximum penalty parameter (λ_{max}) for weak heredity

The smallest value of λ for which the entire parameter vector $(\beta_E, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p, \gamma_1, \ldots, \gamma_p)$ is $\mathbf{0}$ is:

$$\lambda_{max} = \frac{1}{n} \max \left\{ \frac{1}{(1-\alpha) w_E} \left(X_E + \sum_{j=1}^p \gamma_j (X_E \circ \boldsymbol{\Psi}_j) \mathbf{1}_{m_j} \right)^\top R_{(-E)}, \right.$$

$$\max_j \frac{1}{(1-\alpha)w_j} \left\| \left(\mathbf{\Psi}_j + \gamma_j (X_E \circ \mathbf{\Psi}_j)\right)^\top R_{(-j)} \right\|_2,$$
$$\left. \max_j \frac{1}{\alpha w_{jE}} \left((X_E \circ \mathbf{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) \right)^\top R_{(-jE)} \right\}, \quad (57)$$

which reduces to

$$\lambda_{max} = \frac{1}{n(1-\alpha)} \max \left\{ \frac{1}{w_E} \left(X_E\right)^\top R_{(-E)}, \max_j \frac{1}{w_j} \left\| \left(\mathbf{\Psi}_j\right)^\top R_{(-j)} \right\|_2 \right\}.$$

This is the same λ_{max} as the least-squares strong heredity `sail` model.

C. Additional Simulation Results

We visually inspected whether our method could correctly capture the shape of the association between the predictors and the response for both main and interaction effects. To do so, we plotted the true and predicted curves for scenario 1a) only. Figure 5 shows each of the four main effects with the estimated curves from each of the 200 simulations along with the true curve. We can see the effect of the penalty on the parameters, i.e., decreasing prediction variance at the cost of increased bias. This is particularly well illustrated in the bottom right panel where `sail` smooths out the very wiggly component function $f_4(x)$. Nevertheless, the primary shapes are clearly being captured.

To visualize the estimated interaction effects, we ordered the 200 simulation runs by the Euclidean distance between the estimated and true regression functions. Following Radchenko and James 2010, we then identified the 25th, 50th, and 75th best simulations and plotted, in Figures 6 and 7, the interaction effects of X_E with $f_3(X_3)$ and $f_4(X_4)$, respectively. We see that `sail` does a good job at capturing the true interaction surface for $X_E \cdot f_3(X_3)$. Again, the smoothing and shrinkage effect is apparent when looking at the interaction surfaces for $X_E \cdot f_4(X_4)$.

In Figure 8 we visualize the variable selection results from 210 replications of the simulation study for strong hierarchy `sail` using UpSet plots (Conway et al., 2017). Shown are the selected models and their frequencies. We can see that the environment variable is always selected across all simulation scenarios and replications.

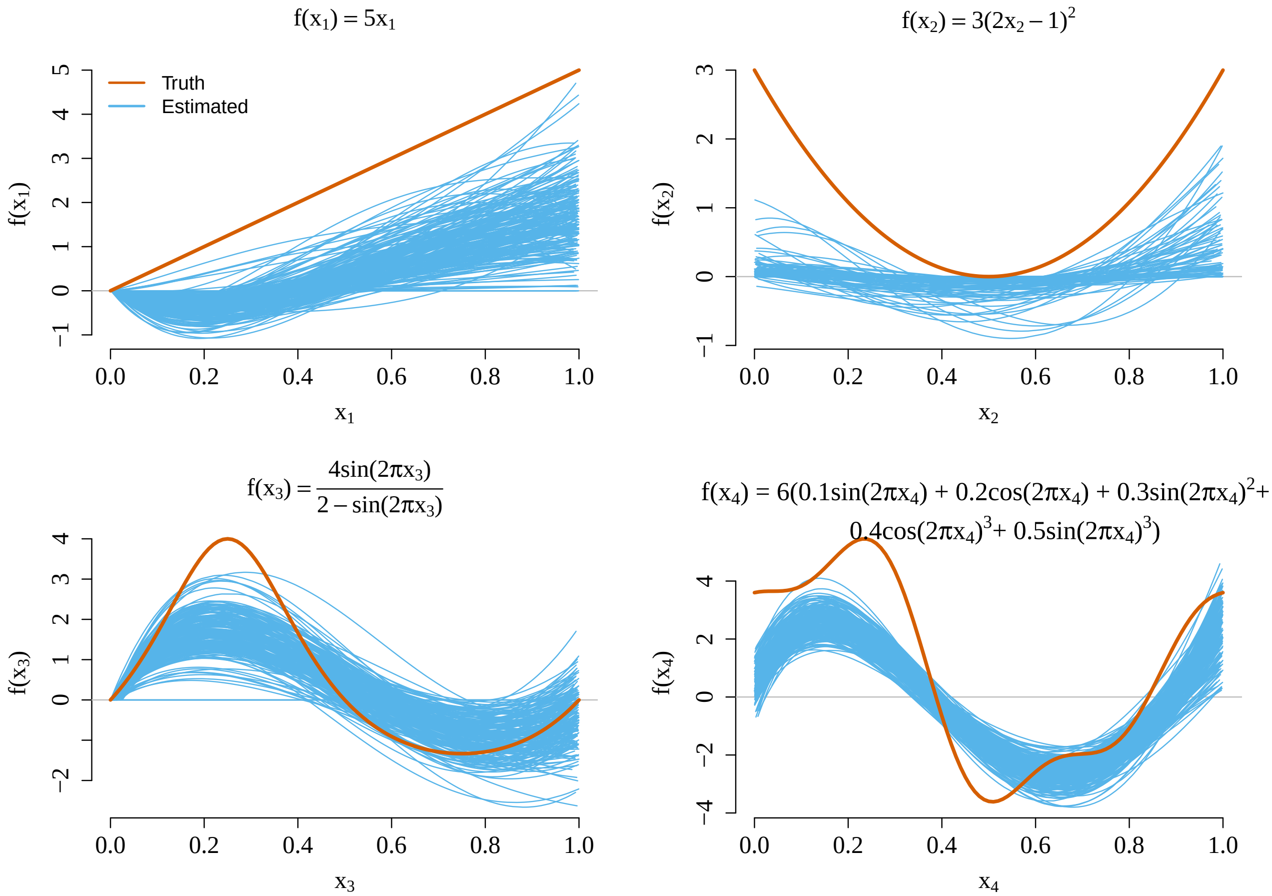


Figure 5: True and estimated main effect component functions for scenario 1a). The estimated curves represent the results from each one of the 200 replications conducted.

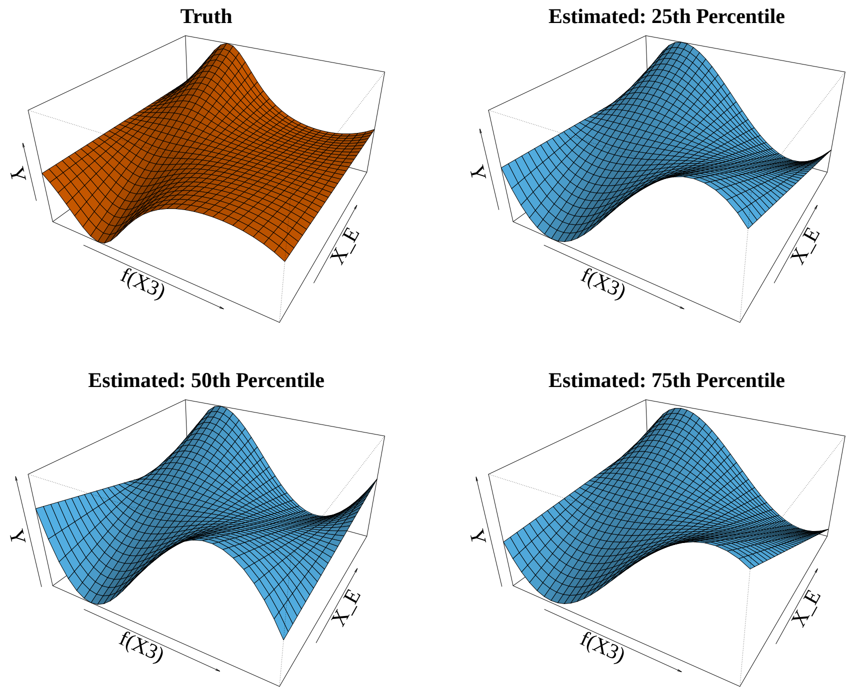


Figure 6: True and estimated interaction effects for $X_E \cdot f_3(X_3)$ in simulation scenario 1a).

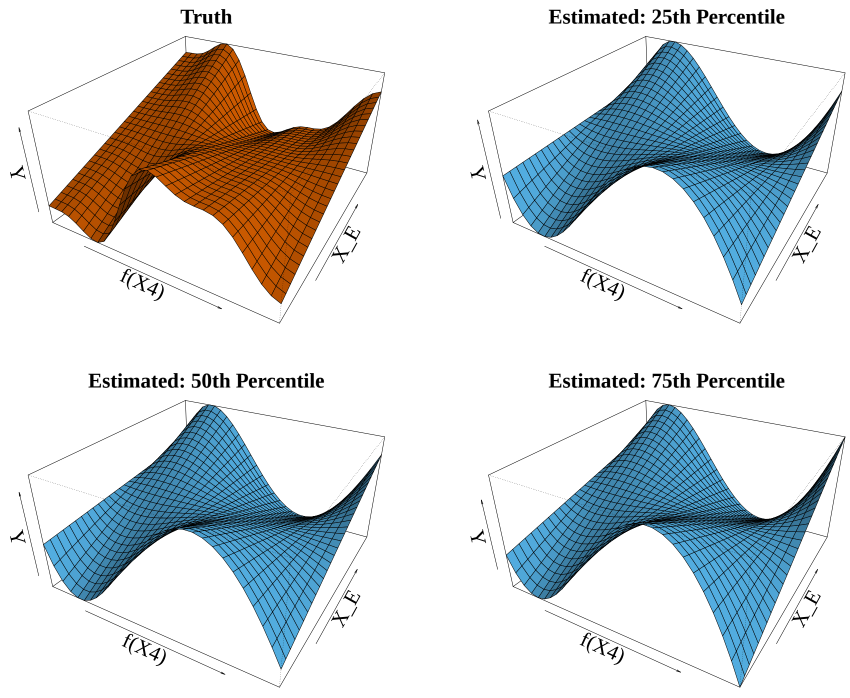
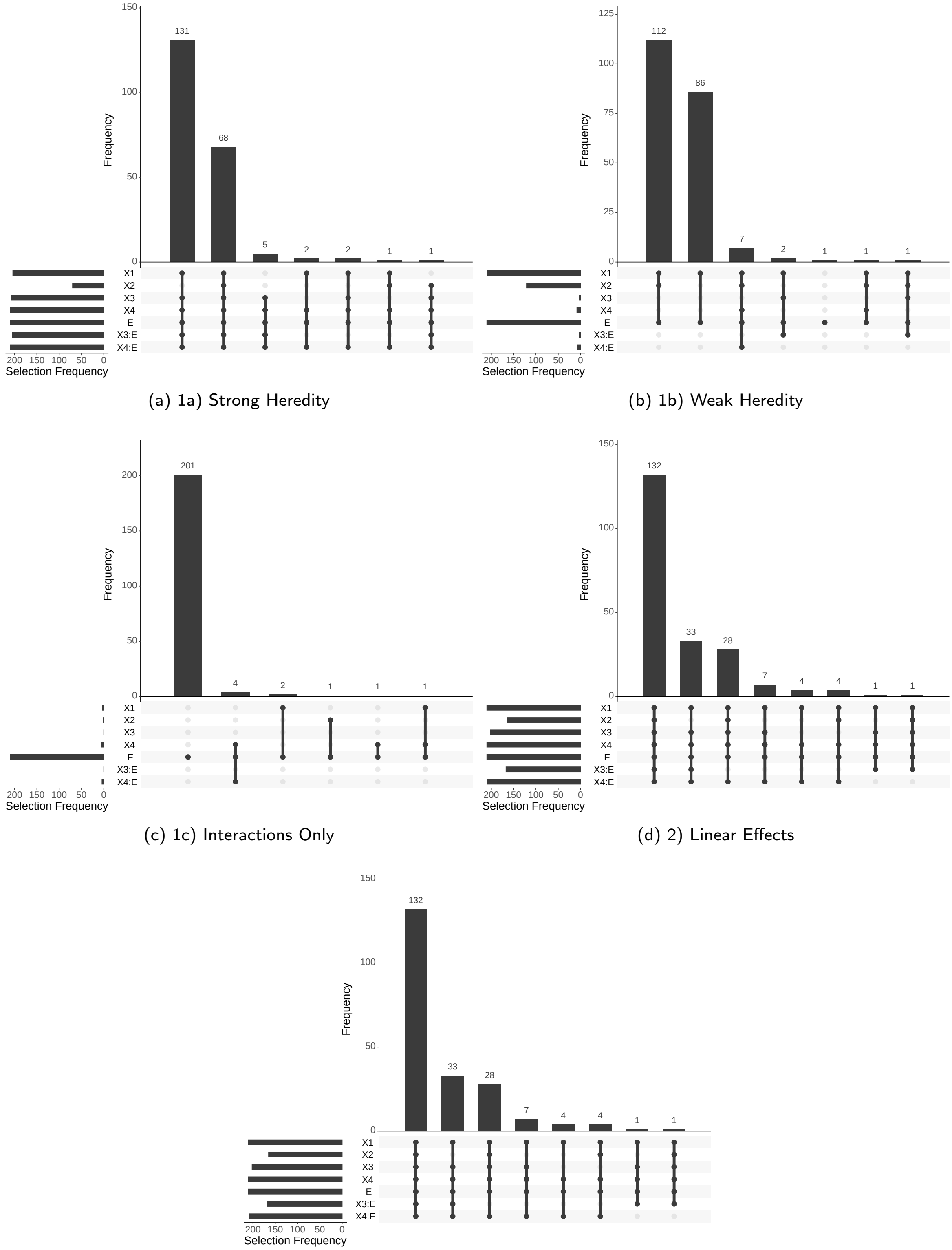


Figure 7: True and estimated interaction effects for $X_E \cdot f_4(X_4)$ in simulation scenario 1a).



(a) 1a) Strong Heredity

(b) 1b) Weak Heredity

(c) 1c) Interactions Only

(d) 2) Linear Effects

(e) 3) Main Effects Only

Figure 8: Variable selection results from 210 replications of the simulation study for strong hierarchy `sail` visualized using UpSet plots (Conway et al., 2017). Shown are the selected models and their frequencies. We can see that the environment variable is always selected across all simulation scenarios and replications.

D. Additional Results on PRS for Educational Attainment

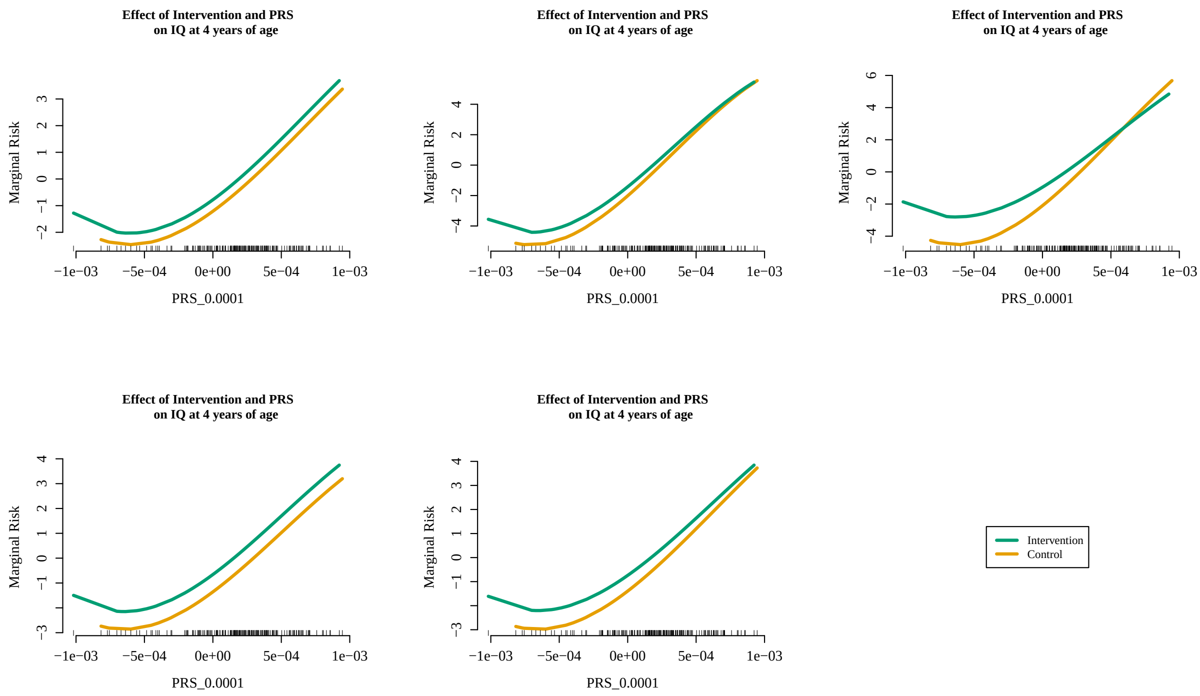


Figure 9: Estimated interaction effect identified by the weak heredity `sail` using cubic B-splines and $\alpha = 0.1$ for the Nurse Family Partnership data for the 5 imputed datasets. Of the 189 subjects, 19 IQ scores were imputed using `mice` (Buuren and Groothuis-Oudshoorn, 2010). The selected model, chosen via 10-fold cross-validation, contained three variables: the main effects for the intervention and the PRS for educational attainment using genetic variants significant at the 0.0001 level, as well as their interaction.

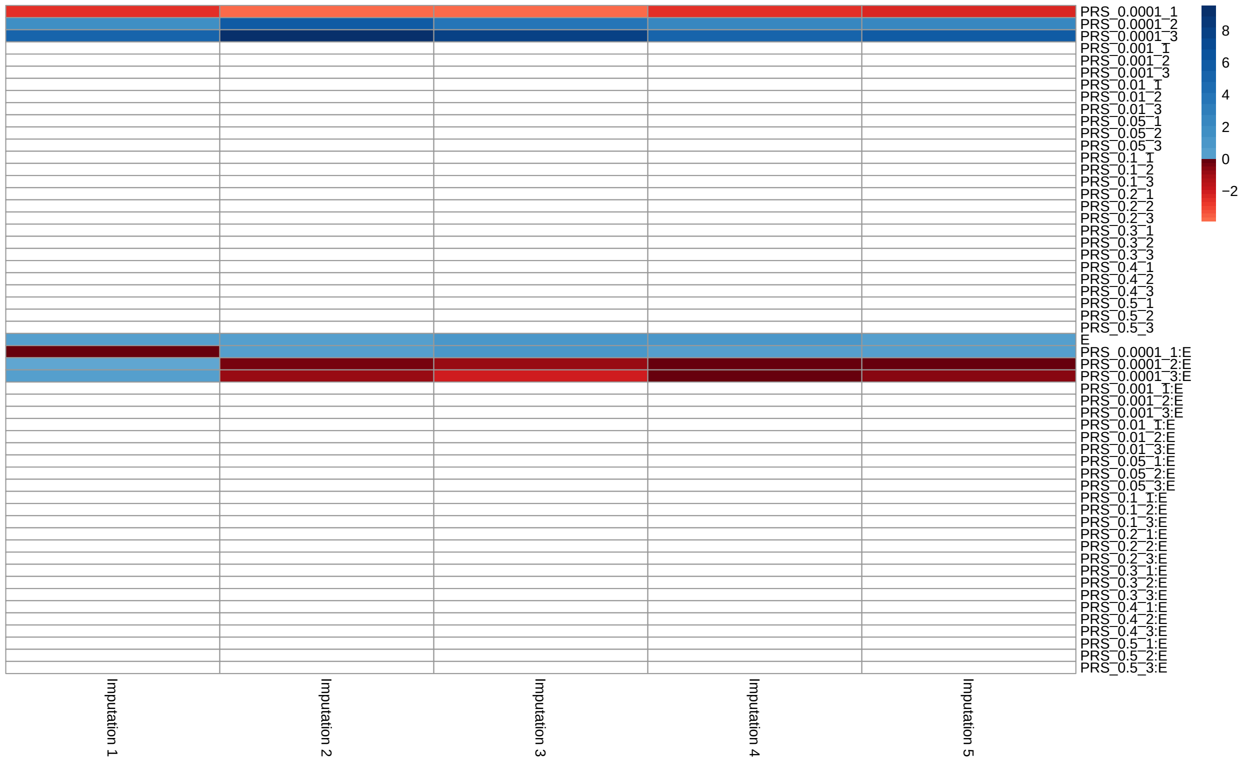


Figure 10: Coefficient estimates obtained by the weak heredity `sail` using cubic B-splines and $\alpha = 0.1$ for the Nurse Family Partnership data for the 5 imputed datasets. Of the 189 subjects, 19 IQ scores were imputed using `mice` (Buuren and Groothuis-Oudshoorn, 2010). The selected model, chosen via 10-fold cross-validation, contained three variables: the main effects for the intervention and the PRS for educational attainment using genetic variants significant at the 0.0001 level, as well as their interaction. This results was consistent across all 5 imputed datasets. The white boxes indicate a coefficient estimate of 0.

E. Data Availability and Code to Reproduce Results

The R scripts used to simulate the data for the simulation studies in Section 4 are provided along with the code for each of the methods being compared. The data used for the two real data analyses in Section 5 are publicly available. The first dataset from the Nurse Family Partnership program is provided by one of the authors of the manuscript (David Olds). The second dataset from the Study to Understand Prognoses Preferences Outcomes and Risks of Treatment (SUPPORT) is publicly available from the Vanderbilt University Department of Biostatistics website.

E.1. Datasets

The datasets are available at `https://github.com/sahirbhatnagar/sail/tree/master/manuscript/raw_data`

1. Nurse Family Partnership program data consists of three files. They are merged together using the script `https://github.com/sahirbhatnagar/sail/blob/master/manuscript/bin/PRS_bootstrap.R`
    - Gen_3PC_scores.txt
    - IQ_and_mental_development_variables_for_Sahir_with_study_ID.txt
    - NFP_170614_INFO08_nodup_hard09_noambi_GWAS_EduYears_Pooled_beta_withaf_5000pruned_noambi_16Jan2018.score
2. The SUPPORT data consists of a single file:
    - `https://github.com/sahirbhatnagar/sail/blob/master/manuscript/raw_data/support2.csv`

All datasets are in `.txt` format. Code used to read in the datasets are provided in the section below. All output from this project published online is available according to the conditions of the Creative Commons License (https://creativecommons.org/licenses/by-nc-sa/2.0/)

E.2. Code

The software which implements our algorithm is available in an R package published on CRAN (https://cran.r-project.org/package=sail) version 0.1.0 with MIT license. The paper itself is written in knitr format, and therefore includes both the code and text in the same `.Rnw` file.

The scripts and data used to produce the results in the manuscript are available at https://github.com/sahirbhatnagar/sail/tree/master/manuscript.

The knitr file which contains both the main text and code is available at: https://github.com/sahirbhatnagar/sail/blob/master/manuscript/source/sail_manuscript_v2.Rnw

The manuscript was compiled using R version 3.6.1 with knitr version 1.25.

The bootstrap analysis was run in parallel on a compute cluster with 40 cores. Though this is not necessary to reproduce the results, it definitely speeds up the computation time.

E.2.1. Instructions for Use

All tables and figures from the paper can be reproduced by compiling the knitr file. The easiest way to reproduce the results is to download the GitHub repository and compile the knitr file from within an R session as follows:

1. Download the GitHub repository https://github.com/sahirbhatnagar/sail/archive/master.zip
2. From within an R session, run the command: `knitr::knit2pdf('sail_manuscript_v2.Rnw')`

Note that to speed up compilation time, we have saved the simulation and bootstrap results in `.RData` files available at https://github.com/sahirbhatnagar/sail/tree/master/manuscript/results. These `.RData` files are called directly by the knitr file.

Note also that the R scripts used to generate the results are called from the knitr file using the 'code externalization' functionality of knitr (https://yihui.org/knitr/demo/externalization/). That is, the actual R code is stored in R scripts and not within the knitr file. These R scripts are available at https://github.com/sahirbhatnagar/sail/tree/master/manuscript/bin.

The expected run time to compile the manuscript is about 5 minutes on a standard desktop machine, assuming that you are using the pre-run simulation and bootstrap results.

E.2.2. R Package Vignette

A website with two vignettes has been created for our sail package available at https://sahirbhatnagar.com/sail/

The 2 vignettes are:

1. https://sahirbhatnagar.com/sail/articles/introduction-to-sail.html
2. https://sahirbhatnagar.com/sail/articles/user-defined-design.html