McGill University
Math 325A: Differential Equations

LECTURE 18: SERIES SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS (III)

BESSEL FUNCTIONS

(Text: Chap. 8)

1 Introduction

In this lecture we study an important class of functions which are defined by the differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where $\nu \geq 0$ is a fixed parameter. This DE is known **Bessel's equation of order** ν. This equation has $x = 0$ as its only singular point. Moreover, this singular point is a regular singular point since

$$xp(x) = 1, \quad x^2q(x) = x^2 - \nu^2.$$

Bessel's equation can also be written

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2}) = 0$$

which for x large is approximately the DE $y'' + y = 0$ so that we can expect the solutions to oscillate for x large. The indicial equation is $r(r-1) + r - \nu^2 = r - \nu^2$ whose roots are $r_1 = \nu$ and $r_2 = -\nu$. The recursion equations are

$$\big[(1+r)^2 - \nu^2\big]a_1 = 0, \quad \big[(n+r)^2 - \nu^2\big]a_n = -a_{n-2}, \quad \text{for } n \geq 2.$$

The general solution of these equations is $a_{2n+1} = 0$ for $n \geq 0$ and

$$a_{2n}(r) = \frac{(-1)^n a_0}{(r+2-\nu)(r+4-\nu)\cdots(r+2n-\nu)(r+2+\nu)(r+4+\nu)\cdots(r+2n+\nu)}.$$

2 The Case of Non-integer μ

If ν is not an integer and $\mu \neq 1/2$, we have the case (I). Two linearly independent solutions of Bessel's equation $J_\nu(x)$, $J_{-\nu}(x)$ can be obtained by taking $r = \pm\nu$, $a_0 = 1/2^\nu\Gamma(\nu+1)$. Since, in this case,

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(r+1)(r+2)\cdots(r+n)},$$

we have for $r = \pm\nu$

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+r}.$$

Recall that the Gamma function $\Gamma(x)$ is defined for $x \geq -1$ by

$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt.$$

For $x \geq 0$ we have $\Gamma(x+1) = x\Gamma(x)$, so that $\Gamma(n+1) = n!$ for n an integer ≥ 0. We have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-t} t^{-1/2} dt = 2\int_0^{\infty} e^{-x^2} dt = \sqrt{\pi}.$$

The Gamma function can be extended uniquely for all x except for $x = 0, -1, -2, \ldots, -n, \ldots$ to a function which satisfies the identity $\Gamma(x) = \Gamma(x)/x$. This is true even if x is taken to be complex. The resulting function is analytic except at zero and the negative integers where it has a simple pole.

These functions are called **Bessel functions of first kind of order ν.**

As an exercise the reader can show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x), \quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \sin(x).$$

3 The Case of $\mu = -m$ with m an integer ≥ 0

For this case, the first solution $J_m(x)$ can be obtained as in the last section. As examples, we give some such solutions as follows:

- The Case of $m = 0$:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

- The case $m = 1$:

$$J_1(x) = \frac{1}{2} y_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(n+1)!} x^{2n}.$$

To derive the second solution, one has to proceed differently. For $\nu = 0$ the indicial equation has a repeated root, we have the case (II). One has a second solution of the form

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty} a'_{2n}(0) x^{2n}$$

where

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2 \cdots (r+2n)^2}.$$

It follows that

$$\frac{a'_{2n}(r)}{a_{2n}} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots \frac{1}{r+2n}\right)$$

so that

$$a'_{2n}(0) = -\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) a_{2n}(0) = -h_n a_{2n}(0),$$

where we have defined
$$h_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Hence
$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} h_n}{2^{2n}(n!)^2} x^{2n}.$$
Instead of y_2, the second solution is usually taken to be a certain linear combination of y_2 and J_0. For example, the function
$$Y_0(x) = \frac{2}{\pi}\Big[y_2(x) + (\gamma - \ln 2)J_0(x)\Big],$$
where $\gamma = \lim_{n\to\infty}(h_n - \ln n) \approx 0.5772$, is known as the **Weber function of order** 0. The constant γ is known as Euler's constant; it is not known whether γ is rational or not.

If $\nu = -m$, with $m > 0$, the the roots of the indicial equation differ by an integer, we have the case (III). Then one has a solution of the form
$$y_2 = aJ_m(x)\ln(x) + \sum_{n=0}^{\infty} b'_{2n}(-m)x^{2n+m}$$
where $b_{2n}(r) = (r+m)a_{2n}(r)$ and $a = b_{2m}(-m)$. In the case $m = 1$ we have $a_0 = 1$,
$$a = b_2(-1) = -\frac{a_0}{2},$$

$$b_0(r) = (r - r_2)a_0$$
and
$$b_{2n}(r) = \frac{(-1)^n a_0}{(r+3)(r+5)\cdots(r+2n-1)(r+3)(r+5)\cdots(r+2n+1)}, \qquad (n \geq 1),.$$
Subsequently, we have
$$b'_0(r) = a_0$$
and for $n \geq 1$,
$$b'_{2n}(r) = -\left(\frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n-1} + \frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n+1}\right) b_{2n}(r).$$
From here, we obtain
$$\begin{aligned} b'_0(-1) &= a_0 \\ b'_{2n}(-1) &= \frac{-1}{2}(h_n + h_{n-1})b_{2n}(-1) \qquad (n \geq 1), \end{aligned}$$
where
$$b_{2n}(-1) = \frac{(-1)^n}{2^{2n}(n-1)!n!}a_0.$$
So that

$$
\begin{aligned}
y_2 &= \frac{-1}{2} y_1(x)\ln(x) + \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n+1}(n-1)!n!} x^{2n}\right] \\
&= -J_1(x)\ln(x) + \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n+1}(n-1)!n!} x^{2n}\right]
\end{aligned}
$$

where, by convention, $h_0 = 0$, $(-1)! = 1$. The **Weber function of order 1** is defined to be

$$
Y_1(x) = \frac{4}{\pi}\Big[- y_2(x) + (\gamma - \ln 2) J_1(x)\Big].
$$

The case $m > 1$ is slightly more complicated and will not be treated here.

The second solutions $y_2(x)$ of Bessel's equation of order $m \geq 0$ are unbounded as $x \to 0$. It follows that any solution of Bessel's equation of order $m \geq 0$ which is bounded as $x \to 0$ is a scalar multiple of J_m. The solutions which are unbounded as $x \to 0$ are called **Bessel functions of the second kind**. The Weber functions are Bessel functions of the second kind.