Summary. Optimal centroid Voronoi tessellations (CVT) have a wide range of applications, such as vector quantization, data analysis, image processing and resource management. A centroid Voronoi tessellation is a tessellation of a region such that the generating points are centroids of their Voronoi regions.

We consider the three dimensional space, with domain being the unit cube Q, uniform density and square distance interaction. A configuration Y is optimal if the CVT associated to Y has the lowest possible energy among all configurations with the same cardinality.

Such configurations are closely related to Gersho's conjecture:

- (Gersho's conjecture) there exists a polytope V with $|V| = 1$ which tiles the space with congruent copies such that the following holds: let $(Y_n)_n$ be a sequence of minimizers, with $Y_n \in \text{argmin}_{\sharp Y=n} E(Y)$, then the Voronoi cells of points Y_n are asymptotically congruent to $n^{-1/3}V$ as $n \to +\infty$.

Gersho's conjecture is trivial in 1D, and has been proven in 2D. For higher dimensions it is still open. My contribution involves giving an explicit upper bound, independent of n, on the geometric complexity of Voronoi regions in optimal CVTs.

Quantization in 3D: energy estimate and Gersho's conjecture

Description of the problem. Optimal centroid Voronoi tessellations (CVT) have a wide range of applications, such as vector quantization, data analysis, image processing and resource management. A centroid Voronoi tessellation is a tessellation of a region such that the generating points are centroids of their Voronoi regions.

We consider the three dimensional space, and restrict our domain to the unit cube Q. Moreover, we will consider the case of uniform density and square distance interaction: that is, the energy associated to a mesh Y is

$$E(Y) := \int_Q d^2(x, Y)dx = \sum_{k=1}^{\sharp Y} E(V_k), \qquad E(V_k) := \int_{V_k} |x - y_k|^2 dx, \qquad k = 1, \cdots, \sharp Y,$$

where V_k denotes the Voronoi region generated by $y_k \in Y$, $k = 1, \cdots, \sharp Y$. The centroid of V_k is a point $y \in V_k$ satisfying

$$\int_{V_k} (x - y) \cdot e_j dx = 0, \qquad j = 1, 2, 3,$$

with e_j denoting the unit vector with "1" in the j-th position. It is straightforward to check that y is a centroid of V_k if and only if

$$\int_{V_k} |x - y|^2 dx = \min_{y' \in V_k} \int_{V_k} |x - y'|^2 dx.$$

A configuration Y is optimal if the CVT associated to Y has the lowest possible energy among all configurations Y' with $\sharp Y' = \sharp Y$.

Such configurations are closely related to Gersho's conjecture:

- (Gersho's conjecture) there exists a polytope V with $|V| = 1$ which tiles the space with congruent copies such that the following holds: let $(Y_n)_n$ be a sequence of minimizers, with $Y_n \in \text{argmin}_{\sharp Y=n} E(Y)$, then the Voronoi cells of points Y_n are asymptotically congruent to $n^{-1/3}V$ as $n \to +\infty$.

Gersho's conjecture is trivial in 1D, and has been proven in 2D: it is known (see for instance [5, 3]) that the optimal CVT is a tessellation with regular hexagons.

For higher dimensions, it is open. The following partial results are known:

(1) Zador's asymptotic formula [6]. For the 3D case, it reads: there exists some constant $\tau > 0$ such that given any sequence $(Y_n)_n$, with $Y_n \in \text{argmin}_{\sharp Y=n} E(Y)$, then $n^{2/3}E(Y_n) \to \tau$. Moreover, it is known that

$$\tau \geq 0.6 \cdot (\frac{3}{4\pi})^{2/3}.$$

(2) Gruber's uniform distribution result [4]. In the 3D case, with domain being the unit cube Q, this reads: let Y_n be a sequence of minimizers, and it holds (for $n \gg 1$):
   (a) there exists $\beta > 1$ such that Y_n is a $(1/\beta n^{1/3}, 1/n^{1/3})$-Delone set,

   (b) Y_n is uniformly distributed in Q, i.e.

$$\sharp(K \cap Y_n) = |K|n + o(n) \qquad \text{as } n \to +\infty$$

   for any Jordan measurable set $K \subseteq Q$.

(3) Barnes and Sloane [1]: the BCC (body centered cubes) lattice is asymptotically optimal among all lattices.

It is worthy noting that, even in view of the above results, Gersho's conjecture is still a nonlocal problem, and the optimal polytope V can still have arbitrarily large number of faces.

Research results. In collaboration with Rustum Choksi (McGill University), we were able to bound the geometric complexity of each Voronoi cell of optimal CVTs. The main result is:

Theorem. There exists an explicit universal constant N, such that given an arbitrary sequence $(Y_n)_n$ of minimizers (i.e. $Y_n \in \operatorname{argmin}_{\sharp Y=n} E(Y)$ for any n), then for sufficiently large n, any optimal CVT is composed of Voronoi cells which are polyhedra with at most N faces.

This result reduces Gersho's conjecture to a local, finite problem: indeed if one can prove the following results:

(1) given a tessellation of Q with convex polyhedra, the average number of faces per polyhedron is m.

(2) The function

$$f(n) := \min_{\substack{V \text{ convex, } |V|=1 \\ V \text{ has at most } n \text{ faces}}} \int_V |x-y|^2 dx, \qquad y = \text{centroid of } V,$$

is convex for all $n \leq N$ (with N as in the Theorem above), and an optimal polytope for $n = m$ is space tiling. In principle this condition can be checked numerically.

Then Gersho's conjecture would follow. This because we can follow the Gruber's idea in the proof (from [2]) of Gersho's conjecture in 2D: let

$$g(\alpha, n) := \min_{\substack{V \text{ convex, } |V|=\alpha \\ V \text{ has at most } n \text{ faces}}} \int_V |x-y|^2 dx, \qquad y = \text{centroid of } V,$$

and g is convex in both variables. Then, for any arbitrary tessellation Y_n (with $\sharp Y_n = n$), of Q, let $\{V_k\}$ be the collection of Voronoi cells, and let α_k be the number of faces of V_k. Thus it follows

$$E(Y_n) = \sum_{k=1}^{n} \int_{V_k} |x-y|^2 dx \geq \sum_{k=1}^{n} g(\alpha_k, |V_k|) \geq ng(m, 1/n) + \text{error due to boundary effects.}$$

Since the error due to boundary effects is a higher order term (compared to $ng(m, 1/n)$, as $n \to +\infty$) it follows that the optimal tessellation (as $n \to +\infty$) consists of congruent copies of a space tiling polyhedron realizing $g(m, 1/n)$.

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