Applications of nonlinear network flow models to market equilibria

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Linear Fisher markets

* B: buyers, G: goods.
* Buyer i has budget m_i, 1 divisible unit of each good j.
* Utility U_{ij} for buyer i on 1 unit of good j.
* Market clearing: prices p_j and allocations x_{ij} if:
  * everything is sold
  * all money is spent
  * only best bang-per-buck purchases: max. U_{ij}/p_j.

$U_{ij}=5$
$p_j=\$3$

$U_{ij}=4$
$p_j=\$2$

$5/3 < 4/2$

Linear Fisher markets

* Formulated by Fisher in 1891.
* Special case of the Arrow-Debreu model.
    * An equilibrium exists under very general conditions (Arrow, Debreu, 1954).
    * Nonconstructive proof based on Kakutani's fixed point theorem.
* The linear Fisher model can be captured by the convex program by Eisenberg and Gale '59.

Eisenberg-Gale convex program, 1959

$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

prices: optimal Lagrange multipliers

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

- Optimal solution corresponds to equilibrium prices.
- There exists a rational optimal solution.

Combinatorial algorithms for linear Fisher markets

* Devanur, Papadimitriou, Saberi, Vazirani '02: polynomial time combinatorial algorithm.
* Several extensions and generalizations studied during the last decade.
* Fisher's market with separable piecewise linear concave utilities: PPAD-complete (Vazirani&Yannakakis '11).

Market equilibria with rational convex programs and combinatorial algorithms

Linear Fisher	DPSV ’02/’08 Orlin ’10: strongly poly.	
Perfect price discrimination	Goel&Vazirani ’10	
Spending constraint utilities	Devanur&Vazirani ’04/ ’10	
Arrow-Debreu Nash bargaining	Vazirani ’11	
Nonsymmetric ADNB		

Convex extensions of classical flow models:

* Concave generalized flows (CGF):
  * Truemper '78, Shigeno '06
  * We give the first combinatorial polytime algorithm.
* Minimum cost flows with separable convex objectives (MCCF):
  * Combinatorial polytime algorithms: Minoux '86, Hochbaum&Shantikumar '92, Karzanov&McCormick '97
  * We give a strongly poly algorithm for certain classes of objectives.

Market equilibria with (rational) convex programs and combinatorial algorithms

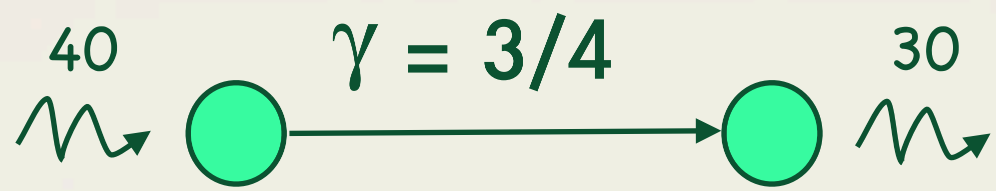
Linear Fisher	DPSV '02/'08 Orlin '10: strongly poly.	CGF	MCCF: strongly poly
Perfect price discrimination	Goel&Vazirani '10	CGF	
Spending constraint utilities	Devanur&Vazirani '04/ '10		MCCF: strongly poly
Arrow-Debreu Nash bargaining	Vazirani '11	CGF	
Nonsymmetric ADNB	?	CGF	

CGF: concave generalized flows V. '12b

MCCF: min. cost separable convex flows V. '12a

Generalized Flows

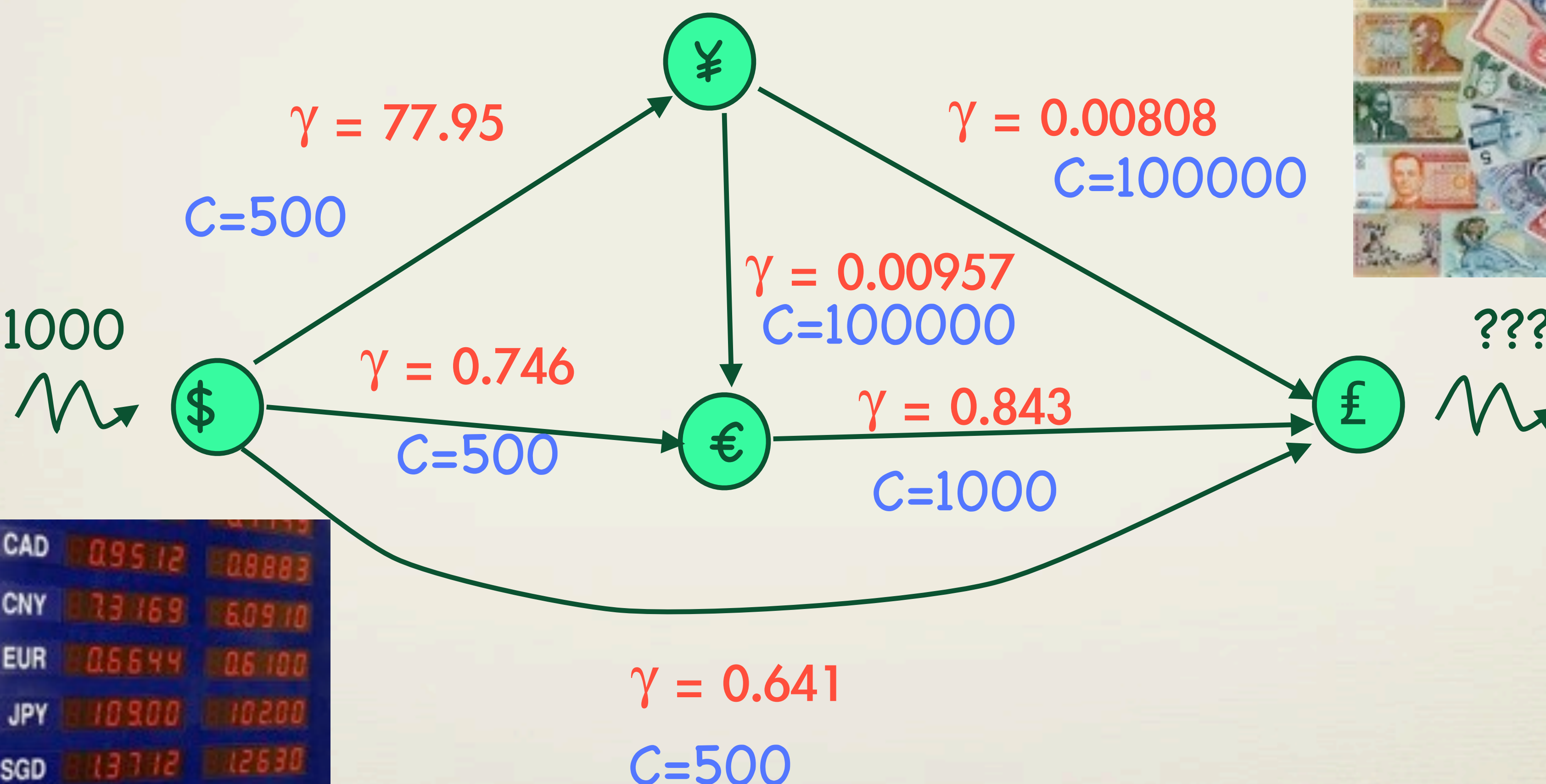
* Network flow model, with gain factors on the arcs.



* Maximize the flow amount reaching the sink t.
* Introduced by Kantorovich '39, Dantzig '62.
* Several applications: financial analysis, transportation, management, etc.

Generalized Flows

* Currency conversion with bounds: obtain the most £ from 1000$.



Generalized Flows

* Linear program.
* Early combinatorial algorithms: Onaga '66, Truemper '77.
* First polynomial time combinatorial algorithm: Goldberg, Plotkin, Tardos '91.
* Followed by Cohen&Megiddo '94, Goldfarb&Jin '96, Goldfarb, Jin&Orlin '97, Tardos&Wayne '98, Wayne '02, Radzik '04, Restrepo&Williamson '09, etc.

Concave Generalized Flows

* Instead of gain factors, concave increasing gain functions.

Convex Program

$$\max \sum_{j:jt\in E} \Gamma_{jt}(f_{jt}) - \sum_{j:tj\in E} f_{tj}$$

$$\sum_{j:ji\in E} \Gamma_{ji}(f_{ji}) - \sum_{j:ij\in E} f_{ij} \geq b_i \quad \forall i \in V - t$$

$$\ell_{ij} \leq f_{ij} \leq u_{ij} \quad \forall ij \in E$$

Concave Generalized Flows

* First defined by Truemper '78.
* Solvable via general purpose convex solver.
* Shigeno '06 gave a combinatorial algorithm that is polynomial for some special classes of gain functions, including piecewise linear.
* We give a polynomial combinatorial algorithm for finding an ε-approximate solution in running time

$$O(m(m + \log n) \log(MUm/\varepsilon))$$

* For problems with a rational optimal solution, we can find it in polynomial time with a final rounding.

Eisenberg-Gale convex program, 1959

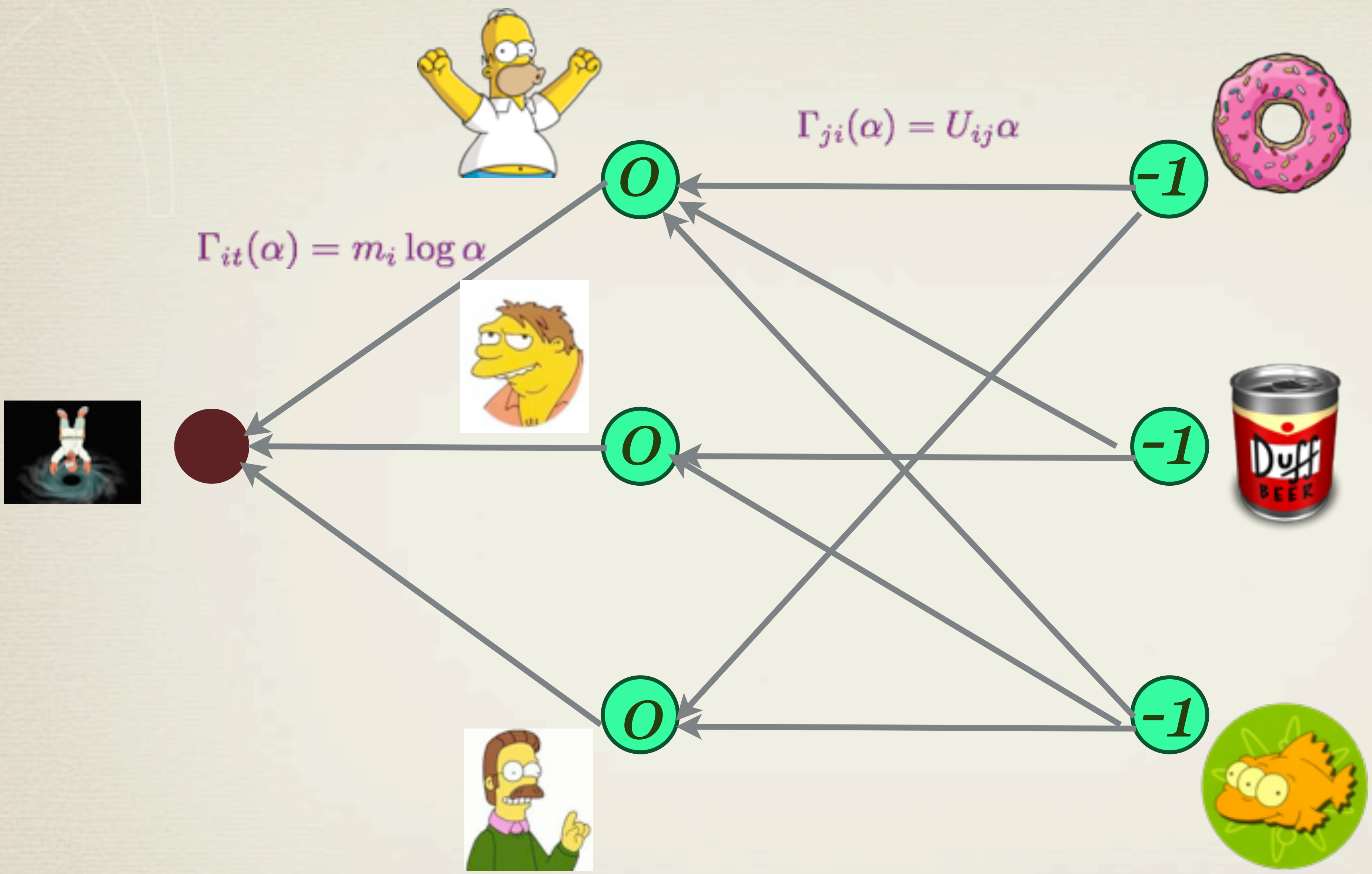
$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

Reduction for linear Fisher market



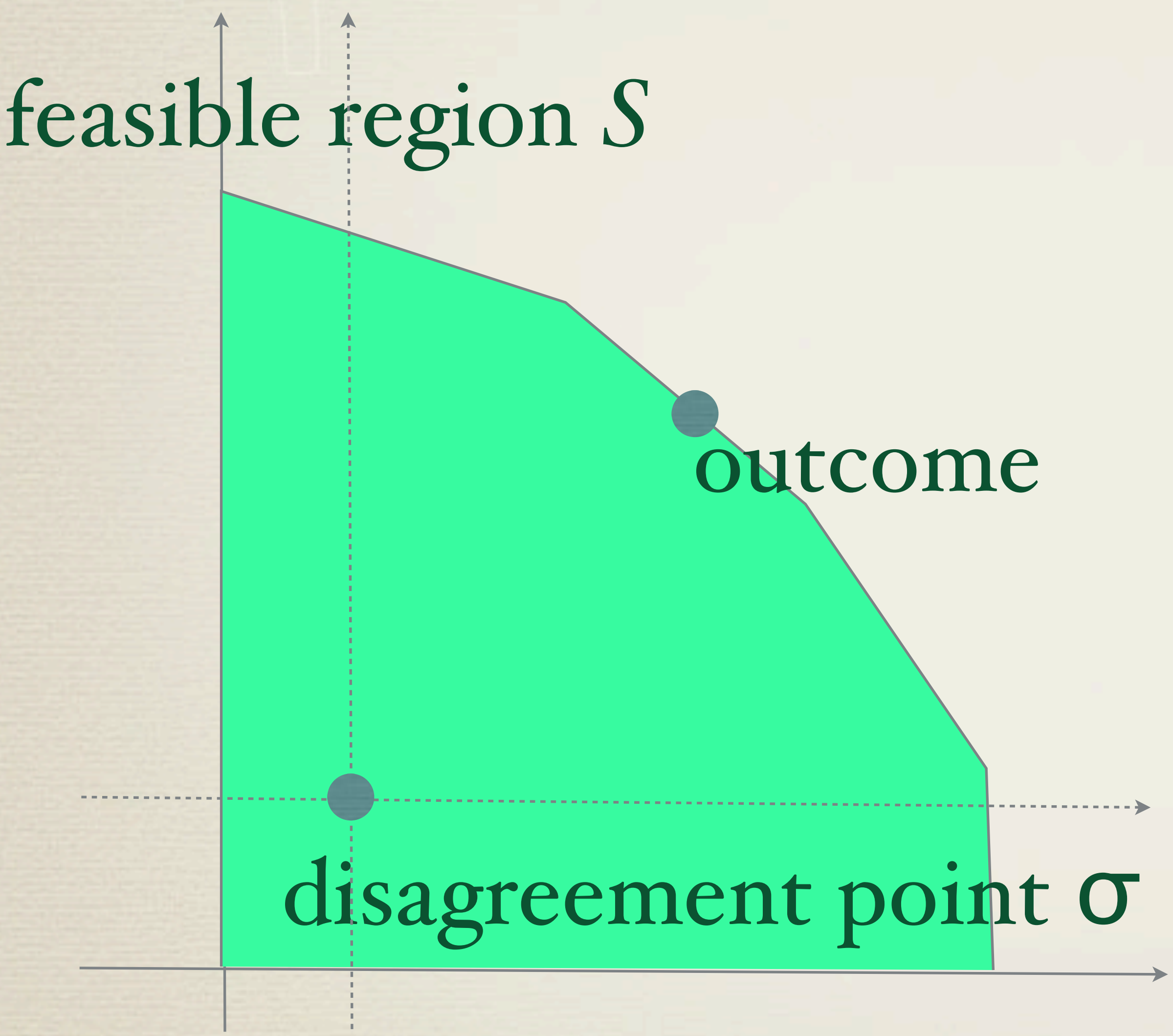
Extensions of linear Fisher markets

- Goel, Vazirani '10: perfect price discrimination
  - (Piecewise linear) concave increasing utilities.
  - Middleman between buyers and firms. He charges different costumers at different rates they are capable of paying.
  - Replace $\Gamma_{ji}(\alpha) = U_{ij}\alpha$ by a piecewise linear concave function.
  - Using our model, it can be replaced by arbitrary concave!

Nash bargaining, 1950

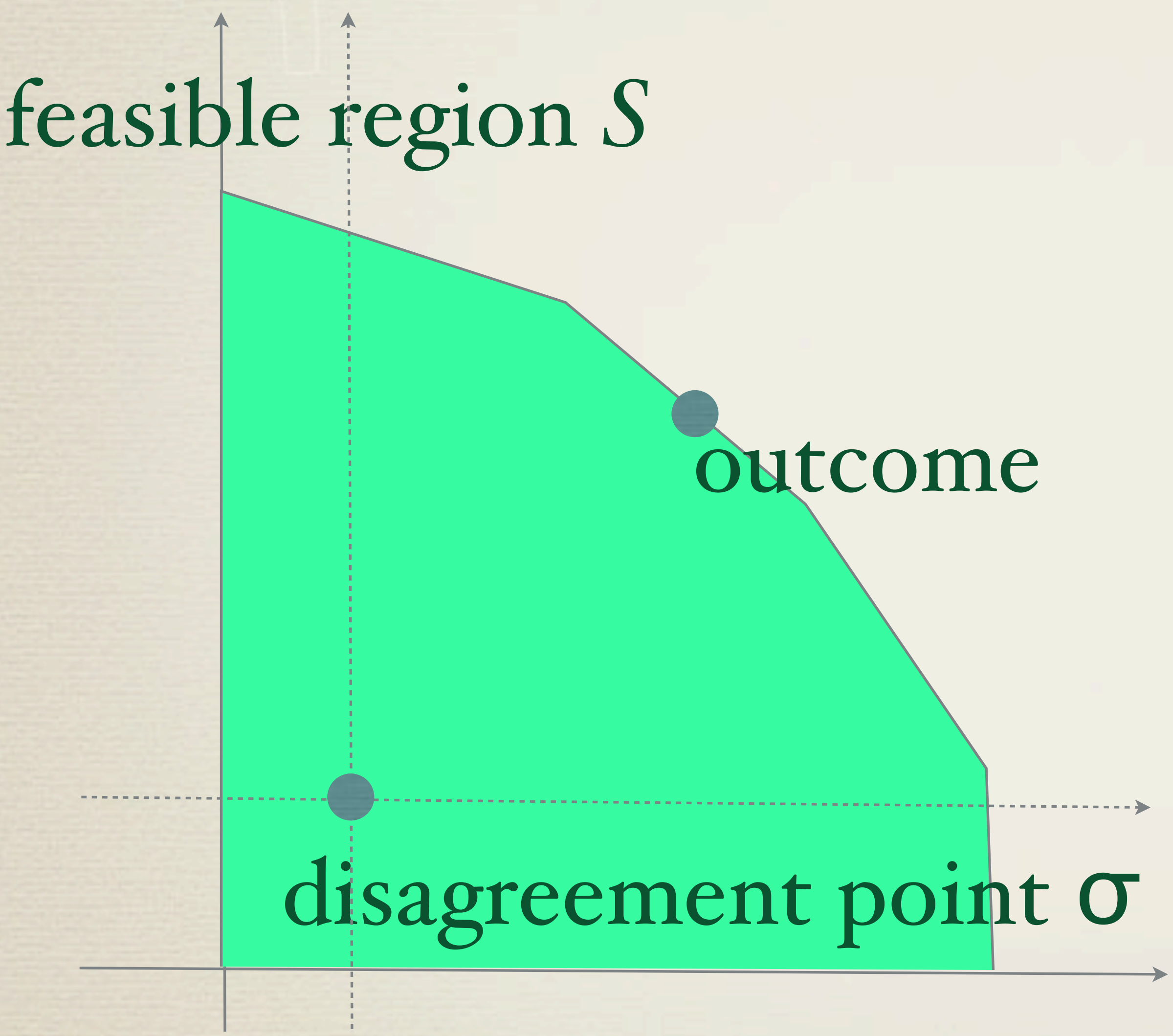
* n players, set of possible outcomes $S \subseteq \mathbb{R}^n_+$
* In outcome $s = (s_1, \ldots, s_n) \in S$, player i gets utility s_i.
* Disagreement point (status quo): $\sigma \in S$
* The players have to agree together in an outcome. If they cannot agree, the status quo remains.

Nash bargaining, 1950



- Which is the best outcome?
- Four criteria:
  - Pareto optimality
  - Invariance under affine transformations
  - Symmetry
  - Indifference of independent alternatives

Nash bargaining, 1950



Theorem (Nash, 1950)

For a convex feasible region, there exists a unique optimal solution, the one maximizing

$$\sum_{i\in[n]} \log(s_i - \sigma_i)$$

Arrow-Debreu Nash bargaining: Vazirani '12

* Nash bargaining between agents, each of them having an initial endowment of goods, giving utility c_i to player i.
* Possible outcomes: distributions of goods.

$$\max \sum_{i \in B} \log(U_i - c_i)$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

Arrow-Debreu Nash bargaining: Vazirani '12

* Vazirani '12: sophisticated two phase algorithm, first deciding feasibility, then optimality.

$$\max \sum_{i \in B} \log(U_i - c_i)$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

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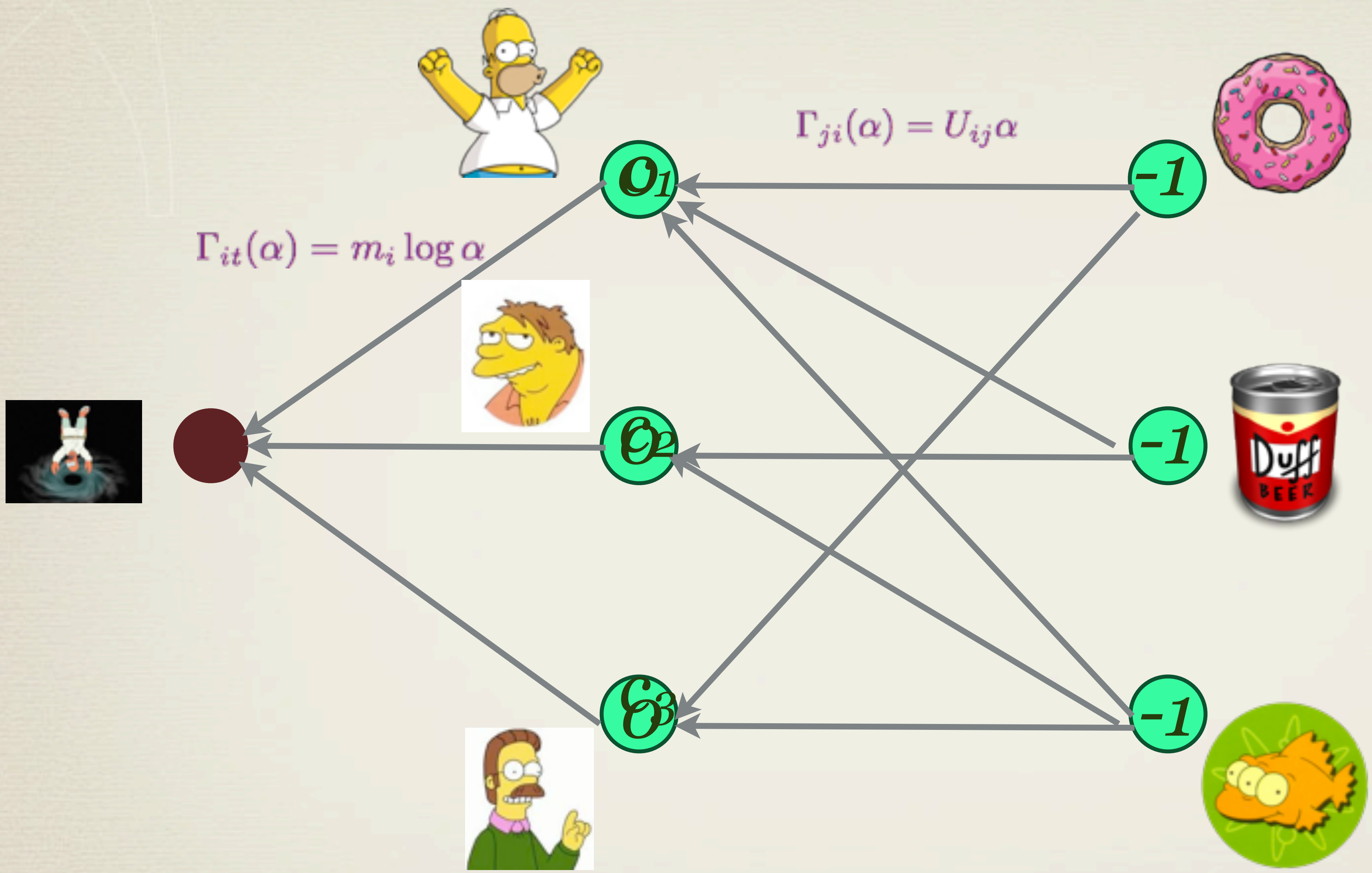
Reduction to Concave Generalized Flows



Arrow-Debreu Nash bargaining: Vazirani, '12

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- Nonsymmetric Nash bargaining: Kalai '77
- Different weights m_i for player i.
- Finding a combinatorial algorithm was left open. Our model also captures this, solving in

$$O(m^2(\log C_{\max} + n \log(nU_{\max} M_{\max})))$$

- Vazirani '12 for symmetric:

$$O(n^8 \log U_{\max} + n^4 \log C_{\max})$$

Further applications of concave generalized flows

* **Jain, Vazirani '10:** single source multiple sink flow markets.
* **Jain '11:** online matching with concave utilities (offline optimum)

Linear and convex flow problems I.

	flows		*generalized flows*
Linear	Minimum cost circulations Edmonds&Karp '72	↔	Generalized flows Goldberg, Plotkin, Tardos'91
	↓		↓
Convex	Minimum cost circulations w. separable convex cost Minoux '86	↔	Concave generalized flows V. '12b

Linear minimum cost flow problem

* $G=(V,E)$ directed graph
* On each arc ij, lower and upper capacities l_{ij}, u_{ij}.
* On each node i, node demand b_i: incoming flow minus outgoing flow should be b_i.
* Minimum cost flow problem: for a cost function c on the arcs, find a minimum cost feasible flow
* First weakly polynomial algorithm: Edmonds, Karp '72
* Strongly polynomial algorithms: Tardos '85, Goldberg, Tarjan '88, Orlin '93, ...

Strongly polynomial algorithms

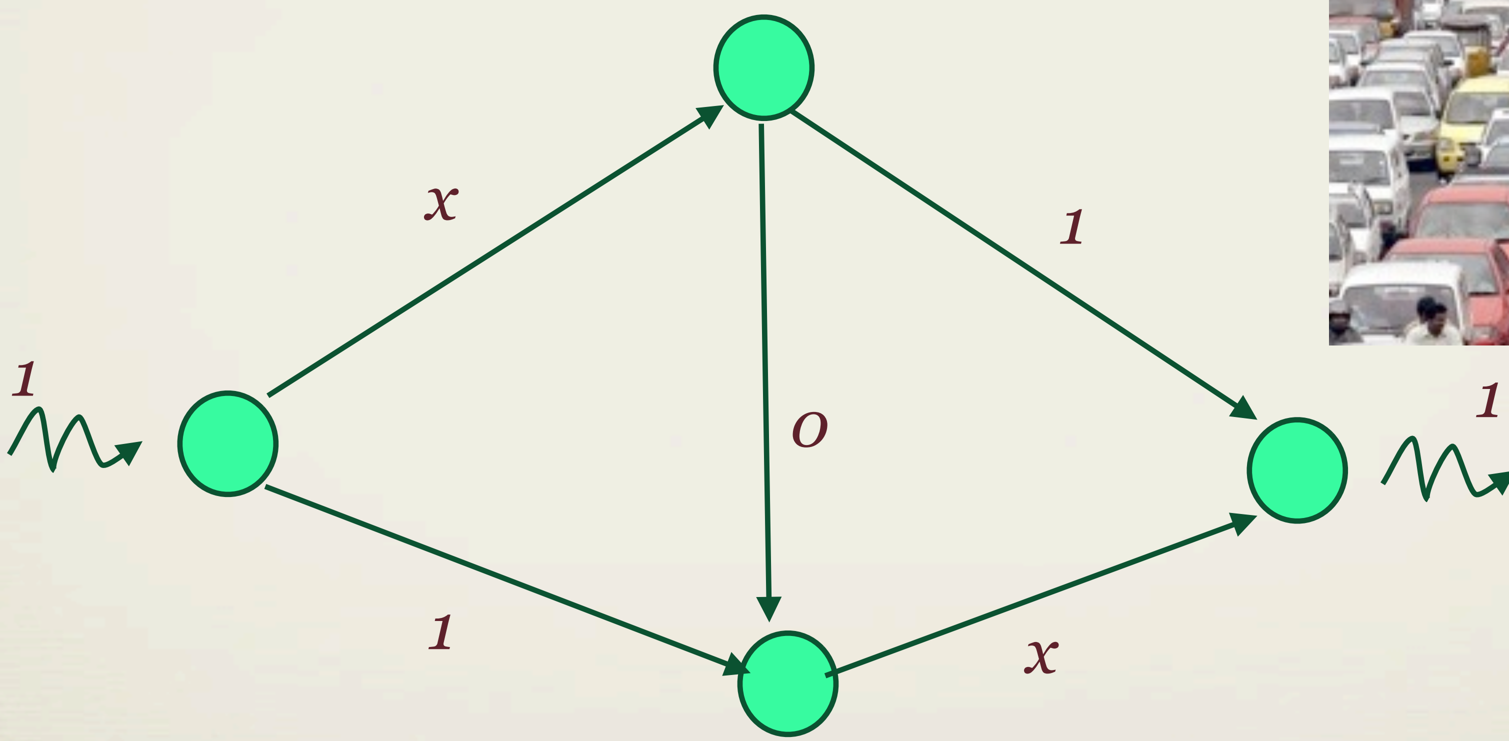
* Problem given by N integers in the input, each at most C.
* (Weakly) polynomial algorithm: the running time is *poly(N, log C)*.
* Strongly polynomial algorithm: the algorithm consists of *poly(N)* elementary arithmetic operations, independent from C.
  * The numbers in the operations are at most *poly(C)*.
  * Alternatively, we may allow computation with real numbers, assuming we can perform basic arithmetic operations in $O(1)$ time.

Minimum cost flows with separable convex objectives

- $G=(V,E)$ directed graph
- On each arc ij, lower and upper capacities l_{ij}, u_{ij}.
- On each node i, node demand b_i: incoming flow minus outgoing flow should be b_i.
- We want to minimize $\Sigma C_{ij}(f_{ij})$ over feasible flows, where on each arc ij, C_{ij} is a convex function.
- Convex program with several applications: traffic management, matrix balancing, stick percolation...

Minimum cost flows with separable convex objectives

* **Selfish routing in urban traffic networks:** transition time on a road is an increasing function of the traffic amount.



Minimum cost flows with separable convex objectives

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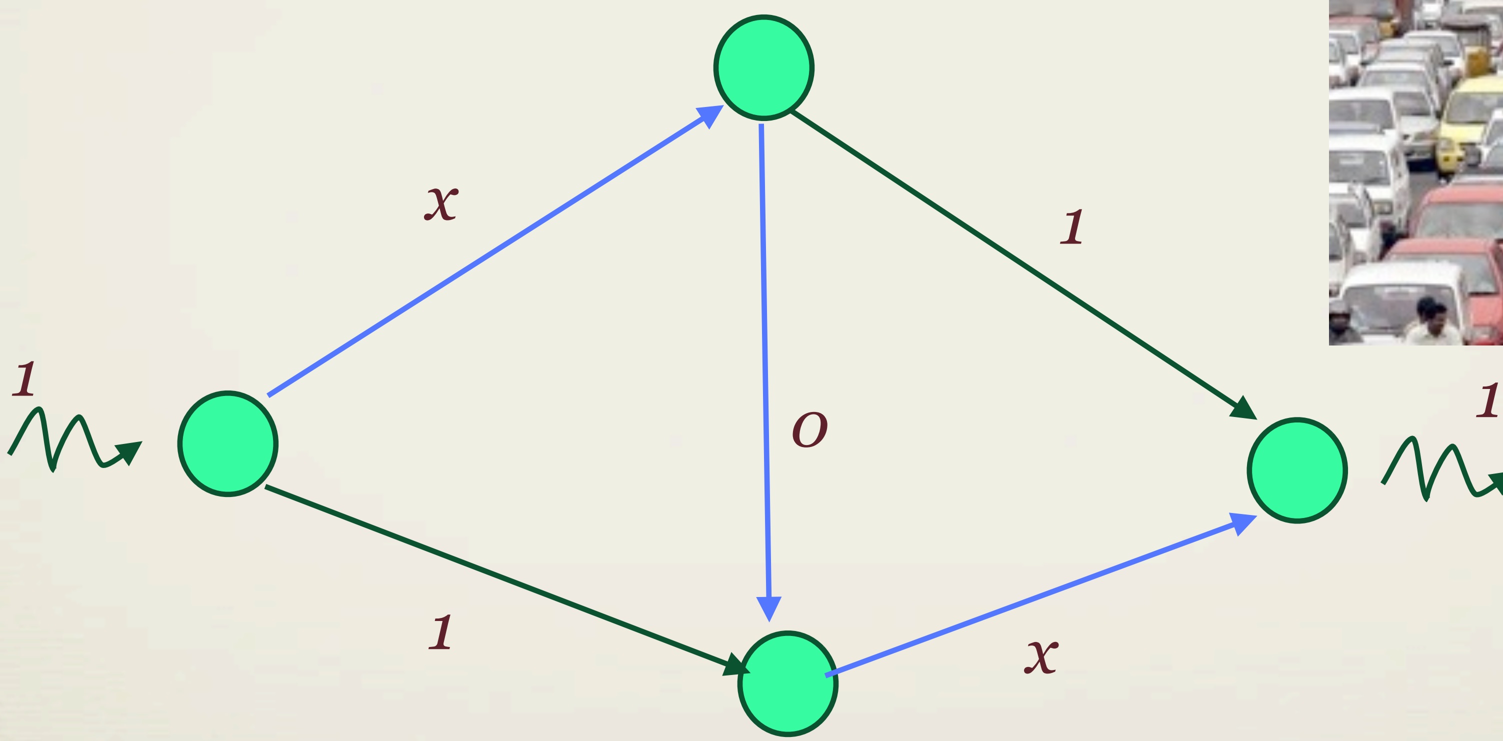


* **Nash equilibrium:** no car may find a shorter route if the others don't change.
* Computing a Nash-equilibrium is a separable convex cost flow problem.

Linear Fisher market: convex formulations

* Eisenberg&Gale, '59

x_{ij} : amount of good j purchased by i

$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x \geq 0$$

* concave generalized flow

* Shmyrev; Devanur '09

y_{ij} : amount of money payed by i for j

$$\min \sum_{i \in G} p_j (\log p_j - 1) - \sum_{ij \in E} y_{ij} \log U_{ij}$$

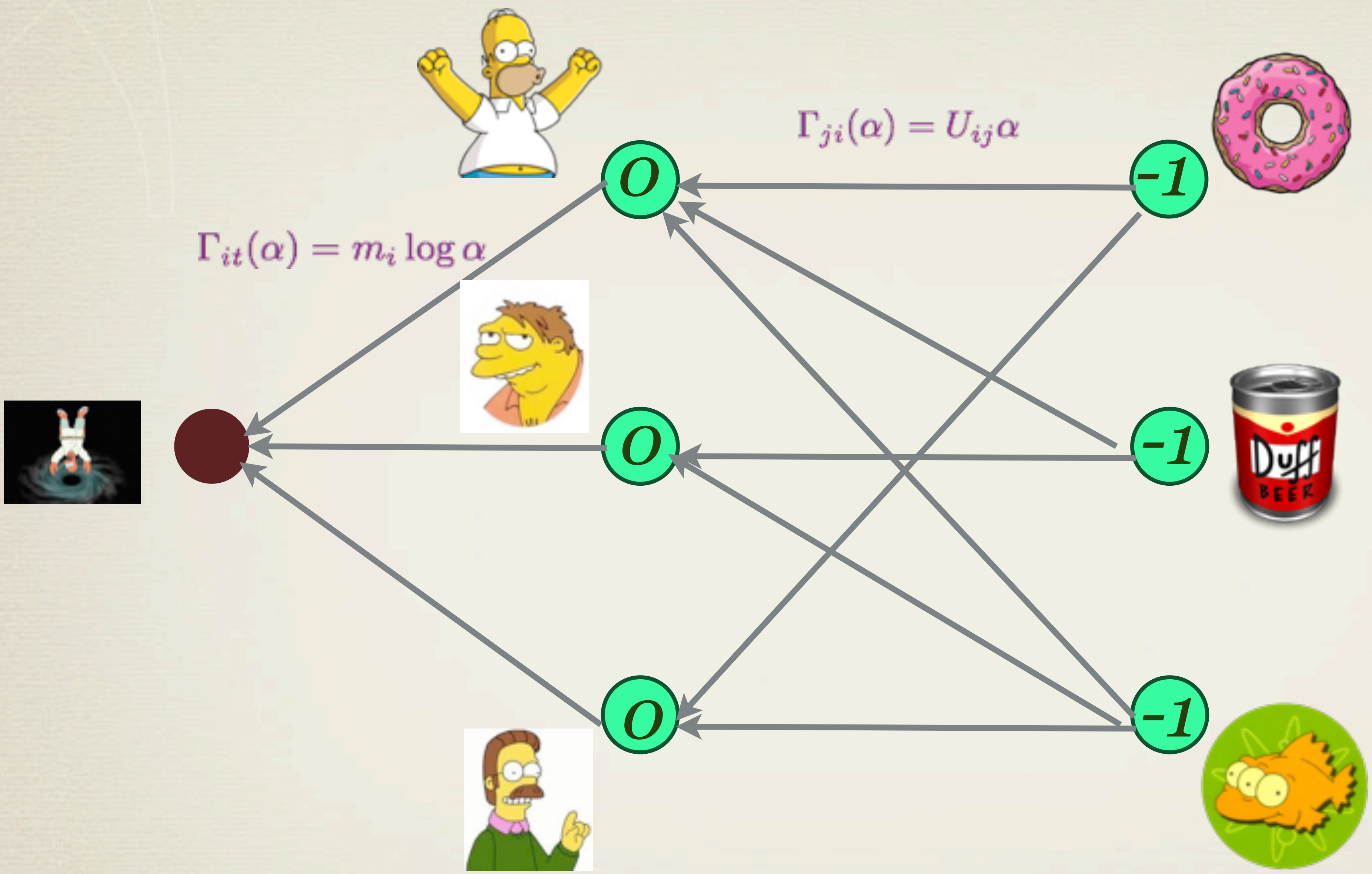
$$\sum_{j \in G} y_{ij} = m_i \quad \forall i \in B$$

$$\sum_{i \in B} y_{ij} = p_j \quad \forall j \in G$$

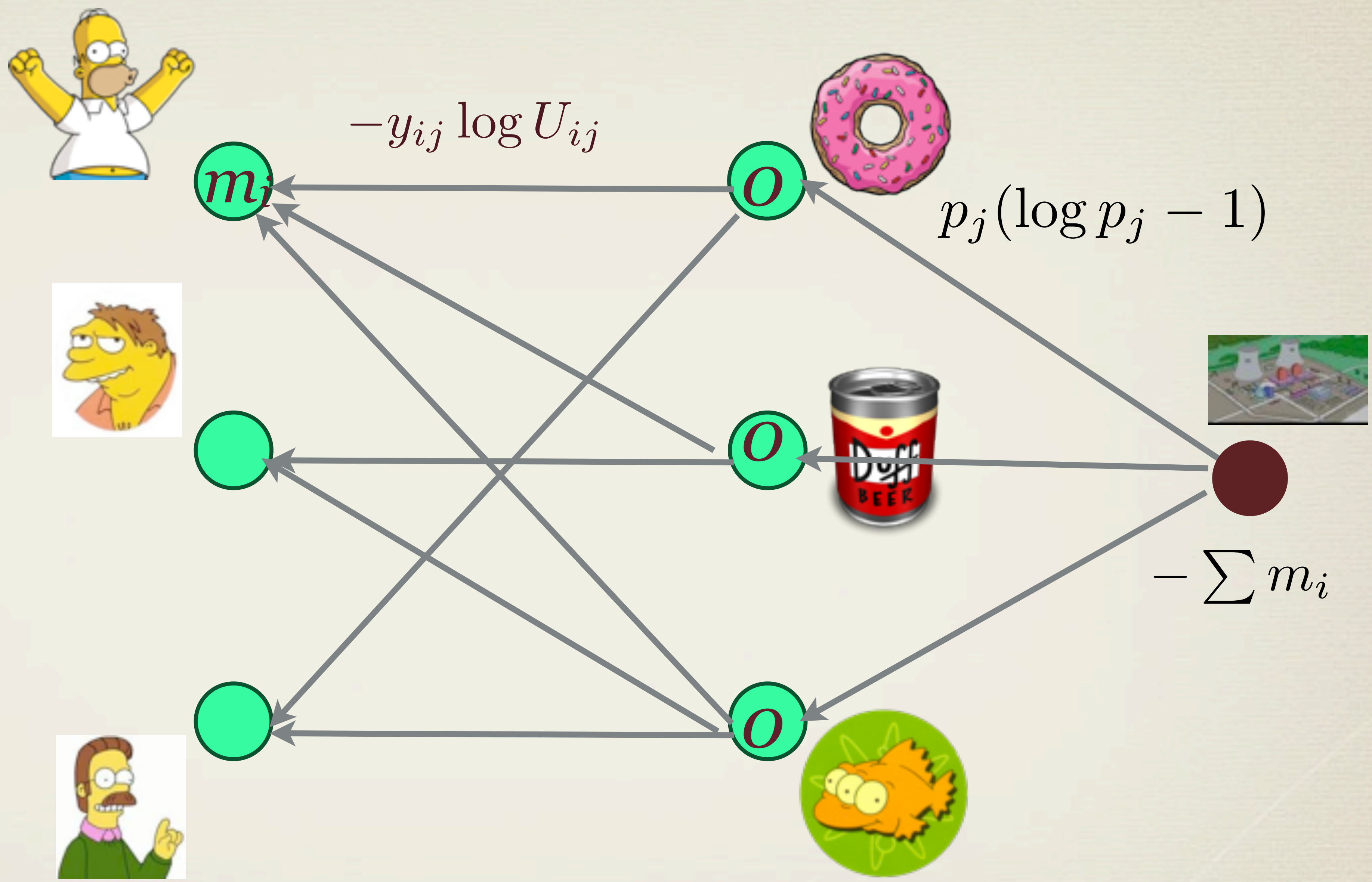
$$y \geq 0$$

* flow with separable convex objective

Reduction for linear Fisher market



Reduction for linear Fisher market



Extensions of linear Fisher markets

* Devanur&Vazirani '04: spending constraint utilities
    * The utility of the buyers is a piecewise linear concave function of the *amount of money spent on the good.*
* Vazirani '10: combinatorial algorithm (extension of DPSV'02)
* Devanur et al. '11: discovered the convex programming relaxation.
* V. '12a: strongly polynomial algorithm

When is there a strongly polynomial algorithm for MCCF?

$$G = (V, E) \text{ directed graph}$$

$$C_{ij} : [\ell_{ij}, u_{ij}] \to \mathbb{R} \text{ convex, (differentiable)}$$

$$\min \sum_{ij \in E} C_{ij}(f_{ij})$$

$$\sum_{j:ji \in E} f_{ji} - \sum_{j:ij \in E} f_{ij} = b_i \quad \forall i \in V$$

$$\ell_{ij} \le f_{ij} \le u_{ij} \quad \forall ij \in E$$

Previously known cases

- Linear costs ($C_{ij}(x)=c_{ij}x$)
- Quadratic costs ($C_{ij}(x)=c_{ij}x^2+d_{ij}x,\ c_{ij}\geq 0$)
  - Series parallel graphs: Tamir '93.
  - Transportation problem with fixed number of sources: Cosares&Hochbaum '94.
- Other nonlinear
  - Fisher's market with linear utilities: Orlin '10.

Negative results for strongly polynomiality

- Optimal solution can be irrational (even non-algebraic!)
- Q: is it possible to find an ε-approximate solution in time polynomial in the input size and *log 1/ε*?
  - Even this is impossible if the C_{ij}'s are polynomials of degree ≥3 (Hochbaum '94)
    - Reason: impossible to ε-approximate roots of polynomials in strongly polynomial time (Renegar '87)
- This does not apply for quadratic objectives!

Our result
(STOC 2012)

- Strongly polynomial algorithm under certain oracle assumptions on the objective.
- Key assumption: *we can compute an optimal solution, provided its support.*
- Special cases include:
  - Convex quadratic objectives.
  - Fisher's market with linear and with spending constraint utilities.

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Convex	Minimum cost circulations w. separable convex cost Minoux '86	Concave generalized flows V. '12b

Linear and convex flow problems I.

	weakly polynomial	*strongly polynomial*
Linear	Minimum cost circulations Edmonds&Karp '72	Orlin'93
Convex	Minimum cost circulations w. separable convex cost Minoux '86	V. '12a

Main algorithmic ideas

* Edmonds, Karp '72: capacity scaling algorithm:
  * Successive shortest paths method, first transporting the huge parts of the excesses.
* Minoux '86: naturally extends to convex costs, with linearizing the cost in Δ chunks in the Δ-phase.

$$\frac{C_{ij}(f_{ij}+\Delta)-C_{ij}(f_{ij})}{\Delta}$$

Main algorithmic ideas

- V '12a: apply Minoux's algorithm, and maintain a subset *F* of arcs guaranteed to be in the support of (the) optimal solution. *F* shall be extended in every *O(log n)* iterations.
- In certain phases, we make a guess: *maybe F is already optimal?* We compute an optimal solution based on the assumption that it's support is *F*.
  - if yes: great!
  - if not: either it still gives a better solution than the current one: Δ decreases radically;
  - or it gives a guarantee that *F* must soon be extended.

Market equilibria with rational convex programs and combinatorial algorithms

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Nonsymmetric ADNB	?	CGF	

CGF: concave generalized flows V. '12b

MCCF: min. cost separable convex flows V. '12a

Further questions

* Concave generalized flows: the algorithm is not strongly polynomial
* No strongly polynomial algorithm exists for linear generalized flows!
* Solving that could help develop strongly poly. alg. for certain concave gain functions.
* Linear Arrow-Debreu markets: no combinatorial algorithm known. Convex programming formulation: Nenakov&Primak '83, Jain '06.

Thank you for your attention!