To Save Or Not To Save: The Fisher Game

Ruta Mehta[1], Nithum Thain[2], László A. Végh[3], and Adrian Vetta[4]

[1] College of Computing, Georgia Institute of Technology. rmehta@cc.gatech.edu
[2] Department of Mathematics and Statistics, McGill University. nithum@gmail.com
[3] London School of Economics. L.Vegh@lse.ac.uk
[4] Department. of Mathematics and Statistics, and School of Computer Science, McGill University. vetta@math.mcgill.ca

Abstract. We examine the Fisher market model when buyers, as well as sellers, have an intrinsic value for money. We show that when the buyers have oligopsonistic power they are highly incentivized to act strategically with their monetary reports, as their potential gains are unbounded. This is in contrast to the bounded gains that have been shown when agents strategically report utilities [5]. Our main focus is upon the consequences for social welfare when the buyers act strategically. To this end, we define the *Price of Imperfect Competition (PoIC)* as the worst case ratio of the welfare at a Nash equilibrium in the induced game compared to the welfare at a Walrasian equilibrium. We prove that the PoIC is at least $\frac{1}{2}$ in markets with CES utilities with parameter $0 \leq \rho \leq 1$ – this includes the classes of Cobb-Douglas and linear utility functions. Furthermore, for linear utility functions, we prove that the PoIC increases as the level of competition in the market increases. Additionally, we prove that a Nash equilibrium exists in the case of Cobb-Douglas utilities. In contrast, we show that Nash equilibria need not exist for linear utilities. However, in that case, good welfare guarantees are still obtained for the Nash dynamics of the game.

1 Introduction

General equilibrium is a fundamental concept in economics, tracing back to 1872 with the seminal work of Walras [26]. Traditionally, the focus has been upon *perfect competition*, where the number of buyers and sellers in the market are so huge that the contribution of any individual is infinitesimal. In particular, the participants are *price-takers*.

In practice, however, this assumption is unrealistic. This observation has motivated researchers to study markets where the players have an incentive to act strategically. A prominent example is the seminal work of Shapely and Shubik [22]. They defined *trading post games* for exchange markets and examined whether Nash equilibria there could implement competitive equilibrium prices and allocations. Another example, and a prime motivator of our research, is the Cournot-Walras market model introduced by Codognato and Gabszewicz [6] and Gabszewicz and Michel [15], which extends oligopolistic competition into the Arrow-Debreu setting. The importance of this model was demonstrated by

Bonniseau and Florig [2] via a connection, in the limit, to traditional general equilibria models under the standard economic technique of agent *replication*. More recently, in the computer science community, Babaioff et al [3] extended Hurwicz's framework [17] to study the welfare of Walrasian markets acting through an auction mechanism.

Our interest is in how robust a pricing mechanism is against strategic manipulation. Specifically, our primary goal is to quantify the loss in social welfare due price-making rather than price-taking behaviour. To do this, we define the *Price of Imperfect Competition (PoIC)* as the ratio of the social welfare at the worst Nash equilibrium to the social welfare at the perfectly-competitive Walrasian equilibrium.

Two remarks are pertinent here. First, we are interested in changes in the welfare produced by the market mechanism under the two settings of price-takes and price-makers. We are not interested in comparisons with the optimum social welfare, which requires the mechanism to possess the unrealistic power to perform total welfare redistribution. In particular, we are not concerned here with the *Price of Anarchy* or *Price of Stability*. Interestingly, though, the groundbreaking Price of Anarchy results of Johari and Tzitsiklis [20] on the proportional allocation mechanism for allocating one good (bandwidth) can be seen as the first Price of Imperfect Competition results. This is because in their setting the proportional allocation mechanism will produce optimal allocations in non-stategic settings; in contrast, for our markets, Walrasian equilibrium can be arbitrarily poor in comparison to optimal allocations.

Second, in some markets the Price of Imperfect Competition may actually be larger than one. Thus, strategic manipulations by the agents can lead to improvements in social welfare! Indeed, we will exhibit examples where the social welfare increases by an arbitrarily large factor when the agents act strategically.

In this paper, we analyze the Price of Imperfect Competition in Fisher markets with strategic buyers, a special case of the Cournot-Walras model. This scenario models the case of an oligopsonistic market, where the price-making power lies with the buyers rather than the sellers (as in an oligopoly).[5] Adsul et al. [1] study Fisher markets where buyers can lie about their preferences. They gave a complete characterization of its symmetric Nash equilibria (SNE) and showed that market equilibrium prices can be implemented at one of the SNE. Later Chen et. al. [5] studied *incentive ratios* in such markets to show that a buyer can gain no more than twice by strategizing in markets with linear, Leontief and Cobb-Douglas utility functions. In upcoming work, Branzei et al [4] study the Price of Anarchy in the game of Adsul et al. and prove polynomial lower and upper bounds for it. Furthermore, they show Nash equilibria always exist.

In the above games (and the Fisher model itself), only the sellers have an intrinsic utility for money. In contrast, we postulate that buyers (and not just sellers) have utility for money. Thus, buyers may also benefit by saving money for

[5] The importance of oligopsonies was recently highlighted by the price-fixing behaviour of massive technology companies in San Francisco.

later use. This incentivizes buyers to withhold money from the market. This defines our *Fisher Market Game*, where agents strategize on the amount of money they wish to spend, and obtain utility one from each unit of saved money. Contrary to the bound of two on gains when strategizing on utility functions [5], we observe that strategizing on money may facilitate unbounded gains (see Appendix A.1). These incentives can induce large variations between the allocations produced at a Market equilibrium and at a Nash equilibrium. Despite this, we prove the Price of Imperfect Competition is at least $\frac{1}{2}$ for Fisher markets when the buyers utility functions belong to the utililty class of Constant Elasticity of Substitution (CES) with the weak gross substitutability property – this class includes linear and Cobb-Douglas functions.

1.1 Overview of Paper

In Section 2, we define the Fisher Game, give an overview of CES utility functions, and present our welfare metrics. In Section 3, we prove that Price of Imperfect Competition is at least $\frac{1}{2}$, for CES utilities which satisfy the weak gross substitutability property. In Section 4, we apply the economic technique of replication to demonstrate that, for linear utilities, the PoIC bound improves as the level of competition in the market increases. In Section 5, we turn our attention to the question of existence of Nash equilibria. We establish that Nash equilibria exist for the subclass of Cobb-Douglas utilities. However, they need not exist for all CES utilities. In particular, Nash equilibria need not exist for linear utilities. To address this possibility of non-existence, in Section 6, we examine the dynamics of the linear Fisher Game and provide logarithmic welfare guarantees.

2 Preliminaries

We now define the Fisher market model and the corresponding game where agents strategize on how much money to spend. We require the following notation. Vectors are shown in bold-face letters, and are considered as column vectors. To denote a row vector we use $\boldsymbol{x}^T$. The i^{th} coordinate of $\boldsymbol{x}$ is denoted by x_i, and $\boldsymbol{x}_{-i}$ denotes the vector $\boldsymbol{x}$ with the i^{th} coordinate removed.

2.1 The Fisher Market

A Fisher market $\mathcal{M}$, introduced by Irving Fisher in his 1891 PhD thesis, consists of a set $\mathcal{B}$ of buyers and and a set $\mathcal{G}$ of goods (owned by sellers). Let $n = |\mathcal{B}|$ and $g = |\mathcal{G}|$. Buyer i brings m_i units of money to the market and wants to buy a bundle of goods that maximizes her utility. Here, a non-decreasing, concave function $U_i : \mathbb{R}_+^g \to \mathbb{R}_+$ measures the utility she obtains from a bundle of goods. Without loss of generality, the aggregate quantity of each good is one.

Given prices $\boldsymbol{p} = (p_1, \ldots, p_g)$, where p_j is price of good j, each buyer demands a utility maximizing (an optimal) bundle that she can afford. The prices $\boldsymbol{p}$ are said to be a *market equilibrium* (ME) if agents can be assigned an optimal bundle such that demand equals supply, *i.e.* the market clears. Formally, let x_{ij} be the amount of good j assigned to buyer i. So $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ig})$ is her bundle. Then,

1. `Supply = Demand`: $\forall j \in \mathcal{G},\ \sum_i x_{ij} = 1$ whenever $p_j > 0$.
2. `Utility Maximization`: $\boldsymbol{x}_i$ is a solution of $\max U_i(\boldsymbol{z})$ s.t $\sum_j p_j z_{ij} \leq m_i$.

We denote by y_{ij} the amount of money player i invests in item j after prices are set. Thus $y_{ij} = p_j x_{ij}$. Equivalently y_{ij} can be thought of as player i's demand for item j in monetary terms.

Utility Functions.
An important sub-class of Fisher markets occurs when we restrict utility functions to what are known as *Constant Elasticity of Substitution (CES)* utilities [24]. These functions have the form:

$$U_i(\boldsymbol{x}_i) = (\sum_j u_{ij} x_{ij}^{\rho})^{\frac{1}{\rho}}$$

for some fixed $\rho \leq 1$ and some coefficients $u_{ij} \geq 0$. The elasticity of substitution for these markets are $\frac{1}{1-\rho}$. Hence, for $\rho = 1$, *i.e.* linear utilities, the goods are perfect substitutes; for $\rho \to -\infty$, the goods are perfect complements. As $\rho \to 0$, we obtain the well-known Cobb-Douglas utility function:

$$U_i(\boldsymbol{x}_i) = \prod_j x_{ij}^{u_{ij}}$$

where each $u_{ij} \geq 0$ and $\sum_j u_{ij} = 1$. In this paper, we will focus on the cases of $0 < \rho \leq 1$ and the case $\rho \to 0$. These particular markets satisfy the property of *weak gross substitutability*, meaning that increasing the price of one good cannot decrease demand for other goods. It is also known that for these particular markets, one can determine the market prices and allocations by solving the Eisenberg-Gale convex program (see [11], [12], [19]):

$$\max\ \big(\sum_i m_i \log U_i(\boldsymbol{x}_i) : \sum_i x_{ij} \leq 1, \forall j;\ x_{ij} \geq 0, \forall i, j.\big) \qquad (1)$$

2.2 The Fisher Game.

An implicit assumption within the Fisher market model is that money has an intrinsic value *to the sellers*, stemming from its potential use outside of the market or at a later date. Thus, money is not just a numéraire. We assume this intrinsic value applies to all market participants including the buyers. This assumption induces a strategic game in which the buyers may have an incentive to save some of their money.

This *Fisher Game* is a special case of the general Cournot-Walras game introduced by Codognato, Gabszewicz, and Michel ([6], [15]). Here the buyers can choose some strategic amount of money $s_i < m_i$ to bring to the market, which will affect their budget constraint. They gain utility both from the resulting market equilibria (with s_i substituted for m_i) and from the money they withhold from the market. Observe, in the Fisher market model, the sellers have no value for the goods in the market. Thus, in the corresponding game, they will place all their goods on sale as their only interest is in money. (Equivalently, we may assume the sellers are non-strategic.)

Thus, we are in an oligopsonistic situation where buyers have indirect price-making power. The set of strategies available to buyer i is $M_i = \{s \geq 0 \mid s \leq m_i\}$. When each buyer decides to spend $s_i \in M_i$, then $\boldsymbol{p}(\boldsymbol{s})$ and $\boldsymbol{x}(\boldsymbol{s})$ are the prices and allocations, respectively, produced by the Fisher market mechanism. These can be determined from the Eisenberg-Gale program (1) by substituting s_i for m_i. Thus, total payoff to buyer i is

$$T_i(\boldsymbol{s}) = U_i(\boldsymbol{x}_i(\boldsymbol{s})) + (m_i - s_i) \tag{2}$$

Our primary tool to analyze the Fisher Game is via the standard solution concept of a Nash equilibrium. A strategy profile $\boldsymbol{s}$ is said to be a *Nash equilibrium* if no player gains by deviating unilaterally. Formally, $\forall i \in \mathcal{B}$, $T_i(\boldsymbol{s}) \geq T_i(s', \boldsymbol{s}_{-i})$, $\forall s' \in M_i$. For the market game defined on market $\mathcal{M}$, let $NE(\mathcal{M})$ denote its set of NE strategy profiles.

The incentives in the Fisher Game can be high. In particular, in Appendix A.1, we show that for any $L \geq 0$, there is a market with linear utility functions where an agent improve his payoff by a multiplicative factor of L by acting strategically.

The Price of Imperfect Competition.
The social welfare of a strategy is the aggregate payoff of both buyers and sellers. At a state $\boldsymbol{s}$, with prices $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s})$ and allocations $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{s})$, the social welfare is:

$$\mathcal{W}(\boldsymbol{s}) \;=\; \sum_{i \in \mathcal{B}} (U_i(\mathbf{x_i}) + m_i - s_i) + \sum_{j \in \mathcal{G}} p_j \;=\; \sum_{i \in \mathcal{B}} U_i(\mathbf{x_i}) + \sum_{i \in \mathcal{B}} m_i \tag{3}$$

Note, here, that the cumulative payoff of sellers is $\sum_{j \in \mathcal{G}} p_j = \sum_{i \in \mathcal{B}} s_i$.

The focus of this paper is how strategic manipulations of the market mechanism affect the overall social welfare. Thus, we must compare the social welfare of the strategic Nash equilibrium to that of the unstrategic market equilibrium where all buyers simply put all of their money onto the market. This latter equilibrium is the *Walrasian equilibrium (WE)*. This comparison gives rise to a welfare ratio, which we term the *Price of Imperfect Competition (PoIC)*, the ratio of the minimum welfare amongst strategic Nash equilibria in the market game to the welfare of the unstrategic Walrasian equilibrium. Formally, for a given market $\mathcal{M}$,

$$\text{PoIC}(\mathcal{M}) = \min_{\boldsymbol{s} \in NE(\mathcal{M})} \frac{\mathcal{W}(\boldsymbol{s})}{\mathcal{W}(\boldsymbol{m})}$$

Thus the Price of Imperfect Competition is a measure of how robust, with respect to social welfare, the market mechanism is against oligopsonist behaviour.

Observe that the Price of Imperfect Competition could be either greater or less than 1. Indeed, the example in Appendix A.1 shows that a Nash Equilibrium may produce arbitrarily higher welfare than a Walrasian Equilibrium. Of course, one may expect that welfare falls when the mechanism is gamed and, in Appendix A.2, we do present an example where the welfare at a Nash Equilibrium is slightly lower than at the Walrasian Equilibrium. This leads to the question of whether the welfare at a Nash can be much worse than at a market equilibrium. We will show that the answer is no; a Nash always produces at least a constant factor of the welfare of a market equilibrium.

3 Bounds on the Price of Imperfect Competition

In this section we establish bounds on the PoIC for the Fisher Game for CES utilities with $0 < \rho \leq 1$ and for Cobb-Douglas utilities. The example in Appendix A.1 shows that there is no upper bound on PoIC for the Fisher Game. Thus, counterintuitively, even for linear utilities, it may be extremely beneficial to society if the players are strategic.

In the rest of this section, we demonstrate a lower bound of $\frac{1}{2}$ on the PoIC. This result distinguishes the Fisher Game from other strategic market models. For example, consider the case of the Proportional Allocation Mechanism applied over a multi-good market (see Feldman et al. [13] for details on this application). In Appendix B, we show that the PoIC may then approach zero in the proportional allocation mechanism with savings. Thus the Fisher Game is, in a sense, more resilient to strategic play than other mechanisms.

So consider a market with Cobb-Douglas or CES utility functions (where $0 < \rho \leq 1$). The key to proving the factor $\frac{1}{2}$ lower bound on the PoIC is the following lemma showing the monotonicity of prices.

Lemma 1. *Given two strategic allocations of money* $\mathbf{s}^* \leq \mathbf{s}$*, then the corresponding equilibrium prices satisfy* $\boldsymbol{p}^* \leq \boldsymbol{p}$*, where* $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{s}^*)$ *and* $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s})$*.*

Proof. We break the proof up into three classes of utility function.
`(i) Cobb-Douglas Utilities`
The case of Cobb-Douglas utility functions is simple. To see this, recall a result of Eaves [10]. He showed that, when buyer i spends s_i, the prices and allocations for the Fisher market are given by

$$p_j = \sum_i u_{ij} s_i \qquad x_{ij} = \frac{u_{ij} s_i}{\sum_k u_{kj} s_k} \tag{4}$$

It follows that if strategic allocations of money increase, then so must prices.

`(ii) CES Utilities with` $0 < \rho < 1$
Recall that market equilibria for CES Utilities can be calculated via the Eisenberg-Gale convex program (1). From the KKT conditions of this program, where p_j

is the dual variable of the budget constraint, we observe that:

$$\begin{array}{lll} \forall j, & p_j > 0 \Rightarrow \sum_i x_{ij} = 1 & \\ \forall (i,j), & \frac{s_i u_{ij}}{U_i(\boldsymbol{x}_i)^\rho x_{ij}^{1-\rho}} \le p_j \quad \text{and} & x_{ij} > 0 \Rightarrow \frac{s_i u_{ij}}{U_i(\boldsymbol{x}_i)^\rho x_{ij}^{1-\rho}} = p_j \end{array} \tag{5}$$

Claim. If players have CES utilities with $0 < \rho < 1$ and $\boldsymbol{s} \ge 0$, then $x_{ij} > 0,\ \forall (i,j)$ with $u_{ij} > 0$.

Proof. Consider the derivative of U_i with respect to x_{ij} as $x_{ij} \to 0$:

$$\lim_{x_{ij} \to 0} \frac{\partial U_i(\boldsymbol{x}_i)}{\partial x_{ij}} = \lim_{x_{ij} \to 0} \frac{u_{ij} U_i(\boldsymbol{x}_i)^{1-\rho}}{x_{ij}^{1-\rho}} = +\infty \tag{6}$$

The claim follows since $p_j \le \sum_i s_i$ and is, thus, finite. □

We may now proceed by contradiction. Suppose $\exists k$ s.t. $p_k < p_k^*$. Choose a good j such that $\frac{p_j}{p_j^*}$ is minimal and therefore less than 1, by assumption. Take any player i such that $u_{ij} > 0$. By the above claim, we have $x_{ij}, x_{ij}^* > 0$. Consequently, by the KKT conditions (5), we have:

$$\frac{u_{ij}}{p_j x_{ij}^{1-\rho}} = \frac{U_i(\boldsymbol{x}_i)^\rho}{s_i} \quad \text{and} \quad \frac{u_{ij}}{p_j^* x_{ij}^{*1-\rho}} = \frac{U_i(\boldsymbol{x}^*)^\rho}{s_i^*} \tag{7}$$

Taking a ratio gives:

$$\frac{p_j x_{ij}^{1-\rho}}{p_j^* x_{ij}^{*1-\rho}} = \frac{U_i(\boldsymbol{x}_i^*)^\rho s_i}{U_i(\boldsymbol{x}_i)^\rho s_i^*} \tag{8}$$

Indeed, this equation also holds for every good $t \in \mathcal{G}$ with $u_{it} > 0$. Next consider the following two cases:

Case 1: $x_{ij} \le x_{ij}^*$ for some player i.

From (8) we must then have that $U_i(\boldsymbol{x}_i) > U_i(\boldsymbol{x}_i^*)$. However, by the minimality of $\frac{p_j}{p_j^*}$, and since (8) holds for every $t \in \mathcal{G}$ with $u_{it} > 0$, we obtain $x_{it} \le x_{it}^*$ for all such t. This implies $U_i(\boldsymbol{x}_i) \le U_i(\boldsymbol{x}_i^*)$, a contradiction.

Case 2: $x_{ij} > x_{ij}^*$ for every player i.

Since $p_j^* > p_j$, we must have $p_j^* > 0$. By (5) it follows that $\sum_i x_{ij}^* = 1$. But now we obtain the contradiction that demand must exceed supply as $\sum_i x_{ij} > \sum_i x_{ij}^* = 1$.

`(iii) Linear Utilities`

We begin with some notation. Let $S_i = \{j \in \mathcal{G} : x_{ij} > 0\}$ be the set of goods purchased by buyer i at strategy $\boldsymbol{s}$. Let $\beta_{ij} = \frac{u_{ij}}{p_j}$ be the *rate-of-return* of good j for buyer i at prices $\mathbf{p}$. Let $\beta_i = \max_{j \in \mathcal{G}} \beta_{ij}$ be the *bang-for-buck* buyer i can obtain at prices $\mathbf{p}$. It can be seen from the KKT conditions of the Eisenberg-Gale program (1) that at $\{\boldsymbol{p}, \boldsymbol{x}\}$, every good $j \in S_i$ will have a rate-of-return equal to the bang-for-buck (see, for example, [25]). Similarly, let S_i^*, β_i^* be correspondingly defined for strategy s^*.

Note that, assuming for each good j, $\exists i,\ u_{ij} > 0$, we have that $\boldsymbol{p}, \boldsymbol{p}^* > 0$. Thus, we can partition the goods into groups based on the *price ratios* $\frac{p_j^*}{p_j}$. Suppose there are k distinct price ratios over all the goods (thus $k \leq g$), then partition the goods into k groups, say $\mathcal{G}_1, \ldots, \mathcal{G}_k$ such that all the goods in a group have the same ratio. Let the ratio in group j be λ_j and let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Thus $\mathcal{G}_1$ are the goods whose prices have fallen the most (risen the least) and $\mathcal{G}_k$ are the goods whose prices have fallen the least (risen the most).

Let $\mathcal{I}_k = \{i : \exists j \in \mathcal{G}_k, x_{ij} > 0\}$ and $\mathcal{I}_k^* = \{i : \exists j \in \mathcal{G}_k, x_{ij}^* > 0\}$. Thus $\mathcal{I}_k$ and $\mathcal{I}_k^*$ are the collections of buyers that purchase goods in $\mathcal{G}_k$ in each of the allocations. Take any buyer $i \in \mathcal{I}_k^*$; so there is some good $j \in S_i^* \cap \mathcal{G}_k$.

If $S_i \cap \bigcup_{\ell=1}^{k-1} \mathcal{G}_\ell \neq \emptyset$ then buyer i would not desire good j at prices p_j^*. To see this, take a good $j' \in S_i \cap \bigcup_{\ell=1}^{k-1} \mathcal{G}_\ell$. Then $\beta_{ij'} = \beta_i \geq \beta_{ij}$. Therefore

$$\begin{aligned}
\beta_i^* \geq \frac{u_{ij'}}{p_{j'}^*} \quad &\geq \quad \frac{u_{ij'}}{\lambda_{k-1} \cdot p_{j'}} \quad > \quad \frac{u_{ij'}}{\lambda_k \cdot p_{j'}} \\
= \frac{1}{\lambda_k} \cdot \frac{u_{ij'}}{p_{j'}} \quad &\geq \quad \frac{1}{\lambda_k} \cdot \frac{u_{ij}}{p_j} \\
= \frac{u_{ij}}{p_j^*} \quad &= \quad \beta_i^*
\end{aligned}$$

This contradiction tells us that $S_i \subseteq \mathcal{G}_k$ and $\mathcal{I}_k^* \subseteq \mathcal{I}_k$. It follows that $\cup_{i \in \mathcal{I}_k^*} S_i \subseteq \mathcal{G}_k$. Putting this together, we obtain that

$$\sum_{i \in \mathcal{I}_k^*} s_i \ \leq \ \sum_{i \in \mathcal{I}_k} s_i \ \leq \ \sum_{j \in \mathcal{G}_k} p_j \tag{9}$$

Now recall that all goods must be sold by the market mechanism (as $\boldsymbol{p}, \boldsymbol{p}^* > 0$). Thus the buyers $\mathcal{I}_k^*$ must be able to afford all of the goods in $\mathcal{G}_k$. Thus

$$\sum_{i \in \mathcal{I}_k^*} s_i^* \ \geq \ \sum_{j \in \mathcal{G}_k} p_j^* \ = \ \lambda_k \cdot \sum_{j \in \mathcal{G}_k} p_j \tag{10}$$

But $s_i^* \leq s_i$ for all i. Consequently, Inequalities (9) and (10) imply that $\lambda_k \leq 1$. Thus no price in $\boldsymbol{p}^*$ can be higher than in $\boldsymbol{p}$. □

First we use Lemma 1 to provide lower bounds on the individual payoffs.

Lemma 2. *Let s_i be a best response for agent i against the strategies $\boldsymbol{s}_{-i}$. Then $T_i(\boldsymbol{s}) \geq \max(\hat{U}_i, m_i)$, where $\hat{U}_i$ is her utility at the Walrasian equilibrium.*

Proof. Clearly $T_i(\boldsymbol{s}) \geq m_i$, otherwise player i could save all her money and achieve a payoff of m_i. For $T_i(\boldsymbol{s}) \geq \hat{U}_i$, let $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{m})$ and $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{m})$ be the prices and allocation at Walrasian equilibrium. If buyer i decides to spend all his money when the others play $\boldsymbol{s}_{-i}$, the resulting equilibrium prices will be less than $\boldsymbol{p}$, by Lemma 1. Therefore, she can afford to buy bundle $\boldsymbol{x}_i$. Thus, her best response payoff must be at least $\hat{U}_i$.

It is now easy to show the lower bound on the Price of Imperfect Competition.

Theorem 1. *In the Fisher Game, with Cobb-Douglas or CES utilities (*$0 < \rho \leq 1$*), we have PoIC* $\geq \frac{1}{2}$*. That is,* $\mathcal{W}(\boldsymbol{s}^*) \geq \frac{1}{2}\mathcal{W}(\boldsymbol{m})$*, for any Nash equilibrium* $\boldsymbol{s}^*$*.*

Proof. Let $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{s}^*)$ and $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{s}^*)$. Let $\boldsymbol{p}$ and $\boldsymbol{x}$ be the Walrasian equilibrium prices and allocations, respectively. At the Nash equilibrium $\boldsymbol{s}^*$ we have $T_i(\boldsymbol{s}^*) \geq \max(m_i, U_i(\boldsymbol{x}_i))$ for each player i, by Lemma 2. Thus, we obtain:

$$2\sum_i T_i(\boldsymbol{s}^*) \geq \sum_i U_i(\boldsymbol{x}_i) + \sum_i m_i \tag{11}$$

Therefore $\mathcal{W}(\boldsymbol{s}^*) \geq \frac{1}{2}\mathcal{W}(\boldsymbol{m})$, as desired. □

4 Social Welfare and the Degree of Competition

In this section, we examine how the welfare guarantee improves with the degree of competition in the market. To model the degree of competition, we apply a common technique in the economics literature, namely *replication* [22]. In a replica economy, we take each buyer type in the market and make N duplicates (the budgets of each duplicate is a factor N smaller than that of the original buyer). The *degree of competition* in the resultant market is N. We now consider the Fisher Game with linear utility functions and show how the lower bound on Price of Imperfect Competition improves with N.

Theorem 2. *Let* $\boldsymbol{s}^*$ *be a NE in a market with degree of competition* N*. Then*

$$\mathcal{W}(\boldsymbol{s}^*) \geq (1 - \frac{1}{N+1}) \cdot \mathcal{W}(\boldsymbol{m})$$

In order to prove Theorem 2, we need a better understanding of how prices adjust to changes in strategy under different degrees of competition. Towards this goal, we need the following two lemmas.

Lemma 3. *Given an arbitrary strategic money allocation* $\boldsymbol{s}$*. If player* i *increases (resp. decreases) her spending from* s_i *to* $(1+\delta)s_i$ *then the price of any good increases (resp. decreases) by at most a factor of* $(1+\delta)$*.*

Proof. We focus on the case of increase; the argument for the decrease case is analogous. Suppose all players increase their strategic allocation by a factor of $(1+\delta)$. Then the allocations to all players would remain the same by the market mechanism and all prices would be scaled up by a factor of $(1+\delta)$. Then suppose each player $k \neq i$ subsequently lowers its money allocation back down to the original amount s_k. By Lemma 1, no price can now increase. The result follows. □

Lemma 4. *Given an arbitrary strategic money allocation* $\boldsymbol{s}$ *in a market with degree of competition* N*. Let buyer* i *be the duplicate player of her type with the smallest money allocation* s_i*. If she increases her spending to* $(1 + N \cdot \delta)s_i$ *then the price of any good increases by at most a factor* $(1+\delta)$*.*

Proof. We utilize the symmetry between the N identical players. Let players $i_1 = i, i_2, ..., i_N$ be the replicas identical to player i. If each of these players increased their spending by a factor of $(1+\delta)$ then, by Lemma 3, prices would go up by at most a factor $(1+\delta)$. From the market mechanism's perspective, this is equivalent to player i increasing her strategic allocation to $s_i + \delta \cdot \sum_k s_{i_k}$. But this is greater than $(1 + N \cdot \delta)s_i$. Thus, by Lemma 1, the new prices are larger by a factor of at most $(1+\delta)$. □

Now let $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{m})$ and $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{s}^*)$. Since we have rational inputs, $\mathbf{x}$ and $\mathbf{x}^*$ must be rational [19]. Therefore, by appropriately duplicating the goods and scaling the utility coefficients, we may assume that there is exactly one unit of each good and that both $\mathbf{x}$ and $\mathbf{x}^*$ are $\{0,1\}$-allocations. Recall from the proof of Lemma 1 our definition of S_i, S_i^* and β_i, β_i^*. Under this assumption, $S_i = \{j \in \mathcal{G} : x_{ij} = 1\}$ and similarly for S_i^*. We are now ready to prove the following welfare lemma.

Lemma 5. *For any Nash equilibrium* $\{\mathbf{s}^*, \mathbf{p}^*, \mathbf{x}^*\}$ *and any Walrasian equilibrium* $\{\mathbf{s} = \mathbf{m}, \mathbf{p}, \mathbf{x}\}$*, we have*

$$\sum_{i \in \mathcal{B}} \sum_{j \in S_i^*} u_{ij} \geq \left(1 - \frac{1}{N}\right) \cdot \sum_{i \in \mathcal{B}} \sum_{j \in S_i} u_{ij} \qquad (12)$$

Proof. To prove the lemma we show that total utility produced by goods at NE, after scaling by a factor $\frac{N}{N-1}$, is at least as much as the utility they produce at the Walrasian equilibrium. We do this by partitioning goods into the sets S_i. We then notice that for each good, the player who receives it at NE must receive utility from it in excess of the price he paid for it. In many cases, this price is more than the utility of the player who receives it in Walrasian equilibrium and we are done. Otherwise we will set up a transfer system where players in NE who receive more utility for the good than the price paid for it transfer some of this excess utility to players who need it. This will ultimately allow us to reach the desired inequality.

For the rest of this proof wlog we will restrict our attention to Nash equilibria where each identical copy of a certain type of player has the same strategy. We are able to do this as the market could treat the sum of these copies as a single player and thus we are able to manipulate the allocations between these players without changing market prices or the total utility derived from market allocations. Thus if our argument holds for Nash equilibria where identical players have the same strategy, it will also hold for heterogeneous Nash equilibria. Now take any player i. There are two cases:
Case 1: $s_i^* = m_i$.
By Lemma 1, we know that

$$\sum_{j \in S_i^* \cap S_i} p_j^* \leq \sum_{j \in S_i^* \cap S_i} p_j \qquad (13)$$

Therefore, by the assumption that $s_i^* = m_i$, we have

$$\sum_{j\in S_i\setminus S_i^*} p_j = m_i - \sum_{j\in S_i^*\cap S_i} p_j = s_i^* - \sum_{j\in S_i^*\cap S_i} p_j \le s_i^* - \sum_{j\in S_i^*\cap S_i} p_j^* = \sum_{j\in S_i^*\setminus S_i} p_j^* \tag{14}$$

Thus buyer i spends more on $S_i^* \setminus S_i$ than she did on $S_i \setminus S_i^*$. But, by Lemma 1, she also receives a better bang-for-buck on $S_i^* \setminus S_i$ than on $S_i \setminus S_i^*$, as $\beta_i^* \ge \beta_i$ (Lemma 1). Let $\beta_i^* = 1 + \epsilon_i^*$. Thus, at the Nash equilibrium, her total utility on $S_i^* \setminus S_i$ is

$$\sum_{j\in S_i^*\setminus S_i} u_{ij} \;=\; \sum_{j\in S_i^*\setminus S_i} \beta_i^*\cdot p_j^* \;=\; (1+\epsilon_i^*)\cdot \sum_{j\in S_i^*\setminus S_i} p_j^*$$

Of this utility, buyer i will allocate p_j^* units of utility to each item $j \in S_i^* \setminus S_i$. The remaining $\epsilon_i^* \cdot p_j^*$ units of utility derived from good j is reallocated to goods in $S_i \setminus S_i^*$.

Consider the goods in S_i. Clearly goods in $S_i \cap S_i^*$ contribute the same utility to both the Walrasian equilibrium and the Nash equilibrium. So take the items in $S_i \setminus S_i^*$. The buyers of these items at NE have obtained at least $\sum_{j\in S_i\setminus S_i^*} p_j^*$ units of utility from them (as $\beta_d^* \ge 1,\ \forall d$). In addition, buyer i has reallocated $\epsilon_i^* \cdot \sum_{j\in S_i^*\setminus S_i} p_j^*$ to goods in $S_i \setminus S_i^*$. So the total utility allocated to goods in $S_i \setminus S_i^*$ is

$$\begin{aligned}\sum_{j\in S_i\setminus S_i^*} p_j^* + \epsilon_i^*\cdot \sum_{j\in S_i^*\setminus S_i} p_j^* &\ge \sum_{j\in S_i\setminus S_i^*} p_j^* + \epsilon_i^*\cdot \sum_{j\in S_i\setminus S_i^*} p_j^* \;=\; (1+\epsilon_i^*)\cdot \sum_{j\in S_i\setminus S_i^*} p_j^* \\ &= \beta_i^*\cdot \sum_{j\in S_i\setminus S_i^*} p_j^* \;\ge\; \sum_{j\in S_i\setminus S_i^*} u_{ij}\end{aligned}$$

Here the first inequality follows by (14) and the final inequality follows as $\beta_i^* \ge \frac{u_{ij}}{p_j^*}$, for any good $j \notin S_i^*$. Thus the reallocated utility on S_i at NE is greater than the utility it provides in the Walrasian equilibrium (even without scaling by $\frac{N}{N-1}$).

Case 2: $s_i^* < m_i$.

Suppose buyer i increases her spending from s_i^* to $(1+N\cdot\delta)\cdot s_i^*$. Then the prices of the goods she buys increase by at most a factor $(1+\delta)$ by Lemma 4. Thus her utility changes by

$$(m_i - (1+\delta\cdot N)\cdot s_i^*) + s_i^*\cdot\beta_i^*\cdot\frac{1+N\cdot\delta}{1+\delta} - (m_i - s_i^*) - s_i^*\cdot\beta_i^* \;\le\; 0$$

where the inequality follows as s^* is a Nash equilibrium. This simplifies to

$$s_i^*\cdot\left(-\delta\cdot N + \beta_i^*\cdot(\frac{1+N\cdot\delta}{1+\delta} - 1)\right) \le 0$$

Now suppose (i) $s_i^* = 0$. In this case we must have $u_{ij}/p_j^* \le 1$ for every good j. To see this, we argue by contradiction. Suppose $u_{ij}/p_j^* = 1+\epsilon$ for some good

j. Notice that if player i changes s_i^* to γ the price of good j can go up by at most γ as we know each price increases by Lemma 1 and the sum of all prices is at most γ higher (by the market conditions). Thus, if player i puts $\gamma < \epsilon$ money onto the market then good j will still have bang-for-buck greater than 1 and so player i will gain more utility than the loss of savings. Thus, s_i^* cannot be an equilibrium, a contradiction.

Thus $u_{ij} \leq p_j^* \leq u_{i^*j}$ where i^* is the player who receives good j at NE. Therefore this player obtains more utility from good j than player i did in the Walrasian equilibrium, even without scaling or a utility transfer.

On the other hand, suppose (ii) $s_i^* > 0$. This can only occur if we have both $\beta_i^* \geq 1$ and

$$\beta_i^* \cdot \frac{(N-1)\cdot\delta}{1+\delta} \leq \delta \cdot N \tag{15}$$

Therefore $1 \leq \beta_i^* \leq (1+\delta)\cdot(1+\frac{1}{N-1})$. Since this holds for all δ, as we take $\delta \to 0$ we must have $\beta_i^* \leq \frac{N}{N-1}$. Thus $\frac{u_{ij}}{p_j^*} \leq \frac{N}{N-1}$ for every good j. Thus if we multiply the utility of the player receiving good j in the Nash equilibrium by $\frac{N}{N-1}$ he will be getting more utility from it than player i did in the Walrasian equilibrium. □

Proof of Theorem 2. Given the other buyers strategies $\boldsymbol{s}^*_{-\mathbf{i}}$ suppose buyer i sets $s_i = m_i$. Then, by Lemma 1, prices cannot be higher for $(m_i, \boldsymbol{s}^*_{-\mathbf{i}})$ than at the Walrasian equilibrium $\boldsymbol{p}(\boldsymbol{m})$. Therefore, by selecting $s_i = m_i$, buyer i could afford to buy the entire bundle S_i at the resultant prices. Consequently, her best response strategy $\boldsymbol{s}_i^*$ must offer at least that much utility. This is true for each buyer, so we have

$$\sum_{i\in\mathcal{B}}\left((m_i - s_i^*) + \sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^*\right) \geq \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} \tag{16}$$

Thus

$$\begin{aligned}\mathcal{W}(\boldsymbol{s}^*) = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^* + \sum_{i\in\mathcal{B}} m_i \;&=\; \sum_{i\in\mathcal{B}}\left((m_i - s_i^*) + \sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^*\right) + \sum_{i\in\mathcal{B}} s_i^* \\ \geq \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} + \sum_{i\in\mathcal{B}} s_i^* & \end{aligned} \tag{17}$$

On the other hand, Lemma 5 implies that

$$\mathcal{W}(\boldsymbol{s}^*) \;=\; \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^* + \sum_{i\in\mathcal{B}} m_i \;\geq\; \left(1-\frac{1}{N}\right)\cdot\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} + \sum_{i\in\mathcal{B}} m_i \tag{18}$$

Taking a convex combination of Inequalities (17) and (18) gives

$$\mathcal{W}(\boldsymbol{s}^*) \geq \left(\alpha\cdot(1-\frac{1}{N}) + (1-\alpha)\right)\cdot\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} + \alpha\cdot\sum_{i\in\mathcal{B}} m_i + (1-\alpha)\cdot\sum_{i\in\mathcal{B}} s_i^*$$

$$\geq \left(\alpha \cdot (1 - \frac{1}{N}) + (1-\alpha)\right) \cdot \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \alpha \cdot \sum_{i \in \mathcal{B}} m_i$$

$$= \left(1 - \frac{\alpha}{N}\right) \cdot \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \alpha \cdot \sum_{i \in \mathcal{B}} m_i \tag{19}$$

Thus plugging $\alpha = \frac{N}{N+1}$ in (19) gives

$$\mathcal{W}(\boldsymbol{s}^*) \;\geq\; \left(1 - \frac{1}{N+1}\right) \cdot \left(\sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \sum_{i \in \mathcal{B}} m_i\right) \;=\; \left(1 - \frac{1}{N+1}\right) \cdot \mathcal{W}(\boldsymbol{m}) \tag{20}$$

This completes the proof. □

5 Existence of Nash Equilibria

We have demonstrated bounds for the Price of Imperfect Competition in the Fisher Game under both CES and Cobb-Douglas utilities. However, these welfare results only apply to strategies that are Nash equilibria. In this section, we prove that Nash equilibria exist for the Cobb-Douglas case, but need not exist for linear utilities. For games without Nash equilibria, we may still recover some welfare guarantees; we show this in Section 6, by examining the dynamics of the Fisher Game with linear utilities.

5.1 Cobb-Douglas Utility Functions

We prove in Appendix C.1 that a Nash equilibrium always exists for Fisher Games with Cobb-Douglas utilities as long as each good provides utility for at least two players.[6]

5.2 Linear Utility Functions

Nash equilibria need not exist in the Fisher Game with linear utilities. We provide an example of this in Appendix C.2.

6 Social Welfare under Best Response Dynamics

Whilst Nash equilibria need not exist in the Fisher Game with linear utilities, we can still obtain a good welfare guarantee in the dynamic setting. Specifically, in the dynamic setting we assume that in every round (time period), each player simultaneously plays a best response to what they observed in the previous

[6] In the absence of this assumption, it is possible for a player who is a monopsonist of a single good to continually decrease their strategic allocation, trivially precluding the possibility of an equilibrium.

round. Dynamics are a natural way to view how a game is played and a well-studied question is whether or not the game dynamics converge to an equilibrium. Regardless of the answer, it is possible to quantify the average social welfare over time of the dynamic process. This method was introduced by Goemans et al in [16] and we show how it can be applied here to bound the *Dynamic Price of Imperfect Competition* - the worst case ratio of the average welfare of states in the dynamic process to the welfare of the Walrasian equilibrium.

For best responses to be well defined in the dynamic Fisher Game, we need the concept of a minimum monetary allocation s_i. Thus we discretize the game by allowing players to submit strategies which are rational numbers of precision up to Φ. This has the added benefit of making the game finite. In Appendix D, we prove the following bound on the Dynamic Price of Imperfect Competition.

Theorem 3. *In the dynamic Fisher Game with linear utilities, the Dynamic Price of Imperfect Competition is lower bounded by $\Omega(1/\log(\frac{M}{\phi}))$ where $M = \max_i m_i$.*

References

1. B. Adsul, C. Babu, J. Garg, R. Mehta, and M. Sohoni, "Nash equilibria in Fisher market", *SAGT*, pp464–475, 2010.
2. J. Bonnisseau and M. Florig, "Existence and optimality of oligopoly equilibria in linear exchange economies", *Economic Theory*, **22(4)**, pp727–741, 2002.
3. M. Babaioff, B. Lucier, N. Nisan, and R. Paes Leme, "On the efficiency of the Walrasian mechanism", *EC*, pp783-800, 2014.
4. S. Branzei, Y. Chen, X. Deng, A. Filos-Ratsikas, S. Frederiksen and J. Zhang, "The Fisher market game: equilibrium and welfare", to appear in *AAAI* , 2014.
5. N. Chen, X. Deng, H. Zhang, and J. Zhang, "Incentive ratios of Fisher markets", *ICALP*, pp464–475, 2012.
6. G. Codognato and J. Gabszewicz, "Equilibre de Cournot-Walras dans une économie d'échange". *Revue économique*, **42(6)**, pp1013–1026, 1991.
7. N. Devanur, J. Garg, and L. Végh, "A rational convex program for linear Arrow-Debreu markets", 2013.
8. N. Devanur, C. Papadimitriou, A. Saberi, and V. Vazirani, "Market equilibrium via a primal-dual algorithm for a convex program", *Journal of the ACM*, **55(5)**, Article 22, 2008.
9. B. Eaves, "A finite algorithm for the linear exchange model", *Journal of Mathematical Economics*, **3(2)**, pp197–203, 1976.
10. B. Eaves, "Finite solution of pure trade markets with Cobb-Douglas utilities", *Economic Equilibrium: Model Formulation and Solution*, pp226–239, 1985.
11. E. Eisenberg and D. Gale, "Consensus of subjective probabilities: The pari-mutuel method", *The Annals of Mathematical Statistics*, **30(1)**, pp165–168, 1959.
12. E. Eisenberg, "Aggregation of utility functions", *Management Sciences*, **7(4)**, pp337–350, 1961.
13. M. Feldman, K. Lei, and L. Zhang, "The proportional-share allocation market for computational resources", *EC*, 2005.
14. D. Gale, "The linear exchange model", *Journal of Mathematical Economics*, **3(2)**, pp205-209, 1976.

15. J. Gabszewicz and P. Michel, "Oligopoly equilibria in exchange economies", *Trade, technology and economics. Essays in honour of Richard G. Lipsey*, pp. 217–240, 1997.
16. M. Goemans, V. Mirrokni, and A. Vetta, "Sink equilibria and convergence", *Foundations of Computer Science*, 2005.
17. L. Hurwicz, "On informationally decentralized systems", In *Decision and Organization: A volume in Honor of Jacob Marschak*, Volume 12 of Studies in Mathematical and Managerial Economics, pp297–336, 1972.
18. K. Jain, "A polynomial time algorithm for computing an Arrow-Debreu equilibrium for linear utilities", *SIAM Journal on Computing*, **37(1)**, pp291–300, 2007.
19. K. Jain and V. Vazirani, "Eisenberg–Gale markets: algorithms and game-theoretic properties", *Games and Economic Behavior*, **70(1)**, pp84–106, 2010.
20. R. Johari and J. Tsitsiklis, "Efficiency loss in network resource allocation game", *Mathematics of Operations Research*, **57(4)**, pp823-839, 2004.
21. J. B. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave N-person Games", *Econometrica*, **33(3)**, pp520534, 1965.
22. L. Shapley and M. Shubik, "Trade using one commodity as a means of payment", *The Journal of Political Economy*, pp937–968, 1977.
23. V. Shmyrev, "An algorithm for finding equilibrium in the linear exchange model with fixed budgets", *SIAM Journal of Applied and Industrial Mathematics*, **3(4)**, pp505–518, 2009.
24. R. Solov, "A contribution to the theory of economic growth", *Quarterly Journal of Economics*, **70**, pp65–94, 1956.
25. V. Vazirani, "Combinatorial Algorithms for Market Equilibria", In *Algorithmic Game Theory*, pp.103-133, 2007.
26. L. Walras, "Principe d'une théorie mathématique de l'échange", 1874.

A Examples of Fisher Games

A.1 A Fisher Game with Unbounded PoIC

In this section we demonstrate a Fisher Game with one good where potential gain in welfare at its only NE is unbounded compared to its WE. Since CES function on one good is essentially a linear function, we show the result for Fisher Game under CES utility function.

Theorem 4. *For any $\Delta > 1$, there exists a Fisher Game under linear utility functions with exactly one NE $\boldsymbol{s}^*$, and $\mathcal{W}(\boldsymbol{s}^*) \geq \Delta\mathcal{W}(\boldsymbol{m})$.*

Proof. Consider the following market with one good a and three buyers $1, 2$ and 3. Buyers 1 has $m_1 = 1$ and $u_{1a} = H$. Buyer 2 is identical: $m_2 = 1$ and $u_{2a} = H$. On the other hand the third buyer has $m_3 = 2L - 2$ and $u_{3a} = 1$. Assuming there is one unit of good j then the market equilibrium is $p_a = 2L$ and $\{x_{1a}, x_{2a}, x_{3a}\} = \{\frac{1}{2L}, \frac{1}{2L}, \frac{2L-2}{2L}\}$. This has a total welfare of

$$\mathcal{W}(\boldsymbol{m}) = \left(\frac{1}{2L} \cdot H + \frac{1}{2L} \cdot H + \frac{2L-2}{2L} \cdot 1\right) + 2L < \frac{H}{L} + 2L + 1$$

There is a Nash equilibrium $\{s_1^*, s_2^*, s_3^*\} = \{1, 1, 0\}$ with $p_j^* = 2$ and $\{x_{1j}^*, x_{2j}^*, x_{3j}^*\} = \{\frac{1}{2}, \frac{1}{2}, 0\}$. For high enough values for H and L, this game has no other equilibrium. The total welfare at this equilibrium is

$$\mathcal{W}(\boldsymbol{s}^*) = \left((\frac{1}{2} \cdot H + 0) + (\frac{1}{2} \cdot H + 0) + (0 \cdot 1 + 2L - 2) \right) + 2 = H + 2L$$

Thus, for any $\Delta > 1$, we can choose H high enough relative to L so that the welfare ratio between the Nash equilibrium and the market equilibrium is greater than Δ.

A.2 A Fisher Game with PoIC < 1

In this section we will demonstrate an example of the Linear Fisher Game where the PoIC is < 1.

Take a four buyer game with two items. There are three units of good 1 and one unit of good 2 ($e_1 = 3, e_2 = 1$). The buyers have $(m_1, m_2, m_3, m_4) = (1, 1, k + 1 - \delta, \delta)$ where k is large and $\delta < \frac{6k}{(6k+1)^2}$. The utility coefficients are $(u_{11}, u_{12}) = (3, 0)$, $(u_{21}, u_{22}) = (3, 0)$, $(u_{31}, u_{32}) = (6, 6k)$ and $(u_{41}, u_{42}) = (0, 1)$. Thus buyer 3 is the only buyer who values both goods.

The market equilibrium is $(p_1, p_2) = (1, k)$ with $(x_{11}, x_{12}) = (x_{21}, x_{22}) = (1, 0)$, $(x_{31}, x_{32}) = (1, \frac{k-\delta}{k})$ and $(x_{41}, x_{42}) = (0, \frac{\delta}{k})$. Total welfare at the equilibrium is then

$$\begin{aligned}
&\sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \sum_{i \in \mathcal{B}} m_i \\
&= \left(3 \cdot 1 + 3 \cdot 1 + (6 \cdot 1 + 6k \cdot \frac{k - \delta}{k}) + 1 \cdot \frac{\delta}{k} \right) + (1 + 1 + (k + 1 - \delta) + \delta) \\
&= 7k + 15 + \frac{\delta}{k} \cdot (1 - 6k) \\
&> 7k + 15 - 6 \cdot \delta
\end{aligned}$$

On the other hand, we claim $(m_1^*, m_2^*, m_3^*, m_4^*) = (1, 1, \sqrt{6k \cdot \delta} - \delta, \delta)$ is a Nash equilibrium. This gives the allocation $(x_{11}^*, x_{12}^*) = (x_{21}^*, x_{22}^*) = (\frac{3}{2}, 0)$, $(x_{31}^*, x_{32}^*) = (0, \frac{\sqrt{6k \cdot \delta} - \delta}{\sqrt{6k \cdot \delta}})$ and $(x_{41}^*, x_{42}^*) = (0, \frac{\sqrt{\delta}}{\sqrt{6k \cdot \delta}})$. The welfare of the equilibrium is

$$\begin{aligned}
&\sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij}^* + \sum_{i \in \mathcal{B}} m_i \\
&= \left(3 \cdot \frac{3}{2} + 3 \cdot \frac{3}{2} + (6 \cdot 0 + 6k \cdot \frac{\sqrt{6k \cdot \delta} - \delta}{\sqrt{6k \cdot \delta}}) + 1 \cdot \frac{\delta}{\sqrt{6k \cdot \delta}} \right) + (1 + 1 + (k + 1 - \delta) + \delta) \\
&= \left(9 + 6k + \frac{\delta}{\sqrt{6k \cdot \delta}} (1 - 6k) \right) + (3 + k) \\
&= 7k + 12 - \sqrt{6k \cdot \delta} + \sqrt{\frac{\delta}{6k}} \\
&< 7k + 12
\end{aligned}$$

As δ is small this is lower welfare than the Market equilibrium. Now we need to confirm this is a Nash equilibrium. Since player 2 is spending 1 and is only interested in good 1 we must have that $p_1^* \geq \frac{1}{3}$. Now for buyer 3 to purchase both goods we must have $p_2^* = k \cdot p_1^*$ and hence $p_2^* \geq \frac{1}{3} \cdot k$. But only buyers 3 and 4 want good 2 and $m_3^* + m_4^* = \sqrt{6k \cdot \delta} < \frac{1}{3} \cdot k$. Thus, for the market to clear, buyer 3 will only purchase good 2.

It follows that we can separate the game in two submarkets. The first has buyers 1 and 2 with good 1, and the second has buyers 3 and 4 with good 2.

Consider the first sub-market. Let's show that buyer 1 is making a best response. She is facing $(m_2^*, m_3^*, m_4^*) = (1, \sqrt{6k \cdot \delta} - \delta, \delta)$ and needs to select m_1^*. When buyer 2 spends $y \leq 1$ dollars, the utility of buyer 1 is $(1 - x) + 3 \cdot 3 \cdot \frac{x}{x+y}$ when she spends $x \leq 1$ dollars. To see this, she wins a $\frac{x}{x+y}$ fraction of the good; there are three units of the good and she gets a utility of 3 per unit.

Taking the derivative we get

$$\begin{aligned} -1 + \frac{9}{x+y} - \frac{9x}{(x+y)^2} &= -1 + \frac{9(x+y) - 9x}{(x+y)^2} \\ &= -1 + \frac{9y}{(x+y)^2} \end{aligned}$$

But this is positive because $y = 1$ and $x \leq 1$. Thus buyer will spend as much as possible, that is $x = 1$ is a best response. By symmetry, buyer 2 is also making a best response.

Now consider the second sub-market. When buyer 4 spends y dollars, the utility of buyer 3 is $(k + 1 - \delta - x) + 6k \cdot \frac{x}{x+y}$ when she spends x dollars. To optimise x we equate

$$\begin{aligned} -1 + \frac{6k}{x+y} - \frac{6kx}{(x+y)^2} &= 0 \\ \therefore \quad 6ky &= (x+y)^2 \\ \therefore \quad \sqrt{6ky} - y &= x \end{aligned}$$

Since buyer 4 is spending δ dollars, it is a best response for buyer 3 to spend $\sqrt{6k \cdot \delta} - \delta$ dollars, as desired.

Now consider buyer 4. When buyer 3 spends x dollars, the utility of buyer 4 is $(1 - y) + 1 \cdot \frac{y}{x+y}$ when she spends $y \leq \delta$ dollars. Taking the derivative we have

$$-1 + \frac{1}{x+y} - \frac{y}{(x+y)^2} = \frac{-(x+y)^2 + x}{(x+y)^2}$$

Since buyer 3 is spending $x = \sqrt{6k \cdot \delta} - \delta$ dollars and $y \leq \delta$, the numerator is at least

$$\begin{aligned} x - (x+\delta)^2 &= \sqrt{6k \cdot \delta} - \delta - (\sqrt{6k \cdot \delta})^2 \\ &= \sqrt{6k \cdot \delta} - (6k+1) \cdot \delta \end{aligned}$$

But this is positive provided

$$6k \cdot \delta > (6k+1)^2 \cdot \delta^2$$
$$\therefore \quad \frac{6k}{(6k+1)^2} > \delta$$

Thus, buyer 4 will spend all his money and we have a Nash equilibrium.

B The Proportional Share Mechanism

In this section we analyze proportional share mechanisms [13] with and without utility for saved money, and compare welfare at corresponding equilibrium. We show that in proportional share mechanisms [13] adding utility for saved money may lead to an unbounded loss in welfare. In other words, the Price of Imperfect Competition may go to zero. This is unlike Fisher Game, where the Price of Imperfect Competition is bounded below by $\frac{1}{2}$ (Theorem 1). In proportional share mechanisms [13] buyer i allocates in advance a specific amount m_{ij} of money to each good j. The key point here is that when we allow unit utility for each unit of saved money, then prices can rise for some goods.

For example. Take three players and two goods. Let the players have budgets $K, K, 1$, respectively. Let $(u_{11}, u_{12}) = (h^{-1}, 0)$, $(u_{21}, u_{22}) = (h^2, h)$, $(u_{31}, u_{32}) = (0, h^3)$, for some large h.

The optimality conditions at an equilibrium in these games are:

$$u_{ij} \cdot \frac{p_j - m_{ij}}{(p_j^*)^2} = 1 + \epsilon_i^* \qquad \text{if } m_{ij} > 0 \tag{21}$$

and

$$u_{ij} \cdot \frac{p_j - m_{ij}}{(p_j^*)^2} \leq 1 + \epsilon_i^* \qquad \text{if } m_{ij} = 0 \tag{22}$$

Without having any value for saved money, we have that buyer 1 allocates all her money to good 1 and buyer 3 allocates all his money to good 2. Thus the optimality conditions state if buyer 2 allocates money to both goods then

$$u_{21} \cdot \frac{K}{(K+m_{21})^2} = u_{22} \cdot \frac{1}{(1+m_{22})^2}$$
$$h^2 \cdot \frac{K}{(K+m_{21})^2} = h \cdot \frac{1}{(1+K-m_{21})^2}$$
$$h \cdot (1+K-m_{21})^2 \cdot K = (K+m_{21})^2$$

But for $h >> K$ this cannot happen and buyer 2 will allocate all her money to good 1. Thus buyer 3 will win all of good 3 fetching social welfare of at least h^3.

On the other hand if each unit of saved money gives unit utility, then buyer 1 will not allocate any money to good 1 unless its price is at most h^{-1}.

Thus player 2 cannot allocate more than h^{-1} to good 1. Thus he allocates at least $K - h^{-1}$ dollars to good 2. Thus the price of good 2 rises! In which case, buyer 3 gets a $\frac{1}{K}$ fraction of good 2. This gives a social welfare of around $\frac{1}{K} \cdot h^3$.

C Existence of Nash Equilibria

C.1 Cobb-Douglas Utility Functions

We will prove that a strategic equilibrium exists if each player has Cobb-Douglas utility functions and each good provides utility to at least two players. Recall that $T_i(\boldsymbol{s})$ is player i's total utility at strategy profile $\boldsymbol{s}$. The first step in this proof is to show that T_i is a concave function with respect to s_i when $\boldsymbol{s}_{-i}$ is fixed.

Lemma 6. *T_i is a concave function of s_i.*

Proof. First, it is enough for us to consider the component of the utility from the market, U_i (as the utility from saving money is always concave). Recall that from (4), we have $y_{ij} = x_{ij} \cdot p_j = s_i \cdot u_{ij}$. Thus, we can easily express U_i as a function of s_i as:

$$U_i \;=\; \prod_j x_{ij}^{u_{ij}} \;=\; \prod_j \left(\frac{s_i \cdot u_{ij}}{\tilde{p}_j + s_i u_{ij}} \right)^{u_{ij}} \tag{23}$$

Here $\tilde{p}_j = \sum_{k \neq i} y_{kj}$. We get the second equality simply by writing each x_{ij} as $\frac{y_{ij}}{p_j}$. Now, note that $\prod_j u_{ij}^{u_{ij}}$ is just a positive constant and so does not affect concavity. Also, $\prod_j s_i^{u_{ij}} = s_i$ by our assumption that $\sum_j u_{ij} = 1$. Thus it is enough to show that the following is concave:

$$\tilde{U}_i = \frac{s_i}{\prod_j (\tilde{p}_j + s_i u_{ij})^{u_{ij}}}.$$

Taking derivatives give us:

$$\tilde{U}_i' = \frac{\prod_j (\tilde{p}_j + s_i u_{ij})^{u_{ij}} - s_i \sum_k u_{ik}^2 (\tilde{p}_k + s_i u_{ik})^{(u_{ik}-1)} \prod_{j \neq k} (\tilde{p}_j + s_i u_{ij})^{u_{ij}}}{\prod_j (\tilde{p}_j + s_i u_{ij})^{2u_{ij}}} \tag{24}$$

Notice that the numerator simplifies considerably, if we take advantage of the the fact that $\sum_j u_{ij} = 1$ to rewrite it as:

$$\sum_k u_{ik} \prod_j (\tilde{p}_j + s_i u_{ij})^{u_{ij}} - s_i \sum_k u_{ik}^2 (\tilde{p}_k + s_i u_{ik})^{(u_{ik}-1)} \prod_{j \neq k} (\tilde{p}_j + s_i u_{ij})^{u_{ij}}$$
$$= \sum_k \tilde{p}_k (\tilde{p}_k + s_i u_{ik})^{(u_{ik}-1)} \prod_{j \neq k} (\tilde{p}_j + s_i u_{ij})^{u_{ij}}$$

Thus, we can simplify to

$$\tilde{U}_i' = \sum_k \frac{\tilde{p}_k}{(\tilde{p}_k + s_i u_{ik}) \prod_j (\tilde{p}_j + s_i u_{ij})^{u_{ij}}} \tag{25}$$

But this is clearly a decreasing function of s_i and so $\tilde{U}_i$ is concave. □

We are now ready to prove the existence of an equilibrium.

Theorem 5. *If for every good at least two players have positive utility for that good, then a Nash equilibrium of the strategic game exists.*

Proof. This proof is similar in structure to that of [13]. Let $\Gamma = (\boldsymbol{U}, \boldsymbol{m})$ be the original market game. For each $\epsilon > 0$, we define the epsilon-market as Γ_ϵ. This market has all of the original players and goods, but will limit the strategy sets of each player by forcing them to put at least ϵ of their money on the market.

It is easy to see that in the epsilon version of the game, utilities are continuous with respect to the strategic variable. This follows from (23). Also, by Lemma 6, we see that the function T_i with respect to s_i is concave. Applying Rosen's theorem [21] we get that a market equilibrium must exist for each epsilon market. Let $\boldsymbol{s}^*_\epsilon$ be this equilibrium.

Notice that, since the strategy sets are compact, there must be a limit point to $\boldsymbol{s}^*_\epsilon$ as $\epsilon \to 0$. Call this point $\boldsymbol{s}^*$. Clearly $\boldsymbol{s}^*$ is a feasible strategy of the original game. We will try to show that $\boldsymbol{s}^*$ is a strategic Nash equilibrium for the original game. Note also that we can take a subsequence of the $\boldsymbol{s}^*_\epsilon$, say $\{\epsilon_1, \epsilon_2, ...\}$ so that each of the corresponding allocations and prices $\boldsymbol{x}^*_{\epsilon_j}$ and $\boldsymbol{p}^*_{\epsilon_j}$ also converge to a limit point, say $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$, respectively, as they also lie on a compact set. Next we show a lower bound on $\boldsymbol{p}^*_{\epsilon_j}$.

Claim. If at least two players have positive utility for good j, then there is some constant $c > 0$ such that for every epsilon game, the strategic equilibrium price $\boldsymbol{p}^*_\epsilon > c$.

Proof. We argue by contradiction. Let us choose some ϵ and some good j for which two players have positive utility and such that the equilibrium price is $p^*_{\epsilon j} \leq c$. We will define c later. Since there are at least two users who have positive utility from good j, there is at least one user, say user i, who has $u_{ij} > 0$ but who is allocated at most half of good j (i.e. $x^*_{ij} \leq 1/2$ and could in fact be 0). Consider two cases.

Case 1: $s^*_i \geq \frac{m_i}{2}$.

In this case, by (4), we must have $p^*_j \geq y_{ij} = s^*_i u_{ij} \geq \frac{m_i u_{ij}}{2}$. Choosing $c < \frac{m_{\min} u_{\min}}{2}$ gives a contradiction.

Case 2: $s^*_i < \frac{m_i}{2}$.

In this case, recall from (25) that:

$$\frac{\partial U_i}{\partial s_i} = \sum_k \frac{\tilde{p}_k}{(\tilde{p}_k + s_i u_{ik}) \prod_j (\tilde{p}_j + s_i u_{ij})^{u_{ij}}} \prod_j u_{ij}^{u_{ij}} \tag{26}$$

Since we are assuming $x^*_{ij} < 1/2$, we must have that $\tilde{p}^*_j > y^*_{ij} = s^*_i u_{ij}$. Then, as all of the terms of the above sum are positive, we can simply focus on the j-th term to get the following inequality at the equilibrium point:

$$\frac{\partial U_i}{\partial s_i} > \frac{1}{2(2\tilde{p}_j)^{u_{ij}} \prod_{k \neq j} (\tilde{p}_k + s_i u_{ik})^{u_{ik}}} \prod_k u_{ik}^{u_{ik}} \tag{27}$$

Now we let $U = \prod_k u_{ik}^{u_{ik}}$ and notice that each term of the product in the denominator is bounded by the total money between all players (which we will call M). Thus, at equilibrium we have:

$$\frac{\partial U_i}{\partial s_i} > \frac{U}{2(2\tilde{p_j})^{u_{ij}} M^m} \tag{28}$$

Thus, by choosing $c < \frac{1}{2}(\frac{U}{2M^m})^{\frac{1}{u_{\max}}}$, we can ensure that $\frac{\partial U_i}{\partial s_i} > 1$. This contradicts the fact that we are at an internal equilibrium of the strategic game. □

By the above claim it is clear that for each epsilon game the prices for each good must be at least c and, thus, in the limit $\boldsymbol{p}^* > c$. From this we will establish that $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$ are in fact valid prices and allocations for the market equilibrium if the players play strategy $\boldsymbol{s}^*$. First, the demands and prices are feasible as, by convergence, we have that $\sum_i x_{ij}^* = 1$ for all j and $\sum_j x_{ij}^* p_j^* = s_i^*$ for all i. It is also clear from the convergence that the allocation $\boldsymbol{x}^*$ must maximize each player's utility amongst all allocations that they can afford. We need only check that if a player has $\boldsymbol{s}_i^* = 0$ that they are allocated no goods which is the only possible discontinuous condition on the game. This follows from the fact that we have guaranteed that $\boldsymbol{p}^* > c > 0$. Thus, $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{s}^*)$ and $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{s}^*)$.

Since the allocations $\boldsymbol{x}_{\epsilon_j}$ of $\boldsymbol{s}_{\epsilon_j}$ converge to the allocation $\boldsymbol{x}^*$ of $\boldsymbol{s}^*$, it must be that, for every $\delta > 0$, there exists some $J > 0$ such that for all $j > J$:

$$\left| T_i(\boldsymbol{s}^*) - T_i(\boldsymbol{s}_{\epsilon_j}) \right| < \delta. \tag{29}$$

We are now ready to show that $\boldsymbol{s}^*$ is a Nash equilibrium for the strategic game. Suppose that it is not. Then there must be some player i who has a payoff improving allocation. In fact, suppose that instead of playing $\boldsymbol{s}^*$, player i deviated to a new strategy $\hat{s}_i$ with strictly greater payoff. Define $\hat{\boldsymbol{s}} = (\boldsymbol{s}_1^*, ..., \hat{s}_i, ..., \boldsymbol{s}_n^*)$ and $\hat{\boldsymbol{s}}_\epsilon = (s_{\epsilon 1}, ..., \hat{s}_i, ..., s_{\epsilon n})$ for sufficiently small ϵ. Again, we partition into two cases.

Case 1: $\hat{s}_i = 0$.

If $\hat{s}_i = 0$ then $s_i^* > 0$. Now consider $s_{\epsilon_1 i}^*, s_{\epsilon_2 i}^*, \ldots$ the set of strategies converging to s_i^*. Since these are at Nash equilibrium, each of these strategies has utility more than $m_i - \epsilon$ (which is the minimum utility obtained if player i only put ϵ in the market in the epsilon game). Thus these must converge to a strategy with utility $\geq m_i$. Thus, defecting with $\hat{s}_i = 0$ which gives utility m_i cannot be a utility increasing move.

Case 2: $\hat{s}_i > 0$.

Suppose $T_i(\hat{\boldsymbol{s}}) - T_i(\boldsymbol{s}^*) = \epsilon' > 0$. Then, for sufficiently small ϵ we must have $T_i(\hat{\boldsymbol{s}}_\epsilon) - T_i(\boldsymbol{s}_\epsilon^*) > 0$ by (29). This contradicts the fact that $\boldsymbol{s}_\epsilon^*$ is a Nash equilibrium. Thus $\boldsymbol{s}^*$ must be a Nash equilibrium for the strategic game as required. □

C.2 A Fisher Game with no Nash Equilibrium

A Nash equilibrium need not exist in a Fisher Game with linear utilities. We show this using the following simple counterexample. Consider a market with two buyers a and b and two goods 1 and 2. Let each player get utility 1 for each good, except that $u_{a2} = 2$. Let the budgets of each player be $m_a = m_b = 4$. Suppose now that each player chooses a strategy $s_a \leq m_a$ and $s_b \leq m_b$. There are four cases.

Case I: $s_a < s_b$.

The market equilibrium in this case is $p_1 = p_2 = \frac{s_a+s_b}{2}$, a taking only good 2 with total utility $U_a = \frac{4s_a}{(s_a+s_b)} + m_a - s_a$, and b taking the full good 1 and the rest of good 2 with utility $\frac{2s_b}{s_a+s_b} + m_a - s_a$. Now U_a is a concave function in s_a, its derivative is $\frac{4s_b}{(s_a+s_b)^2} - 1$, and the s_a value maximizing it must satisfy $4s_b = (s_a + s_b)^2$, hence this must hold in NE. Similarly, for b, we get $2s_a = (s_a + s_b)^2$ in NE. This gives $s_a = 2s_b$, a contradiction to $s_a < s_b$.

Case II: $s_a = s_b = s$.

Now $s = 0$ cannot be NE, because a buyer putting a tiny amount of money on the market could get the utility 3 or 2, resp. If $s > 0$ then the market equilibrium prices are $p_1 = p_2 = s$, a buying the full unit of 2, b buying the full unit of 1. This cannot be NE, since if b's utility is $1+m_b-s_b$ then if he puts in a little less money he will still get the full unit of good 1, giving utility 1 (see next case).

Case III: $s_b < s_a \leq 2s_b$.

At the market equilibrium, a only buys 2 and b only buys 1. Hence $p_1 = s_b, p_2 = s_a$. This clearly cannot be a NE: a's utility is $2+m_a-s_a$, b's utility $1+m_b-s_b$, i.e. they get the full utility of the corresponding good for infinitesimal money. In particular, a could decrease s_a.

Case IV: $2s_b < s_a$.

At the market equilibrium, $p_1 = \frac{s_a+s_b}{3}$ and $p_2 = \frac{2(s_a+s_b)}{3}$. Buyer a takes the full good 2, b spends all his money on 1. So

$$U_a = \frac{3s_a}{s_a + s_b} - s_a, \quad U_b = \frac{3s_b}{s_a + s_b} - s_b$$

Then the same way as in Case I, if $0 < 2s_b < s_a < m_a$, then we must have that if it's a NE then $3s_a = 3s_b = (s_a + s_b)^2$. This again contradicts $2s_b < s_a$.

If $s_b = 0$, then a gets all goods with utility $3 + m_a - s_a$, and could get it for less. If $0 < 2s_b < s_a = m_a = 4$, then again we must have $3s_a = (s_a + s_b)^2$ for b to be optimal, giving $s_b = 2\sqrt{3} - 4 < 0$.

D Social Welfare under Best Response Dynamics

We now prove the logarithmic lower bound in the Dynamic Price of Imperfect Competition for Fisher games with linear utilities. To prove Theorem 3, we first notice that if a player puts a certain fraction of his budget onto the market, he is guaranteed at least that fraction of his utility in the Walrasian equilibrium.

Lemma 7. *In strategy profile $\boldsymbol{s}$ suppose player i has played strategy $s_i > \frac{m_i}{K}$ for some K. Then $U_i(\boldsymbol{x}_i(\boldsymbol{s})) \geq \frac{\hat{U}_i}{K}$ where $\hat{U}_i$ is that player's utility in the Walrasian equilibrium.*

Proof. Let β_i and β_i^W be the bang-for-buck of player i at the current strategy and at Walrasian equilibrium, then using Lemma 1 we have $U_i(\boldsymbol{x}_i(\boldsymbol{s})) = s_i\beta_i \geq \frac{m_i}{K}\beta_i \geq \frac{m_i}{K}\beta_i^W = \frac{\hat{U}_i}{K}$. □

Next, we will show that if a player is not receiving much utility in the current strategy state, then in his next move he will either dramatically decrease or dramatically increase his allocation of money to the market.

Lemma 8. *Suppose at time t, the players have chosen strategies $\boldsymbol{s}^t$. If for player i, $T_i(\boldsymbol{s}^t) < \frac{\hat{U}_i}{K}$ then $s_i^{t+1} \geq K s_i^t$.*

Proof. Notice that if for his next move, player i were to put in $s_i^{t+1} = m_i$ then he would get utility at least $\hat{U}_i$ (Lemma 2). Thus his best response must lead him to expect at least this amount. Since increasing s_i from s_i^t will only worsen his bang per buck and reduce the savings, the only way to get at least $\hat{U}_i$ is to put in at least K times what he previously did. □

Lemma 9. *Suppose at time t, the players have chosen strategies $\boldsymbol{s}^t$. If for player i, $T_i(\boldsymbol{s}^t) < \frac{m_i}{K}$, then $s_i^{t+1} \leq \frac{s_i^t}{K}$.*

Proof. Since $T_i(\boldsymbol{s}^t) < \frac{m_i}{K}$, player i's bang-for-buck at $\boldsymbol{s}^t$ is less than $\frac{1}{K}$. Notice that if for his next move, he were to put in $s_i^{t+1} = 0$ then he would get total utility at least m_i. Thus his best response must lead him to expect at least this amount. By Lemma 3, the only way he can expect to increase his bang-for-buck to 1 is by decreasing his allocation of money by a factor of at least $\frac{1}{K}$. □

We observe that it is not possible for the conditions of Lemmas 8 and 9 to be satisfied simultaneously for $K > 1$. We are now ready to prove Theorem 3.
Proof of Theorem 3. Let us fix some constant $K > 1$. We will argue that any player i will receive aggregate utility at least $\frac{\max(\hat{U}_i, m_i)}{K}$ in any sequence of $C \cdot \log(M/\phi)$ moves, for some constant C. Note that sum of these agregates is at least $O(\sum_i m_i)$, and therefore the utility of sellers is also taken care of with an additional factor of 2.

Let β_i^W be the bang-for-buck that player i achieves in the Walrasian equilibrium, and let β_i^t be her bang-for-buck in round t. From Lemma 1 we have $\beta_i^t \geq \beta_i^W$, $\forall i$, $\forall t$, and $\hat{U}_i = m_i\beta_i^W$ and $T_i(\boldsymbol{s}^t) = U_i(\boldsymbol{s}^t) + m_i - s_i^t = s_i^t\beta_i^t + m_i - s_i^t$. We will consider 4 cases:
Case I: $1 \leq \beta_i^W \leq K$**.** In this case, player i's bang-for-buck will always be at least 1 in each round. Thus $\forall t,\ T_i(\boldsymbol{s}^t) \geq m_i \Rightarrow T_i(\boldsymbol{s}^t) \geq \frac{\hat{U}_i}{K}$ using $\hat{U}_i = m_i\beta_i^W$.
Case II: $\frac{1}{K} \geq \beta_i^W \geq 1$**.** As $\beta_i^t \geq \beta_i^W,\ \forall i$, we have that she will receive at least $\hat{U}_i$ total payoff which is $\beta_i^W m_i \geq \frac{m_i}{K}$.
Case III: $\beta_i^W > K$**.** Since $\beta_i^t \geq \beta_i^W > K$, we will have that $T_i(\boldsymbol{s}^t) \geq m_i,\ \forall t$. So we need only show that at least once in every $C \cdot \log(M/\phi)$ moves, player i receives utility at least $\hat{U}_i/K$. We argue by applying Lemma 8. If player i is not receiving the desired utility, then in the next time period she will increase her allocation by a factor of K. Thus within $O(\log(M/\phi))$ time periods either she

receives $\frac{\hat{U}_i}{K}$ payoff or she allocates at least m_i/K. In the latter case too she will receive $\hat{U}_i/K$ payoff due to Lemma 7.

Case IV: $\beta_i^W < \frac{1}{K}$. Since $\beta_i^W < \frac{1}{K}$, we will have that $T_i(\boldsymbol{s}^t) \geq \hat{U}_i,\ \forall t$. So we need only show that at least once in every $O(\log(M/\phi))$ moves, player i receives utility at least m_i/K. In this case, we argue by applying Lemma 9. If player i is not receiving the desired utility, then in the next time period she will decrease her allocation by a factor of $1/K$. Thus, in the next time period she will receive a utility of at least m_i/K which is sufficient. □