Bounds on the Profitability of a Durable Good Monopolist

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Abstract
A durable good is a long-lasting good that can be consumed repeatedly over time, and a duropolist is a monopolist in the market of a durable good. Theoretically, less is known about durable goods than their more well-studied counterparts, consumable and perishable goods. It was quite startling, therefore, when Ronald Coase (1972) conjectured that a duropolist has no monopoly power at all! Specifically, a duropolist who lacks commitment power cannot sell the good above the competitive price if the time between periods approaches zero. The Coase conjecture was first proved by Gul et al. (1986) under an infinite time horizon model with non-atomic consumers. Remarkably, the situation changes dramatically for atomic consumers and an infinite time horizon. Bagnoli et al. (1989) showed the existence of a subgame perfect Nash equilibrium where the duropolist extracts all the consumer surplus, provided the discount factor is large enough. Thus, for atomic consumers, the duropolist may have perfect price discriminatory power! Observe that, in these cases, duropoly profits are either arbitrarily smaller or arbitrarily larger than the corresponding static monopoly profits – the profit a monopolist for an equivalent consumable good could generate. Neither situation accords in practice with the profitability of durable good producers. Indeed we show that the results of Gul et al. (1986) and Bagnoli et al. (1989) are driven by the infinite time horizons. For finite time horizons, duropoly profits for any equilibrium satisfying the standard skimming property closely relate to static monopoly profits. In particular, for atomic agents, we prove that duropoly profits are always at least as large as static monopoly profits, but never exceed double the static monopoly profits.

1 Introduction

A *durable good* is a long-lasting good that can be consumed repeatedly over time. Theoretically less is known about durable goods than their more well-studied counterparts, consumable and perishable goods. However, on the practical side, durable goods abound and are very familiar to us. For example, many of the most important consumer items are (at least to some extent) durable, such as land, housing, cars, etc. A *duropolist* is a monopolist in the market of a durable good – topically, duropolists include several well-known purveyors of digital goods. Indeed, Amazon has recently been awarded a patent (8,364,595) for establishing market places for second-hand digital-content items, and Apple has recently applied for a similar patent (20130060616).

Pricing a durable good is not as simple as it may appear. Specifically, whilst durable goods are more desirable to the consumer, it is questionable whether a duropolist has additional monopoly power beyond that of an equivalent monopolist for a perishable good. Indeed, quite the opposite may be true. In 1972, Nobel recipient Ronald Coase made the startling conjecture that, in fact, a duropolist who lacks commitment power cannot sell the good above the competitive price if the time between periods approaches zero (Coase, 1972). The intuition behind the Coase conjecture is that if the monopolist charges a high price then consumers anticipate a future price reduction (as they expect the duropolist to later target lower value consumers) and therefore they prefer to wait. The duropolist, anticipating this consumer behaviour, will then drop prices down to the competitive level. In essence, the argument is that a duropolist is not a monopolist at all: the firm does face stiff competition – not from other firms but, rather, from future incarnations of itself! This is known as the *commitment problem*: the duropolist cannot credibly commit to charging a high price.

1.1 Previous Work

The Coase conjecture was first proven by Gul et al. (1986) under an infinite time horizon model with non-atomic consumers. Ausubel and Deneckere (1989) later showed that if the stationary condition is relaxed, the duropolists profits at subgame perfect Nash equilibria (SPNE) can range from Coasian profits to the static monopoly profit: the optimal price the duropolist charges in a one-period game.[1] Stokey (1979) studied pricing mechanisms for duropolist that *possess* commitment power in a continuous time model. She showed that the duropolist can then attain the static monopoly profit by committing to a fixed price; all sales are then made at the beginning of the game.

Underlying the results of Gul et al. (1986) lies the assumption of an infinite time horizon. This assumption is clearly unrealistic in several settings. Furthermore, there are often even situations where trade must take place before a hard deadline. For example, consider a TV network selling advertising space a week-ahead for a show. Theoretical and empirical evidence of the strong effects of deadlines have been observed in many bargaining contexts such as in contract negotiations and civil case settlements (see, e.g., Cramton and Tracy (1992) , Williams (1983) and Fuchs and Skrzypacz (2011)). When there is a finite time horizon and non-atomic consumers, Guth and Ritzberger (1998) showed that there exists a subgame perfect equilibrium, as period lengths approach zero, in which the duropolist profits converge to the static monopoly profits discounted to the beginning of the game.

The above results all assume non-atomic consumers. Bagnoli et al. (1989) studied the duropoly problem with atomic consumers. When the time horizon is

[1] However, profits larger than the Coasian value occur only in the case where there is no gap between the lowest customer value and the marginal cost of production (e.g. for $c = 0$, the lowest customer value is 0).

infinite, they proved another surprising result: the existence of a subgame perfect Nash equilibrium, dubbed *pacman*, in which the duropolist extracts all the consumer surplus. Under pacman, the duropolist announces a price in each period equal to the highest valuation amongst consumers who have not yet bought. The consumers' best response, dubbed *get-it-while-you-can*, is to buy whenever the price is less than or equal to their valuation. This equilibrium refutes the Coase conjecture when there is a finite set of atomic consumers and an infinite time horizon. Indeed, it suggest that a duropolist may have perfect price discriminatory power! Moreover, it shows there exist subgame perfect Nash equilibria where duropoly profits exceed the static monopoly profits by an unbounded factor.[2] Later, von der Fehr and Kuhn (1995) showed that under certain conditions Pacman is the only equilibrium.

Another setting, in which the extremely large profits of the Pacman equilibrium of Bagnoli et al. (1989) cannot exist, was proposed by Cason and Sharma (2001). Instead of assuming a duropolist with perfect information, the authors constructed a two buyers and two-valuation model with infinite time periods in which the duropolist does not know exactly whether a customer is of high or of low type. They later showed that in these games there exists a unique equilibrium that is Coasian (i.e. the first price is equal to the valuation of the low type consumer). Recently, Montez (2013) studied the duropoly problem under infinite horizon with atomic consumers that have two-types (high value consumers and low value consumers) and exactly two consumers are of the high-value type. He showed that there are sometimes inefficient equilibria where the time at which the market clears does not converge to zero as the length of the trading periods approaches zero.

Bagnoli et al. (1989) also presented very small examples with a finite time horizon. They showed that in such games with two or three consumers, it is possible to obtain subgame perfect equilibria where the duropolist extracts more revenue than the price commitment strategy of Stokey (1979). These examples again refute the Coase conjecture but they also suggest that, even for finite time horizons, the duropolist may have **more** monopoly power than the equivalent static monopolist. This is very interesting because, whilst duropolists are not believed to be powerless in practice, the standard assumption is that duropolists are weaker than monopolists for consumable goods. Indeed this argument has been accepted by the Federal Courts in the United States; see Orbach (2004) for a history of the law with regards to duropolies.

1.2 Our Results

Several questions arise immediately from the work of Bagnoli et al. (1989). Does this phenomenon (of duropoly profits exceeding static monopoly profits) arise for more natural games where the number of consumers and the number of time

[2] For example, consider a game with N consumers where buyer i has a valuation of $1/i$. Then, as N gets large, duropoly profits under the Pacman strategy approach $\log N$ whilst the static monopoly profit is clearly 1.

periods is much larger than three? If so, how would a duropolist actually compute a profit maximizing strategy? Finally, from an optimization perspective, can we quantify *exactly* how much more profit a duropolist can obtain at equilibria in comparison to a static monopolist?

Our main contributions is to answer those three questions. To achieve this, we first characterize, in Sect. 3, the class of subgame perfect equilibria that satisfy the standard skimming property: high-value consumers buy before lower-valued consumers.[3] We do this both in the complete information setting as well as a setting where market participants have limited information about other consumer valuations. Our main result, proven in Sect. 4, is then that, at equilibria, duropoly profits are *at least* static monopoly profits but *at most twice* static monopoly profits, regardless of the number of consumers, their values, and the number of time periods. We also prove that this factor two bound is tight: we construct a (infinite) family of examples where duropoly profits approach double the static monopoly profit as the number of consumers goes to infinity.

We believe that our main result sheds light into this classical problem in at least four ways. To begin, this is the first theoretical result that concurs with the practical experience that duropolists and static monopolists have comparable profitability. (Recall that previous theoretical works have suggested that the duropolist either has no monopoly power or has perfect price discriminatory power).

Second, the result that a duropolist can do up to twice as well using a (threat based) strategy rather than a price commitment strategy is actually best viewed from the opposite direction. Specifically, a duropolist can obtain at least half the optimum profit by mimicking a static monopolist via a price commitment strategy. From a practical perspective this is important because a price commitment strategy can generally be implemented by the duropolist very easily, even with limited consumer information. Furthermore, price commitment strategies can be popular with customers as they are typically introduced within a money back guarantee or envy-free pricing framework. In contrast, a threat based optimization strategy is harder to implement and can antagonize consumers.

Thirdly, the standard view in the literature is that the surprising and well-known result of Bagnoli et al. (1989), namely that the duropolist can extract all consumer surplus, is due to the assumption of atomic consumers. Our results showed that this is not true in general – their result is driven by the infinite time horizon. For finite time horizons, the power of a duropolist is limited. This is true even when the Pacman strategy is an equilibrium; indeed, we show that the *Pacman* strategy can be an equilibrium in finite time horizon games only under very specific conditions – see Section 5.

Finally, the main result highlights a distinction in how the time horizon affects bargaining power. With non-atomic consumers, a finite time horizon increases the bargaining power of the duropolist. In Guth and Ritzberger (1998), a finite time-horizon increases duropolist profits from the Coasian result to the static

[3] Bagnoli et al. (1989) state that such a characterization would be extremely interesting.

monopoly profits. With atomic consumers, the finiteness of the time horizon, reduces duropolist profits for total consumer surplus to, at most, twice the static monopoly profits.

To conclude the paper, we examine (in Section 6) subgame perfect equilibria in which the skimming-property does not hold. We provide the first example of a subgame perfect equilibrium in which the skimming-property does not hold. Furthermore, we conjecture that amongst all equilibria that maximize duropoly profits at least one satisfies the skimming property. We prove this conjecture is true for the case of $T = 2$.

2 The Model

In this section, we present the durable good monopoly model of Bagnoli et al. (1989) that we will analyze in the subsequent sections. Consider a durable good market with one seller (a duropolist), N customers and a finite horizon of T time periods. The N customers have valuations $v_1 \geq v_2 \geq \cdots \geq v_N$[4] and the firm can produce units of the good at a unitary cost of c dollars. Here we assume, without loss of generality, that $c = 0$. Consequently, profit and revenue are interchangeable in this setting.

We can view this as a sequential game over T periods. At time t, $1 \leq t \leq T$ the firm will select a price μ_t to charge for the good. The duropolist seeks a credible pricing strategy that maximizes her profits, namely $\sum_{t=1}^{T}(x_t \cdot \mu_t)$, where x_t denotes the number of customers who buy in period t.

Each consumer i desires at most one item and seeks to maximize her *utility*, which is $v_i - \mu_t$ if she buys the good in period t.[5] The customers decide simultaneously if they will buy an item for μ_t. The game then proceeds to period $t+1$. If a consumer doesn't buy an item before the end of period T her utility is zero.

For such a sequential game, the solutions we examine are pure subgame perfect Nash equilibria that satisfy the standard *skimming property* defined below.

Definition 2.1 (Skimming property). *An equilibrium satisfies the* skimming property *if whenever a buyer with value v is willing to buy at price μ_t, given the previous history of prices h_t, then a buyer with value $w > v$ is also willing to buy at this price given the same history.*

For SPNEs that satisfy the skimming property, consumers' strategies can be characterized using a cutoff function. Given a history of prices h_t and the current offered price μ_t, customers with valuations above cutoff $\kappa(h_t, \mu_t)$ buy and customers with valuations below the cutoff do not buy (see Fudenberg and Tirole (1991) for a discussion). When consumers are non-atomic it can be shown that all subgame perfect equilibria satisfy the skimming property (Fudenberg et al., 1985). In the case of atomic consumers and an infinite time horizon, the monopolist can extract all consumer surplus using the *pacman* strategy, in which case

[4] We also use notation $v(y_j)$ instead of v_{y_j} in certain cases to avoid nested subscripts
[5] Discount factors can easily be introduced into the model.

the skimming property is clearly satisfied. Intuitively, the skimming property says that higher value consumers pay a higher (or at least equal) price compared to consumers with a lower valuation. In real markets, this phenomenon is widely observed.[6]

We study the model in both a complete and an incomplete information setting. In the complete information setting, the values of each consumer are known to all participants in the market. In our incomplete information setting, the duropolist knows the distribution of values, but not which value corresponds to which consumer. For example, she may know the consumer values are $\{100, 50, 30\}$, but does not know which of these three is the value of Consumer 1. Similarly, each consumer knows the set of values and which is their own value, but does not know which values correspond to the other consumers.

An Example: We now present a small example to illustrate the model and and the concepts involved for the complete information setting. Consider a two-period game with the valuations as shown in Table 1.

Table 1. Threat prices

Consumer	**1**	**2**	**3**	**4**
Consumer value	100	85	80	50
Threat price	80	80	50	50

Denote by Π^D and Π^M the maximum profits (revenue) of the duropolist and the corresponding static monopolist, respectively. Then the static monopoly profit Π^M is equal to 240, obtained by selling to the top three customers. However, the duropolist can, in fact, extract 260. Furthermore, the corresponding equilibrium satisfies the skimming property: no customer will buy earlier than another customer with a higher valuation. To understand SPNEs in this game, let's begin with a subgame comprising of only the final (second) time period. In such a subgame, it is a dominant strategy for all customers who have not yet bought to pay any price less than or equal to their value. Consequently, it is a dominant strategy for the duropolist to charge the static monopoly price *as calculated with respect to the set of customers who have not yet bought.* Note that these strategies satisfy the skimming property as everyone remaining with value above the price will buy and everyone else will not buy.

Now consider the first time period. If the skimming property is satisfied, then there will be a cut-off point j_1 at which customers $j \leq j_1$ buy and customers $j > j_1$ wait until period 2. In order for this to be an equilibrium, the customer j_1 must prefer buying in period 1 to period 2. Therefore, the duropolist can charge no more than the static monopoly price as calculated if all customers $j \geq j_1$ (*including* j_1) wait until period 2. We call this price the "threat price"

[6] In experimental economics this is tested via screening techniques.

for customer j_1 (the general definition of threat prices is given in Section 3.1). The threat prices for this example are also listed in Table 1.

The consumers' strategies then correspond to "buy in period 1 if and only if μ_1 is at most their threat price", whilst the duropolist's strategy is to charge the threat price which maximizes total profit. The period 2 strategies are the dominant strategies described above: the remaining consumers pay up to their value, while the duropolist charges the static monopoly price calculated for the set of consumers that are left.

Charging $\mu_1 = 80$ means that the top two consumers would buy in period 1, while the last two consumers would wait until period 2 and buy at $\mu_2 = 50$ (the static monopoly price for the remaining two consumers is 50). The total profit would therefore be 260. The reader can verify that no other choice for the μ_i yields higher profit and therefore $\Pi^D = 260$. So duropoly profits are greater than static monopoly profits. For additional comparisons, Coasian profits are $\Pi^C = 200$ since the competitive price is 50, and Price Discriminatory profits are $\Pi^{PD} = 315$, that is, the consumer surplus.

Observe that, for this example, these strategies satisfy the skimming property as threat prices are monotonically increasing with consumer value. Furthermore, it is easy to see that this is an equilibrium. Consumers 3 and 4 would be no better off buying in the first period, while both consumers 1 and 2 would still pay 80 if either deviated by waiting until the last period. Similarly, the duropolist would find no buyers if she charged more than 80 in period 1, turning the game into a one-period static game and giving her profit Π^M, while charging anything between 80 and 50 would yield the same sales schedule but with lower profit. Charging 50 or lower would result in all consumers buying in period 1 for profit at most 200.

We remark that the equilibrium property for the consumers arises from a simple property of static monopoly prices. Take the set of consumers $\{j \geq j_1\}$ and compute the static monopoly price for these consumers. This is j_1's threat price. Now if we take j_1 and replace her by a higher valued consumer then the static monopoly price can only rise. Hence, as long as μ_1 is j_1's threat price, any customer of higher value that refuses to buy in period 1 would be charged at least μ_1 in period 2, ensuring it is a best response to buy in period 1.

In Sect. 3, we show how these arguments can be extended to give subgame perfect equilibria conditions for games with more than two time periods and explain how, given these constraints, a duropolist can efficiently maximize profits.

3 Subgame Perfect Equilibria Conditions

We now characterize the subgame perfect equilibria that satisfy the skimming property. To do this we reason backwards from the final time period T. It is easy to determine the behaviour of rational consumers and a profit maximizing duropolist at time T. Given this information, we can determine the behaviour of rational consumers at time $T - 1$, etc.

We begin with the characterization of subgame perfect equilibria when there is complete information, that is when all participants know the which player has each value and the duropolist can observe who buys in each period. Then, we show how the model can be extended to an incomplete information setting.

3.1 Complete Information

To formalize this, let $\mathcal{G}_i$ denote the subgame consisting of customers $\{i, i+1, \ldots, N\}$, and let $\Pi(i,t)$ denote the maximum profit obtainable in $\mathcal{G}_i$ if we begin in time period t. Thus $\Pi^D = \Pi(1,1)$. Now set $\Pi(i, T+1) = 0$ for all customers i. Let $p(i,t)$ be the profit maximizing price at period t in the subgame $\mathcal{G}_i$ beginning in t. First, consider the last period, T. Any customer i (who has not yet bought the good) will buy in period T if and only if this final price is at most v_i. Therefore, for the subgame $\mathcal{G}_i$, starting at time T, a profit maximizing duropolist will simply set $\mathbf{p}(i,T)$ to be the static monopoly price p_i for the subgame $\mathcal{G}_i$: $p(i,T) = p_i \equiv v_{j^*(i,T)}$, where

$$j^*(i,T) = \arg\max_{j \geq i} \, (j - i + 1) \cdot v_j.$$

Thus, $j^*(i,T)$ denotes the consumer with the lowest valuation who buys in the subgame $\mathcal{G}_i$ beginning at period T.

The profit is then

$$\Pi(i,T) = (j^*(i,T) - i + 1) \cdot v_{j^*(i,T)}$$

In general we will denote by $j^*(i,t)$, the consumer with the lowest valuation who buys (under our proposed strategy) at period t in the subgame $\mathcal{G}_i$ beginning at period t. Now, suppose we are at time period $T-1$ in the subgame $\mathcal{G}_i$. If the duropolist at period $T-1$ wishes to sell to consumers $\{i, i+1, \ldots, k\}$, then the announced price has to be at most k's threat price, $p(k,T) \equiv v_{j^*(k,T)}$. To see this, suppose that the price announced at $T-1$ is higher and the duropolist still expects to sell the item to consumers $\{i, i+1, \ldots, k\}$. Then, if consumer k refuses to buy while all consumers above her buy, the duropolist would, in the final time period T be in the subgame $\mathcal{G}_k$, and announce price $p(k,T)$, meaning that consumer k would have benefited from deviating. So, the optimal strategy for the duropolist would be to sell to $k - i + 1$ consumers at period $T-1$ at price $v_{j^*(k,T)}$, choosing the value of k such that the profits from periods $T-1$ and T are maximized:

$$j^*(i, T-1) = \arg\max_{k \geq i} \{(k - i + 1) \cdot p(k,T) + \Pi(k+1, T)\}$$

$$\Pi(i, T-1) = (j^*(i,T-1) - i + 1) \cdot p(j^*(i,T-1), T) + \Pi(j^*(i,T-1) + 1, T).$$

The price announced at period $T-1$ can then be written as

$$p(i, T-1) = p(j^*(i, T-1), T)$$

Observe then, that in the final period, we will be in the subgame composed of consumers $\{j^*(i, T-1)+1, \ldots, N\}$.

Iterating this argument backwards in terms of the periods, we have that

$$\begin{aligned} \Pi(i,t) &= (j^*(i,t)-i+1)\cdot p(j^*(i,t),t+1)+\Pi(j^*(i,t)+1,t+1) \\ j^*(i,t) &= \arg\max_{j\geq i} \left((j-i+1)\cdot p(j,t+1)+\Pi(j+1,t+1)\right) \\ p^*(i,t-1) &= p(j^*(i,t-1),t) \end{aligned} \tag{1}$$

We can generalize the concept of the threat price from our two-period example using the above recursion

Definition 3.1 (Threat Prices). *The* threat price $\tau(i,t)$ *for player* i *at period* $t < T$ *under the recursive scheme given in (1) is the price* i *is offered in the subgame* $\mathcal{G}_i$ *starting at period* $t+1$, $\tau(i,t) := p(i,t+1)$. *That is, the price offered if* i *and all consumers of lower value do not buy in period* t.

We can now define the strategy of the duropolist and the consumers in any subgame. Consider a subgame whose remaining consumers are the set S and let there be $T-t+1$ periods remaining (i.e we are starting in period t). Then by re-indexing the consumers, the duropolist can treat the subgame as a full game $\mathcal{G}'$ with $T-t+1$ total periods and S as the set of all consumers. She then calculates, for all i and t, the prices $p^*_{\mathcal{G}'}(i,t)$ from the recursion relationship (1) and chooses the sales schedule which maximizes her profits for $\mathcal{G}'$. The price announces at period t would then be $p^*_{\mathcal{G}'}(1,1)$. The consumers buy if and only if the price is less than or equal to their threat price as calculated for $\mathcal{G}'$.

The following series of lemmas establish basic monotonicity results for static monopoly prices, threat prices, and the prices $p^*(i,t)$ which form the duropolist's equilibrium strategy.

Lemma 3.1. *The static monopoly prices on the subgames* $\mathcal{G}_i$ *are non-increasing in* i: $p_i \geq p_{i+1}$ *for* $i = 1, \ldots, N-1$.

Proof. Without loss of generality we can restrict to the case $i = 1$. Let $a = y_1$ and $b = y_2$. Then $p_1 = v_a$ and $p_2 = v_b$ and therefore $a \geq 1$ and $b \geq 2$. As a first case, consider that $a \leq b$. Given that valuations are non-increasing, it follows that $p_1 = v_a \geq p_2 = v_b$. As the second case, suppose that $a > b \geq 2$. We know that

$$av_a \geq bv_b,$$

by definition of a. Now if $a > b$, in the game $\mathcal{G}_2$ the static monopolist has the option of selling to exactly the consumers in $[2, a]$. Therefore, as v_b is the static monopoly price for $\mathcal{G}_2$, the game with consumers $[2, N]$, it follows that

$$(b-1)v_b \geq (a-1)v_a.$$

Combining these two inequalities gives us $v_a \geq v_b$. But $a > b$ implies $v_b \geq v_a$, so we conclude that $v_a = v_b$. In either case, $v_a \geq v_b$.

□

The following lemma shows that the consumers' strategies defined above satisfies the skimming property.

Lemma 3.2. *In any game $\mathcal{G}$ with T periods, the threat prices are non-increasing in i: for all $i \leq k$ and all $t < T$, $\tau(i,t) \geq \tau(k,t)$.*

Proof. We proceed by backwards induction on t. For $t = T-1$, $\tau(i,t) = p_i$, the static monoply price for the game $\mathcal{G}_i$, for all i. By Lemma 3.1, $p_i \geq p_k$ whenever $i \leq k$. Now consider any earlier period t. $p(i,t+1) = p(j^*(i,t+1),t+2)$ for all i. By the inductive hypothesis, if $j^*(i,t+1) \leq j^*(k,t+1)$, then we are done. Recall that $j^*(k,t+1)$ is determined by the sales schedule which maximizes reveue earned in $\mathcal{G}_i$ for the remaining periods. In particular, for all $l \geq j^*(k,t+1)$,

$$(j^*(k,t+1) - k + 1) \cdot p(j^*(k,t+1),t+2) + \Pi(j^*(k,t+1)+1,t+2) \\ \geq (l-k+1) \cdot p(l,t+2) + \Pi(l+1,t+2)$$

Again, by the inductive hypothesis, $p(j^*(k,t+1),t+2) \geq p(l,t+2)$. Now multiply this inequality by $k-i$ (which is non-negative) and add it to the above to get

$$(j^*(k,t+1) - i + 1) \cdot p(j^*(k,t+1),t+2) + \Pi(j^*(k,t+1)+1,t+2) \\ \geq (l-i+1) \cdot p(l,t+2) + \Pi(l+1,t+2),$$

for every $l \geq j^*(k,t+1)$. Since $j^*(i,t+1)$ satisfies

$$j^*(i,t+1) = \arg\max_{j \geq i} \left((j-i+1) \cdot p(j,t+2) + \Pi(j+1,t+2)\right)$$

it follows that $j^*(i,t+1) \leq j^*(k,t+1)$, which gives us our result. □

In the next lemma, we show that a no consumer has an advantage in delaying a purchase from the proposed price path.

Lemma 3.3. *Consider two duropoly games, $\mathcal{G}$, with T periods and a set S of consumers, and $\mathcal{G}'$, with T periods and a set S' of consumers such that only the top valued consumer in S and S' differ, and the top valued consumer in S' has the higher value. If we use $p^*_{\mathcal{G}}(1,1)$ and $p^*_{\mathcal{G}'}(1,1)$ to denote the first period prices as calculated by the recursion relationship above, then $p^*_{\mathcal{G}'}(1,1) \geq p^*_{\mathcal{G}}(1,1)$.*

Proof. It is easily seen that in the case $T = 1$, the result holds (the price either remains the same or increases to v_x). Consider the optimal sales schedule for the game $\mathcal{G}$, and assume this schedule sells to more than one person in period 1. But prices for a schedule which sells to more than one person in period 1 do not depend on the value of the top consumer in S (the price in period 1 depends on the threat price of a lower-valued consumer, and later prices depend only on the consumers left). Therefore, we can achieve the same profit from such a schedule in $\mathcal{G}'$ with the same prices. So in $\mathcal{G}'$, the optimal sales and pricing schedule either is the same as the optimal for $\mathcal{G}$, in which case we are done, or involves selling only to x in the first period. If the durpolist sells to to x in the first period, it is at a price $p^*_{\mathcal{G}'}(1,1)$ equal to x's threat price. This is, by definition, the same price as for the game $\mathcal{G}''$ with consumers S' but with $T-1$ periods instead of T periods. By the induction hypothesis, this price is higher than the corresponding optimal first period price p_S for the game with consumers S but with $T-1$ periods. But p_S, by definition, is the threat price for the top consumer in S in $\mathcal{G}$, and therefore at least as high as the period 1 threat price under the optimal sales schedule in $\mathcal{G}$ (if $v_i \geq v_j$, i's threat price is $\geq j$'s threat price: see Lemma 3.2) . But the optimal period 1 threat price is $p^*_{\mathcal{G}}(1,1)$, so $p^*_{\mathcal{G}'}(1,1) \geq p_S \geq p^*_{\mathcal{G}}(1,1)$.

It remains to prove the case where the optimal sales schedule in $\mathcal{G}$ sells to just the top consumer in S. In this case, the price charged in period 1 is p_S. But by the same argument as above, the threat price for x is at least as high as p_S. Therefore if the optimal sales schedule in $\mathcal{G}'$ sells to just x in the first period, the first period price is at least as high as in $\mathcal{G}$. But note that if the duropolist sells to more than one person in period 1 of $\mathcal{G}'$, she achieve the same profit as a sub-optimal sales schedule in $\mathcal{G}$. But she can clearly beat that reveue by selling to x in period 1 and then following the optimal sales schedule in $\mathcal{G}$ from period 2 onwards. Therefore, whatever the optimal sales schedule in $\mathcal{G}'$, it must involve selling exactly one item in period 1. Therefore we have $p^*_{\mathcal{G}'}(1,1) \geq p_S = p^*_{\mathcal{G}}(1,1)$.

We have covered all cases, so the lemma is proved. □

We now show that if a consumer deviates from the proposed price path by buying earlier, her utility does not increase.

Lemma 3.4. *In any game $\mathcal{G}$ with T periods, if the duropolist and consumers follow the strategies described above, then prices are non-increasing in time.*

Proof. If both duropolist and consumer follow the strategies described in Sect.3, the duropolist will select an initial sales path x_t , such that, for the last consumer j_t scheduled to buy in period t ($j_t = \sum_{i \leq t} x_i$), we have the recursive relationship:

$$p(j_t + 1, t + 1) = \tau(j_{t+1}, t + 1) = p(j_{t+1}, t + 2)$$

In other words, in period t the duropolist plans to sell to consumers $j_t + 1, j_t + 2, \ldots, j_{t+1}$ at j_{t+1}'s threat price. By Lemma 3.2, this is less than or equal the threat price of everyone in the set $\{j_t+1, j_t+2, \ldots, j_{t+1}\}$. So under the consumer strategies specified, all x_t consumers in $\{j_t + 1, j_t + 2, \ldots, j_{t+1}\}$ buy in period t, and the duropolist's strategy never deviates from the initial sales path. So the price she charges in each period t is $p(j_{t-1} + 1, t)$ (with $j_0 \equiv 0$).

As part of Lemma 3.2 we showed that $j^*(i, t+1) \leq j^*(k, t+1)$ for all $i \leq k$. Using this and the result of Lemma 2, we have

$$\begin{aligned} p(j_{t-1}+1,t) &= p(j_t,t+1) \\ &= \tau(j^*(j_t,t+1),t+1) \\ &\geq \tau(j^*(j_t+1,t+1),t+1) \\ &= \tau(j_{t+1},t+1) \\ &= p(j_t+1,t+1), \end{aligned}$$

where in the fourth line, we use the fact that $j^*(j_t+1,t+1) = j_{t+1}$ from the definition of the j_t's and the argmax condition of the recursion relation (1). So these prices are non-increasing in time. □

Theorem 3.1. *The strategies defined above constitute a SPNE.*

Proof. Since we can treat any subgame as an instance of a full game with a different set of consumers, there is no loss of generality in assuming that the deviation occurs in the first period of the full game. Let $j^*(1,1)$ be the lowest value consumer sold to in equilibrium and assume that a consumer $x \leq j^*(1,1)$ deviates by not buying in period 1. If $x = j^*(1,1)$, then x is charged her threat price in the next period, which by definition is $p^*(1,1)$, so there is no advantage in a deviation. If $x < j^*(1,1)$, then the remaining consumers for period 2 are $\{x, j^*+1, ..., N\}$. We know that if the set of consumers was $\{j^*, j^*+1, ..., N\}$, then the price would be $p^*(1,1)$. But by Lemma 3.3, the price with consumers $\{x, j^*+1, ..., N\}$ must be at least as high as $p^*(1,1)$. Therefore x cannot gain by delaying her purchase for one period. One may wonder whether consumer x could benefit from delaying the purchase by more than one period. But this is not the case, since by construction, the price at period $t+1$ in this subgame is $p(j^*(1,t),t+1) = p(j^*(j^*(1,t),t+1),t+2)$ if x is the lowest value consumer supposed to buy. This means that the price at $t+2$ will be the same as the price at $t+1$ if consumer x doesn't buy and everyone else follows the equilibrium path. If, on the other hand, consumer x is not the lowest valuation consumer that is supposed to buy, the price at period $t+2$ could only increase or stay equal to $p(j^*(j^*(1,t),t+1),t+2)$ by Lemma 3.3. By repeated use of this argument we conclude that, at equilibrium, no consumer would benefit from delaying its purchase.

Lastly, observe that no consumer can benefit from buying early. If a consumer deviates from the equilibrium path by buying early, she pays a price $p^*(1,t)$ when she could have bought in period $t' > t$ at price $p^*(1,t')$. But since prices are non-increasing as a function of time along the proposed sales path (Lemma 3.4), she cannot do any better.

So we conclude that we have a strategy profile which is an equilibrium in every subgame. □

To compute such an equilibrium, we can compute $j^*(i,t)$ for each (i,t) going backwards from period T, and choose the sales path x_t which maximizes profit. The prices μ_t are then computed by "passing back" the next period's threat price. We may solve the corresponding dynamic program to find the maximum profit Π^D for the duropolist.

3.2 Incomplete information

So far we provided a characterization of subgame perfect equilibria when there is complete information, that is when all participants know which player has each value and the duropolist can observe who buys in each period. We now introduce an incomplete information setting and show that the same SGPNEs characterization applies.

We consider the setting in which the market participants can see who buys in period t, and know the distribution of values, but do not know which consumer has which value. [7] [8] Since we are interested in studying equilibria that satisfy the skimming property, regardless of the values of consumers who bought at period t, the duropolist off-path belief is that the k consumers who bought in period t are those with the k highest valuations (among those remaining). We will show that the same conditions as in Section 3 characterizing the subgame perfect equilibria apply.

We first define the strategy of the duropolist and the consumers in any subgame under this incomplete information setting. Let $\mathcal{G}_S$ denote the subgame at period t where the $|S|$ remaining consumers have valuations $w_1 \geq w_2 \ldots \geq w_{|S|}$. Due to the off-path belief(i.e., the belief that consumers follow the skimming property), the duropolist would behave as if it were in the subgame $\mathcal{G}'_1$ with $T - t + 1$ periods in which the consumers are $v'_1 \geq v'_2 \geq \ldots, v'_{|S|}$ where $v'_i = v_{i+N-|S|}$. Observe that $v'_i \leq w_i$ for all $i \in [|S|]$. The monopolist strategy is to then announce the price $p^*_{\mathcal{G}'}(1,1)$ which is obtained by solving the recursion relationship (1). The consumers strategy remains the same as in the complete information setting, i.e. each of them would buy if and only if the price is less than or equal to their threat price as calculated for $\mathcal{G}'$. We now prove the following result.

Theorem 3.2. *The strategies defined above constitute a SPNE in the incomplete information setting.*

Proof. We consider the subgame $\mathcal{G}' = \mathcal{G}_S$ (of the original game $\mathcal{G}$) that begins at period t in which the remaining consumers consists of the set S. These consumers have valuations $w_1 \geq w_2 \ldots \geq w_{|S|}$. Due to the off-path belief(i.e., the belief that consumers follow the skimming property), the duropolist would behave as if it were in the subgame $\mathcal{G}'_1$ with $T - t + 1$ periods in which the consumers are $v'_1 \geq v'_2 \geq \ldots, v'_{|S|}$ where $v'_i = v_{i+N-|S|}$. Observe that in this subgame $\mathcal{G}'_1$, consumer i's real value is actually $w_i \geq v'_i$.

The announced price in $\mathcal{G}'_1$ would then be $p(1,t) = p(j^*(1,t), t+1)$. Suppose now that some consumer x that was supposed to buy under the proposed equilibrium, i.e. $i \leq x \leq j^*(1,t)$ deviates and chooses not to buy at time t. The number of sales at period t would then be $j^*(1,t) - 1$, i.e., one less than the expected. The duropolist then, who observes the total number of sales, and

[7] Each player is still aware of their own value.

[8] Note that this is equivalent to the participants knowing which consumer has which value, but not seeing who buys, only the total number of sales in each period.

assumes consumers follow the skimming property, would behave as if the remaining subgame starting at $t+1$ is $\mathcal{G}'_{j^*(1,t)}$. Again, this means that the announced price would then be $p(j^*(1,t), t+1)$ and consumer x would not have benefited from delaying the purchase by one period. One may, again, wonder whether consumer x could benefit from delaying the purchase by more than one period. But this is not possible since the price at period $t+1$ in this subgame is $p(j^*(1,t), t+1) = p(j^*(j^*(1,t), t+1), t+2)$, which means the price will remain constant over time, as long as the number of transactions is one less than the expected. By repeated use of this argument we conclude that, at equilibrium, no consumer would benefit from delaying its purchase.

Lastly, observe that no consumer can benefit from buying earlier. If a consumer deviates from the equilibrium path by buying earlier, she pays a price $p^*(1,t)$ when she could have bought in period $t' > t$ at price $p^*(k,t')$ for some $k \geq 1$. But since prices are non-increasing as a function of time along the proposed sales path (Lemma 3.4), she cannot do any better.

So we conclude that we have a strategy profile which is an equilibrium in every subgame. □

Note that since the equilibrium path is the same in both our complete and incomplete information setting, all our results also apply to the incomplete information setting.

4 A Relationship Between Duropoly Profits and Static Monopoly Profits

In this section, we will prove our main result: the profits of the duropolist in a skimming property-satisfying SPNE of the duropoly game are at least the profits of the corresponding static monopolist, but at most double.

Theorem 4.1. *$\Pi^M \leq \Pi^D$ for any game $\mathcal{G}$.*

Proof. The proof is by induction on the number of periods. The base case is trivial. Consider the (possibly sub-optimal) sales schedule where we sell at p_1 in period 1, and then follow the duropolists' equilibrium strategy for the remaining subgame $\mathcal{G}'$. Let k_1 be the number of consumers sold to in period 1 under this schedule. Let $\Pi^M_{\mathcal{G}} = j \cdot v_j \equiv j \cdot p_1$ and $\Pi^M_{\mathcal{G}'} = (k-k_1)v_k$. Since $j = |\{i | v_i \geq v_j\}|$, $j \geq k_1$. But $k = \arg\max_{i \geq k_1} (i-k_1)v_i$, therefore $(k-k_1)v_k \geq (j-k_1)v_j$. So

$$\Pi^D \geq k_1 p_1 + \Pi^D_{\mathcal{G}'} \geq k_1 p_1 + \Pi^M_{\mathcal{G}'} \geq k_1 p_1 + (j-k_1)v_j = k_1 p_1 + (j-k_1)p_1 = \Pi^M,$$

where, in the second inequality, we used the induction hypothesis. □

We proceed now to prove the following upper bound.

Theorem 4.2. *$\Pi^D \leq (\Pi^M + v_1) \leq 2 \cdot \Pi^M$*

We prove Theorem 4.2 in two steps. Recall that we have N consumers with valuations $v_1 \geq v_2 \geq \cdots \geq v_N$ and that $\mathcal{G}_i$ is the subgame consisting of customers $\{i, i+1, \ldots, N\}$ with p_i is the static monopoly price for the subgame $\mathcal{G}_i$. Each p_i is equal to some consumer y_i's value so we will define $p_i := v(y_i)$. First we show that $\Pi^D \leq \sum_{i=1}^N p_i$, and second we show that $\Pi^M + p_1 \geq \sum_{i=1}^N p_i$. The result follows since $v_1 \geq p_1$.

Theorem 4.3. *The maximum profit of the duropolist satisfies* $\Pi^D \leq \sum_{i=1}^N p_i$.

To prove this we require the following three lemmas.

Lemma 4.1. *In equilibrium, consumer i never pays more than p_i whenever she buys before the last period.*

Proof. We proceed by induction in the number of time periods. For $T = 2$, let A be the set of consumers that buy at $t = 1$. Suppose for the purpose of contradiction that some consumer $i \in A$ pays a price higher than p_i. Because of the skimming property, we have that $A = \{1, \ldots, k\}$ for some k. By Lemma 3.1, this price is also more than p_k, so consumer k pays more than p_k. But if consumer k refuses to buy at $t = 1$, then at $t = 2$ the duropolist would charge p_k which is a contradiction since consumer k would have obtained a higher profit by waiting.

Now suppose that the lemma is true for all games of 1 to T periods and consider a game $\mathcal{G}$ of $T+1 > 2$ periods. Let E denote a SPNE with the skimming property in $\mathcal{G}$ and let i be smallest value consumer that pays more than p_i, say at some time period t ($t < T+1$). Again, by Lemma 3.1 consumer i is the lowest valuation consumer that buys at period t. Thus, if consumer i refuses to buy at period t we end in the subgame $\mathcal{G}_i$ with $T+1-t$ periods. If $T+1-t = 1$ (i.e. t was the second to last period), the duropolist will charge the price p_i at the last period. If $T + 1 - t > 1$, it holds by the induction hypothesis that consumer i would never pay more than p_i. Thus, we can conclude that such equilibrium E cannot exist as consumer i would have obtained a higher profit by waiting. □

Lemma 4.2. *The maximum profit of the duropolist satisfies*

$$\Pi^D \leq \max_{m \leq N} \left((y_m - m + 1) \cdot v(y_m) + \sum_{i=1}^{m-1} p_i \right)$$

Proof. In the final time period T, customer i is willing to pay up to v_i. In all other periods, by Lemma 4.1, customer i is willing to pay up p_i, the static monopoly price for the subgame $\mathcal{G}_i$.

Suppose that in the optimal solution, the duropolist sells to customers $\{m, m+1, \ldots, y_m\}$ where $1 \leq m \leq y_m \leq N$ in the final period T. The final period profit then is exactly $(y_m - m+1) \cdot v(y_m)$. By Lemma 4.1, customers who buy in earlier periods, that is customers $\{1, 2, \ldots m-1\}$, pay at most their static monopoly prices. Therefore, the maximum profit is upper bounded by

$$(y_m - m + 1) \cdot v(y_m) + \sum_{i=1}^{m-1} p_i.$$

The result follows by taking the maximum over all customers m. □

Lemma 4.3. *The static monopoly profit for the subgame $\mathcal{G}_m$ is at most*

$$\sum_{j=m}^{N} p_j.$$

Proof. Without loss of generality, by re-indexing so that $m = 1$, it suffices to show that

$$\Pi^M = y_1 \cdot v(y_1) \leq \sum_{j=1}^{N} p_j \tag{2}$$

Let $C = \{p_j : j = 1, \ldots, N\}$. We proceed by induction on $|C|$, that is, the number of distinct static monopoly prices over all the subgames $\mathcal{G}_j$. For the base case, $|C| = 1$, we have $p_1 = p_j$ for all j. Thus $p_1 = p_N = v_N$ and $y_1 = N$. Every customer then pays v_N and the total profit is

$$y_1 \cdot v(y_1) = N \cdot v_N = \sum_{j=1}^{N} p_j$$

Assume the proposition holds for $|C| = k-1 \geq 1$. Now take the case $|C| = k$. Let customer l be the highest index customer in the original game with $p_l = p_1$. Thus $p_{l+1} < p_1 = v(y_1)$. By the induction hypothesis, applied to the subgame $\mathcal{G}_{l+1}$ on customers $\{l+1, l+2, \ldots, N\}$, we have

$$\sum_{i=l+1}^{N} p_i \geq (y_{l+1} - l) \cdot v(y_{l+1}) \tag{3}$$

Consequently,

$$\begin{aligned}
\sum_{i=1}^{N} p_i &= l \cdot p_1 + \sum_{i=l+1}^{N} p_i \\
&\geq l \cdot p_1 + (y_{l+1} - l) \cdot v(y_{l+1}) \\
&\geq l \cdot p_1 + (y_l - l) \cdot v(y_l) \\
&= l \cdot v(y_l) + (y_l - l) \cdot v(y_l) \\
&= y_l \cdot v(y_l) \\
&= y_1 \cdot v(y_1)
\end{aligned}$$

Here the first equality follows by definition of l. The first inequality follows by (3). The second inequality holds as $v(y_{l+1})$ is the static monopoly price for the subgame $\mathcal{G}_{l+1}$. The final three equalities follow by definition of l. That is $p_1 = p_l$ and so $y_l = y_1$.

This shows that (2) holds as desired. □

Proof of Theorem 4.3. Combine Lemma 4.2 and Lemma 4.3. □

Theorem 4.4. $\sum_{i=1}^{N} p_i \leq \Pi^M + p_1$

Proof. We proceed by induction in N. For games with a single player, the statement is trivially true. Recall that $p_i = v(y_i)$ is the static monopoly price for the subgame $\mathcal{G}_i$ on customers $\{i, i+1, \ldots, N\}$ and y_i is the index of the lowest value consumer whose value is not less than p_i. Consider now a game $\mathcal{G}$ with $N+1$ consumers. It remains to show that

$$\sum_{i=1}^{N+1} p_i \leq v(y_1) + y_1 \cdot v(y_1).$$

We proceed as follows:

$$\begin{aligned} \sum_{i=1}^{N+1} p_i = & \\ &= v(y_1) + \sum_{i=2}^{N+1} p_i \\ &= v(y_1) + v(y_2) + (y_2 - 1) \cdot v(y_2) \qquad (4) \\ &= v(y_1) + y_2 \cdot v(y_2) \\ &\leq v(y_1) + y_1 \cdot v(y_1) \qquad (5) \end{aligned}$$

where the equation (4) follows by the induction hypothesis and the inequality (5) comes from the fact that $v(y_1)$ is the static monopoly price of $\mathcal{G}_1$. □

4.1 A Tight Example

The factor 2 bound is tight. To see this, assume that $T = 2$ and set $v_j = v_H$ for $1 \leq j \leq k$ and $v_j = v_L = \frac{1}{n-k+1} \cdot v_H$ for all $k+1 \leq j \leq n$. The optimal solution is to charge v_H in the first period and v_L in the last period. The high value customers will buy in the first period and the low value customers in the last period. It can be verified that this solution satisfies the equilibria conditions. The total profit is

$$k \cdot v_H + (n-k) \cdot v_L \;=\; k \cdot v_H + \frac{n-k}{n-k+1} \cdot v_H \;=\; |A_1| + \left(1 - \frac{1}{n-k+1}\right) \cdot v_H.$$

Thus in the limit $n \gg k$ we obtain $v_{\max} + |A_1|$. In the case $k = 1$ we have that $|A_1| = v_{\max}$ and, thus, duropoly profits are twice static monopoly profits.

5 Consumer Surplus

As discussed, Bagnoli et al. (1989) proved that a duropolist who faces atomic consumers with an infinite time horizon can always extract all consumer surplus. They left open the case of finite time horizons. It turns out that although such equilibria may still exist under a finite horizon, the conditions required for their existence are very restrictive. Indeed, applying the techniques we have developed, we characterize in this section necessary and sufficient conditions for this phenomenon to happen.

Theorem 5.1. *Consider a duropoly game $\mathcal{G}$ with $M \leq N$ distinct valuations. There exists an equilibrium at which the duropolist extracts all consumer surplus if and only if $M \leq T$ and $v_i = p_i$ for all $i \in [N]$.*

To prove Theorem 5.1, we require some definitions and three lemmas.

Let $\mathcal{G}$ be a game with N consumers with valuations $v_1 \geq v_2, \ldots, v_N$ and let p_i be the static monopoly price of the subgame consisting of customers $\{i, i+1, \ldots, N\}$. More generally, given a subset $S \subseteq [N]$ we define $p(S)$ to be the static monopoly price of the subgame consisting of consumers $S \subseteq [N]$.

Lemma 5.1. *$\mathcal{G}$ satisfies that $p_i = v_i$ for all $i \in [N]$ if and only if $p(S) = \max\{v_x : x \in S\}$ for all $S \subseteq [N]$.*

Proof. The "if" direction of the proof is trivially true. For the other direction, suppose there exists a subset $S \subseteq [N]$ such that $p(S) < v_i$ where $i = \arg\max\{v_x : x \in S\}$. Let the valuations of the consumers that are in S to be $v_i \geq w_2 \geq w_2 \geq \ldots \geq w_{|S|}$. Then, we have that $j \cdot w_j > v_i$ for some $j > 1$. But then $p_i < v_i$ since by setting a price of w_j in the subgame with consumers $\{i, i+1, \ldots, N\}$ yields a profit of at least $j \cdot w_j > v_i$. □

Let $w_1 > w_2 > \ldots > w_M$ denote the M *distinct* consumer valuations sorted in decreasing order. Let n_i denote the number of consumers with value w_i. The following lemma is required to prove the main result of this section. We set $w_i = n_i = 0$ for all $i > M$.

Lemma 5.2. *If $p_i = v_i$ for all $i \in [N]$ and $M \geq T$, the following inequality holds for all $k = 2, \ldots, T$ and all $i = 1, \ldots, k-1$,*

$$n_i \cdot w_i - n_i \cdot w_k - n_{T+i} w_{T+i} \geq 0.$$

Proof.

$$\begin{aligned}(1 + n_{i+1} + \ldots + n_k) \cdot w_k &\leq w_i \\ w_k &\leq \frac{w_i}{1 + n_{i+1} + \ldots + n_k} \\ w_k &\leq \frac{w_i}{k - i + 1} \leq \frac{w_i}{2} \qquad (6)\end{aligned}$$

$$
\begin{aligned}
(1 + n_{k+1} + +n_{k+2} + \ldots + n_{T+i}) \cdot w_{T+i} &\leq w_k \\
w_{T+i} &\leq \frac{w_k}{1 + n_{k+1} + +n_{k+2} + \ldots + n_{T+i}} \\
n_{T+i} \cdot w_{T+i} &\leq \frac{n_{T+i} \cdot w_k}{1 + n_{k+1} + +n_{k+2} + \ldots + n_{T+i}} \\
n_{T+i} \cdot w_{T+i} &\leq \frac{n_{T+i} \cdot w_k}{n_{T+i} + T + i - k} \\
n_{T+i} \cdot w_{T+i} &\leq \frac{n_{T+i} \cdot w_k}{n_{T+i} + 1} \leq w_k \\
n_{T+i} \cdot w_{T+i} &< \frac{w_i}{k - i + 1} \leq \frac{w_i}{2} \qquad (7)
\end{aligned}
$$

Combining (6) and (7) we have

$$
\begin{aligned}
n_i \cdot w_i - n_i \cdot w_k - n_{T+i} w_{T+i} &\geq n_i \cdot w_i - n_i \frac{w_i}{2} - \frac{w_i}{2} \\
&\geq 0 \qquad (8)
\end{aligned}
$$

as desired.

We recall that the duropolist strategy named *Pacman* is to announce at each time period, a price equal to the valuation of the consumer with the highest value who has yet to buy, and that the consumer strategy known as *get-it-while-you-can* is to buy the first time it induces a non-negative utility.The following lemma gives sufficient conditions for such strategies to be under equilibrium.

Lemma 5.3. *If $p_i = v_i$ for all $i \in [N]$, there exists an equilibrium in which the duropolist uses the* pacman *strategy and consumers follow the* get-it-while-you-can *strategy.*

Proof. We proceed by induction in the number of time periods. For games with a single period (i.e., $T = 1$) the Lemma holds because the duropolist announces the optimal static monopoly price, which is p_1. Suppose now that the lemma holds for all games with at most $T-1$ periods and consider a game with T periods. Let A be the set of consumers that buy at period 1 under an equilibrium $\mathcal{E}$. Observe that by Lemma 5.1, the subgame that begins at period 2, with consumers $[N]-A$ satisfies the that $p_i = v_i$ and therefore there exist an equilibrium where the duropolist uses the Pacman strategy thereon. This means that consumers expect zero profits whenever they don't buy in the first time period, and therefore they would buy in the first time period at any price that is not above their valuation. If $M \leq T$ the duropolist may announce at $t = 1$ the price $\mu_1 = v_1$, all consumers with a valuation of v_1 would buy and by the inductive hypothesis the duropolist would be able to extract all consumer surplus. Thus, pacman is an optimal strategy. We now analyze the case where $M > T$. Observe that because consumers will buy in the first period if and only if the price is not above their value, the duropolist space strategy can be restricted without loss of generality to announcing a first price equal to the valuation of some consumer. Let $\Pi(k)$ denote the duropolist profit along all the game if the first price is w_k. Since the Pacman strategy is an equilibrium after the first period, we have that

$$\Pi(k) = \sum_{i=1}^{k} n_i \cdot w_k + \sum_{j=k+1}^{T+k-1} n_j \cdot w_j .$$

We want to show that

$$\Pi(1) \geq \Pi(k)$$

for all $k = 1, \dots, M$.

By Lemma 5.2 we have

$$\begin{aligned} n_i \cdot w_i - n_i \cdot w_k - n_{T+i} w_{T+i} &\geq 0 \qquad (9) \\ \sum_{i=1}^{k-1} n_i \cdot w_i - n_i \cdot w_k - n_{T+i} w_{T+i} &> 0 \\ \sum_{i=1}^{T} n_i \cdot w_i - \sum_{i=1}^{k} n_i \cdot w_k - \sum_{j=k+1}^{T+k-1} n_j w_j &\geq 0 \\ \Pi(1) - \Pi(k) &\geq 0 \\ \Pi(1) &\geq \Pi(k) \end{aligned}$$

We thus conclude that independently on whether $M \geq T$ or $M \leq T$, there exists an equilibrium where the duropolist uses the pacman strategy.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Suppose first that there is an equilibrium in $\mathcal{G}$ where each consumer i pays v_i. By Lemma 4.1 consumer i never pays more than p_i, therefore $v_i = p_i$ for $i \in [N]$. In addition, it must be the case that the number of time periods is at least equal to the number of different valuations as otherwise the duropolist will not be able to extract the value of every consumer before the end of the game, thus we have $M \leq T$ as desired.

Now suppose that in game $\mathcal{G}$, $M \leq T$ and $v_i = p_i$ for all $i \in [N]$. By Lemma 5.3 there exists an equilibrium in which the duropolist uses the pacman strategy. Thus, whenever the number of time periods is greater or equal to the number of different valuations, the duropolist would obtain all the consumer surplus. □

6 Non-Skimming Equilibria

In games with non-atomic consumers, all equilibria satisfy the skimming property (see, e.g. Fudenberg and Tirole (1991)). Surprisingly, we found an example of an equilibrium in a game with atomic consumers that does not satisfy the skimming property. Consider the following three-customer, two period problem:

Table 2. An Example with a Non-Skimming Equilibrium

Customer valuation	Period 1 Strategy	Period 2 Strategy
80	Buy if $\mu_1 \leq 45$	Buy if $\mu_2 \leq 80$
70	Buy if $\mu_1 \leq 70$	Buy if $\mu_2 \leq 70$
45	Buy if $\mu_1 \leq 45$	Buy if $\mu_2 \leq 45$
Duropolist	$\mu_1 = 70$	Charge Static Monopoly Price (for remaining customers)

Because period 2 is the last period of the game, all the customers who did not buy in period 1 have a dominant strategy which is "buy if and only if $\mu_2 \leq v$". The duropolist's best response to this is to charge a value μ_2 equal to the static monopoly price for the subgame with all customers who did not buy in period 1. It is simple to check that all of the customers' strategies are mutual best responses. The period 1 price is also the price which maximizes profit for the duropolist, given the customer strategies. So this is a SPNE. However, in this equilibrium, the customer with value 70 buys in period 1, while the customer with value 80 buys in period 2 (where $p_2 = 45$). Therefore the skimming property is not satisfied. Also note that the consumer with the second highest value pays her full value, which is significantly more than p_2, the static monopoly price in the subgame $\mathcal{G}_2$ ($p_2 = 45$).

Note, however, that if we swapped the period 1 strategies of the top two customers, we would still have an equilibrium, and one which satisfies the skimming property. This new equilibrium would produce the same profit for the duropolist.

A natural question is: how does the duropolist profit in non-skimming equilibria compare with those equilibria that satisfy it? We can prove the following result.

Theorem 6.1. *In a 2 period game, for every equilibrium which does not satisfy the skimming property, there is a corresponding equilibrium at the same prices which satisfies the skimming property with the same profit for the duropolist.*

To prove Theorem 6.1, we need some results about static monopoly prices.

Claim. Let $V = \{v_1, v_2, \ldots, v_N\}$ be a set of valuations $v_1 \geq v_2 \ldots \geq v_N$. Let $k = \arg\max_i iv_i$ and $p_V^{sm} = v_k$, the static monopoly price calculated for the set V. Now let V' be the set V with some v_j replaced by a value $v_j > v' \geq v_k$, and re-index the set so $v'_1 \geq v'_2 \ldots \geq v'_n$. Then $p_{V'}^{sm} = v_k$. If, instead, $v' \geq v_j > v_k$ and we re-index the set so $v'_1 \geq v'_2 \ldots \geq v'_N$, then we still have $p_{V'}^{sm} \geq v_k$.

Proof. **Case $\mathbf{v' < v_j}$**: Note that $v'_i \leq v_i$ for every i. Therefore, by definition of k, $kv_k \geq iv_i \geq iv'_i$ for all i. For all i such that $v_i \leq v'$, we still have $v'_i = v_i$, so therefore and $v_k = v'_k$ and $kv'_k \geq iv'_i$ for all i. Therefore $k = \arg\max_i iv'_i$ and

$p^{sm}_{V'} = v_k$.
Case $\mathbf{v'} \geq \mathbf{v_j}$: Note that, because $v_j > v_k$, we still have $v'_i = v_i$ for every $i \geq k$, and as $kv_k \geq iv_i$ for all $i \geq k$, it follows that the index which maximizes lv_l can be no larger than k. Therefore $p^{sm}_{V'} \geq v'_k = v_k$

Claim. Let $V = \{v_1, v_2, \ldots, v_N\}$ be a set of valuations $v_1 \geq v_2 \ldots \geq v_N$. Let $k = \arg\max_i iv_i$ and $p^{sm}_V = v_k$. Now add a value $v' \geq v_k$ to the set of valuations, call the new set V' and re-index the set so $v'_1 \geq v'_2 \ldots \geq v'_{N+1}$. Then $p^{sm}_V \leq p^{sm}_{V'} \leq v'$.

Proof. Let $v' = v_l$. We have that $v'_i = v_{i-1}$ for $i > l$. The result follows immediately as for all $j < l$, $jv_j \leq kv_k < (k+1)v_k = (k+1)v'_{k+1}$, so regardless of what the maximizing index actually is, it cannot be less than l, and therefore $p^{sm}_{V'} \leq v_l = v'$. But we also have that $(k+1)v'_{k+1} = (k+1)v_k \geq (j+1)v_j = (j+1)v'_{j+1}$ for all $j \geq k$. Therefore the maximizing index cannot be more that $k+1$, and thus $v'_{k+1} = v_k = p^{sm}_V \leq p^{sm}_{V'}$.

Consider an equilibrium $\mathcal{E}$ which violates the skimming property. The violation must occur in period 1, as in the second period the dominant strategy followed by the customers ensures that there is a cut-off value above which all remaining customers buy. We must then have at least one pair of customers which we denote by their values v, w with $w > v$, where v buys in period 1 and w does not buy in period 1. Without loss of generality, let w be the highest valued customer who doesn't buy in period 1, and v be the lowest valued customer who buys with value below w. We denote by E the set of customers under the equilibrium $\mathcal{E}$ who did not buy in period 1. Then $w \in E$ and $v \notin E$.

The period 2 price, $\mu_2(E)$, is fixed by the fact that it is a dominated strategy to charge the static monopoly price of the remaining customers. For a customer $y \notin E$ who buys in period 1, we will also need to compare $\mu_1(E)$ against its threat price, $\mu_2(E^y)$. Here $E^y = E \cup \{y\}$, and so $\mu_2(E^y)$ is the period 2 price y would face if she did not buy.

Lemma 6.1. *At the equilibrium $\mathcal{E}$, we have*

$$w > v \geq \mu_2(E^v) \geq \mu_1(E) \geq \mu_2(E) \tag{10}$$

Proof. By assumption $w > v$. Consumer v wants to buy in the first period. The equilibrium conditions then imply that (i) $\mu_1(E) \leq v$ and (ii) $\mu_1(E) \leq \mu_2(E^v)$. Consumer w does not want to buy in the first period. But since the first period price is less than her value, she would be willling to buy in the first period if $\mu_2(E) > \mu_1(E)$. The equilibrium conditions then imply that $\mu_2(E) \leq \mu_1(E) < w$ (and w does buy in the second period). It only remains to prove that $v \geq \mu_2(E^v)$. We have seen that $v \geq \mu_2(E)$. But then, by Claim 6, we obtain $v \geq \mu_2(E^v)$.

To prove Theorem 6.1, we will show that with the same period 1 price, the strategy profile $\mathcal{S}$ which consists of v and w swapping actions from $\mathcal{E}$ (and all other customer strategies remaining the same) is also an equilibrium. We will also show that the period 2 price stays the same after swapping. By repeated swapping, we will remove all pairs where the lower valued customer buys before the higher valued customer, until we reach an equilibrium where the skimming property is satisfied, while the prices are unchanged. So let $S = \{E - w\} \cup \{v\}$ be the set of customers who did not buy in period 1 under the swapped strategy $\mathcal{S}$.

Lemma 6.2. *For the strategy profiles $\mathcal{E}$ and $\mathcal{S}$, we have*

$$w > v \geq \mu_2(E^v) = \mu_2(S^w) \geq \mu_1(E) = \mu_1(S) \geq \mu_2(E) = \mu_2(S) \qquad (11)$$

Proof. The inequalities follow from Lemma 6.1. Thus it remains to prove the three equalities. Since S and E differ in exactly the pair $\{w, v\}$ we have that $S^w = E^v$. The equilibrium constraints imply the duropolist applies static monopolist pricing in the final period; thus $\mu_2(S^w) = \mu_2(E^v)$. Next, in the proposed solution corresponding to S, the duropolist is, by choice, setting the same price in period 1 for S as for E; thus $\mu_1(S) = \mu_1(E)$. Finally, we know that $v \geq \mu_2(E)$. But, then, if we reduce the value of customer w to v, the static monopoly price will not change by the first case of Claim 6; equivalently $\mu_2(S) = \mu_2(E)$.

Lemma 6.3. *The solution corresponding to S is an equilibrium.*

Proof. We prove that the strategy profile consisting of v and w swapping actions and all other customers acting the same way is still an equilibrium by considering the best responses of every customer separately.
• **Customer v**: Customer v buys in the second period for S. If this is a best response strategy then we need $\mu_2(S) \leq \min[v, \mu_1(S)]$. This is true, by Lemma 6.2.
• **Customer w**: Customer w buys in the first period for S. If this is a best response strategy then we need $\mu_1(S) \leq \min[w, \mu_2(S^w)]$. This is true, by Lemma 6.2.
• **A customer $\mathbf{u} \neq \mathbf{w}$ who buys in period 1**: We wish to show that u still wishes to buy in the first period after we swap: that is, $\mu_1(S) \leq \min[u, \mu_2(S^u)]$.

Since v is the lowest valued customer who buys in period 1, we must have $u \geq v$. We know, by Lemma 6.2 that $\mu_1(S) \leq v \leq u$. Thus it suffices to prove that $\mu_1(S) \leq \mu_2(S^u)$. We now have two possibilities:

(a) $\mathbf{u} < \mathbf{w}$: By Lemma 6.1, we have $p_2(E^v) \leq v$. Now set $V = E^v = S^w$ and apply the first case of Claim 6 with $v' = u$ and $v_j = w$. Thus, $V' = S^u$ and $\mu_2(E^v) = \mu_2(S^u)$. By the equilibrium conditions, we know $\mu_1(E) \leq \mu_2(E^v)$. Therefore $\mu_1(S) = \mu_1(E) \leq \mu_2(E^v) = \mu_2(S^u)$.

(b) $\mathbf{u} \geq \mathbf{w}$: By Lemma 6.2, we have $\mu_2(E) = \mu_2(S) < w$. Now set $V = S$ and apply Claim 6 by adding a new consumer with value $v' = w$; thus $V' = S^w$ and $\mu_2(S^w) \leq w$. Applying the second case of Claim 6 with $V = S^u$ and $V' = S^w$ and $v' = u$, then gives $\mu_2(S^u) \geq \mu_2(S^w)$. Furthermore, by Lemma 6.2, we know $\mu_1(S) \leq \mu_2(S^w)$; thus $\mu_1(S) \leq \mu_2(S^u)$, as desired.

• **Any other customer $\mathbf{u} \neq \{\mathbf{w}, \mathbf{v}\}$**: Such a customer u either buys in period 2 or does not buy at all. In the former case, $\mu_2(E) \leq \min[u, \mu_1(E)]$. In the latter case, $u \leq \mu_2(E) \leq \mu_1(E)$. But, by Lemma 6.2, we have $\mu_1(S) = \mu_1(E)$ and $\mu_2(S) = \mu_2(E)$. Therefore, in either case, the same best response conditions hold for S as they did for E.

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1.By Lemma 6.3, we know that $\mathcal{S}$ gives an equilibrium. From Lemma 6.2, we have that $\mu_1(S) = \mu_1(E)$ and $\mu_2(S) = \mu_2(E)$. Since the same number of customers buy in each period under $\mathcal{S}$ and $\mathcal{E}$, the duropolist earns the same profit in $\mathcal{S}$ as in $\mathcal{E}$. By repeatedly swapping pairs which are out of order, we obtain an equilibrium with the same profit which satisfies the skimming property. $\square$

We conjecture that Theorem 6.1 applies for any number of periods. If so, no SPNE (with or without the skimming property) extracts more than twice static monopoly profits.

Conjecture 6.1. The profits of the duropolist in any SPNE are at least the profits of the corresponding static monopolist, but at most double.

Bibliography

Lawrence M Ausubel and Raymond J Deneckere. Reputation in bargaining and durable goods monopoly. *Econometrica*, 57:511–531, 1989.

Mark Bagnoli, Stephen W Salant, and Joseph E Swierzbinski. Durable-goods monopoly with discrete demand. *The Journal of Political Economy*, 97:1459–1478, 1989.

Timothy N Cason and Tridib Sharma. Durable goods, Coasian dynamics, and uncertainty: Theory and experiments. *Journal of Political Economy*, 109: 1311–1354, 2001.

Ronald H Coase. Durability and monopoly. *Journal of Law and Economics*, 15: 143–149, 1972.

Peter Cramton and Joseph S Tracy. Strikes and holdouts in wage bargaining. *The American Economic Review*, 82:100–121, 1992.

William Fuchs and Andrzej Skrzypacz. Bargaining with deadlines and private information. Technical report, Mimeo, University of California Berkley, 2011.

Drew Fudenberg and Jean Tirole. *Game Theory*. MIT Press, 1991.

Drew Fudenberg, David Levine, and Jean Tirole. Infinite horizon models of bargaining with one-sided incomplete information. In A. Roth, editor, *Game Theoretic Models of Bargaining*. Cambridge University Press, 1985.

Faruk Gul, Hugo Sonnenschein, and Robert Wilson. Foundations of dynamic monopoly and the coase conjecture. *Journal of Economic Theory*, 39:155–190, 1986.

Werner Guth and Klaus Ritzberger. On durable goods monopolies and the Coase-conjecture. *Review of Economic Design*, 3:215–236, 1998.

João Montez. Inefficient sales delays by a durable-good monopoly facing a finite number of buyers. *The RAND Journal of Economics*, 44:425–437, 2013.

Barak Orbach. The duropolist puzzle: Monopoly power in durable-goods market. *Yale Journal on Regulation*, 21:67–118, 2004.

Nancy L Stokey. Intertemporal price discrimination. *The Quarterly Journal of Economics*, 93:355–371, 1979.

Nils-Henrik Morch von der Fehr and Kai-Uwe Kuhn. Coase versus pacman: Who eats whom in the durable-goods monopoly? *Journal of Political Economy*, 103: 785–812, 1995.

Gerald R Williams. *Legal negotiation and settlement*. West Publishing Company, 1983.

Acknowledgements

This work was partially supported by an NSERC posdoctoral fellowship award. This support is gratefully acknowledged. We are very grateful to Juan Ortner and Sven Feldmann for their helpful comments and suggestions. The authors are also very grateful to an anonymous conference referee for suggestions that greatly simplified the proof of Theorem 4.4.