

Polylogarithmic Supports are required for Approximate Well-Supported Nash Equilibria below $2/3$

Yogesh Anbalagan¹, Sergey Norin², Rahul Savani³, and Adrian Vetta⁴

¹ School of Computer Science, McGill University.

`yogesh.anbalagan@mail.mcgill.ca`

² Department of Mathematics and Statistics, McGill University.

`snorin@math.mcgill.ca`

³ Department of Computer Science, University of Liverpool.

`rahul.savani@liverpool.ac.uk`

⁴ Department of Mathematics and Statistics, and School of Computer Science,
McGill University.

`vetta@math.mcgill.ca`

Abstract. In an ϵ -approximate Nash equilibrium, a player can gain at most ϵ *in expectation* by unilateral deviation. An ϵ -*well-supported* approximate Nash equilibrium has the stronger requirement that every pure strategy used with positive probability must have payoff within ϵ of the best response payoff. Daskalakis, Mehta and Papadimitriou [8] conjectured that every win-lose bimatrix game has a $\frac{2}{3}$ -well-supported Nash equilibrium that uses supports of cardinality at most three. Indeed, they showed that such an equilibrium will exist *subject to* the correctness of a graph-theoretic conjecture. Regardless of the correctness of this conjecture, we show that the barrier of a $\frac{2}{3}$ payoff guarantee cannot be broken with constant size supports; we construct win-lose games that require supports of cardinality at least $\Omega(\sqrt[3]{\log n})$ in any ϵ -well supported equilibrium with $\epsilon < \frac{2}{3}$. The key tool in showing the validity of the construction is a proof of a bipartite digraph variant of the well-known Caccetta-Haggkvist conjecture [4]. A probabilistic argument [13] shows that there exist ϵ -well-supported equilibria with supports of cardinality $O(\frac{1}{\epsilon^2} \cdot \log n)$, for any $\epsilon > 0$; thus, the polylogarithmic cardinality bound presented cannot be greatly improved. We also show that for any $\delta > 0$, there exist win-lose games for which no pair of strategies with support sizes at most two is a $(1 - \delta)$ -well-supported Nash equilibrium. In contrast, every bimatrix game with payoffs in $[0, 1]$ has a $\frac{1}{2}$ -approximate Nash equilibrium where the supports of the players have cardinality at most two [8].

1 Introduction

A Nash equilibrium of a bimatrix game is a pair of strategies in which the supports of both players consist only of best responses. The apparent hardness of computing an exact Nash equilibrium [6,5] even in a bimatrix game has led to work on computing approximate Nash equilibria, and two notions of approximate Nash equilibria have been developed. The first and more widely studied notion is of an ϵ -approximate Nash equilibrium (ϵ -Nash). Here, no restriction is placed upon the supports; any strategy can be in the supports provided each player achieves an expected payoff that is within ϵ of a best response. Therefore, ϵ -Nash equilibria have a practical drawback: a player might place probability on a strategy that is arbitrarily far from being a best response. The second notion, defined to rectify this problem, is called an ϵ -well supported approximate Nash equilibrium (ϵ -WSNE). Here, the content of the supports are restricted, but less stringently than in an exact Nash equilibrium. Specifically, both players can only place positive probability on strategies that have payoff within ϵ of a pure best response. Observe that the latter notion is a stronger equilibrium concept: every ϵ -WSNE is an ϵ -Nash, but the converse is not true.

Approximate well-supported equilibria recently played an important role in understanding the hardness of computing Nash equilibria. They are more useful in these contexts than ϵ -Nash equilibria because their definition is more combinatorial and more closely resembles the *best response condition* that characterizes exact Nash equilibria. Indeed, approximate well-supported equilibria were introduced in [12,6] in the context of PPA reductions that show the hardness of computing (approximate) Nash equilibria. They were subsequently used as the notion of approximate equilibrium by Chen et al. [5] that showed the PPA-hardness of computing an exact Nash equilibrium even for bimatrix games.

Another active area of research is to investigate the best (smallest) ϵ that can be guaranteed in polynomial time. For ϵ -Nash, the current best algorithm, due to Tsaknakis and Spirakis [18], achieves a 0.3393-Nash equilibrium; see [8,7,3] for other algorithms. For the important class of win-lose games – games with payoffs in $\{0,1\}$ – [18] gives a $\frac{1}{4}$ -Nash equilibrium. For ϵ -WSNE, the current best result was given by Fearnley et al. [10] and finds a $(\frac{2}{3} - \zeta)$ -WSNE, where $\zeta = 0.00473$. It builds on an approach of Kontogiannis and Spirakis [13], which finds a $\frac{2}{3}$ -WSNE in polynomial time using linear programming. The algorithm of Kontogiannis and Spirakis produces a $\frac{1}{2}$ -WSNE of win-lose games in polynomial time, which is best-known (the modifications of Fearnley et al. do not lead to an improved approximation guarantee for win-lose games).

It is known that this line of work cannot extend to a fully-polynomial-time approximation scheme (FPTAS). More precisely, there does not exist an FPTAS for computing approximate Nash equilibria unless PPAD is in P [5]. Recall, an FPTAS requires a running time that is polynomial in both the size of the game input and in $\frac{1}{\epsilon}$. A polynomial-time approximation scheme (PTAS), however, need not run in time polynomial in $\frac{1}{\epsilon}$. It is not known whether there exists a PTAS for computing an approximate Nash equilibrium and, arguably, this is the biggest open question in equilibrium computation today. While the best-known approximation guarantee for ϵ -Nash that is achievable in polynomial time is much better than that for ϵ -WSNE, the two notions are polynomially related: there is a PTAS for ϵ -WSNE if and only if there is a PTAS for ϵ -Nash [5,6].

1.1 Our Results

The focus of this paper is on the combinatorial structure of equilibrium. Our first result shows that well-supported Nash equilibria differ structurally from approximate Nash equilibria in a significant way. It is known that there are $\frac{1}{2}$ -Nash equilibria with supports of cardinality at most two [12], and that this result is tight [11]. In contrast, we show in Theorem 2 that for any $\delta > 0$, there exist win-lose games for which no pair of strategies with support sizes at most two is a $(1 - \delta)$ -well-supported Nash equilibrium.⁵

With supports of cardinality three, Daskalakis et al. conjectured, in the first paper that studied algorithms for finding ϵ -WSNE [8], that $\frac{2}{3}$ -WSNE are obtainable in every win-lose bimatrix game. Specifically, this would be a consequence of the following graph-theoretic conjecture.

Conjecture 1 ([8]). Every digraph either has a cycle of length at most 3 or an undominated set⁶ of 3 vertices.

The main result in this paper, Theorem 1, shows that one cannot do better with constant size supports. We prove that there exist win-lose games that require supports of cardinality at least $\Omega(\sqrt[3]{\log n})$ in any ϵ -WSNE with $\epsilon < \frac{2}{3}$. We prove this existence result probabilistically. The key tool in showing correctness is a proof of a bipartite digraph variant of the well-known Caccetta-Häggkvist conjecture [4]. A polylogarithmic cardinality bound, as presented here, is the best

⁵ Random games have been shown to have exact equilibria with support size 2 with high probability; see Bárány et al. [2].

⁶ A set S is undominated if there is no vertex v that has an arc to every vertex in S .

we can hope for – a probabilistic argument [13] shows that there exist ϵ -WSNE with supports of cardinality $O(\frac{1}{\epsilon^2} \cdot \log n)$, for any $\epsilon > 0$.⁷

2 A Lower Bound on the Support Size of Well Supported Nash Equilibria

We begin by formally defining bimatrix win-lose games and well-supported Nash equilibria. A *bimatrix game* is a 2-player game with $m \times n$ payoff matrices A and B ; we may assume that $m \leq n$. The game is called *win-lose* if each matrix entry is in $\{0, 1\}$.

A pair of mixed strategies $\{\mathbf{p}, \mathbf{q}\}$ is a *Nash equilibrium* if every pure row strategy in the support of \mathbf{p} is a best response to \mathbf{q} and every pure column strategy in the support of \mathbf{q} is a best response to \mathbf{p} . A relaxation of this concept is the following. A pair of mixed strategies $\{\mathbf{p}, \mathbf{q}\}$ is an ϵ -*well supported Nash equilibrium* if every pure strategy in the support of \mathbf{p} (resp. \mathbf{q}) is an ϵ -approximate best response to \mathbf{q} (resp. \mathbf{p}). That is, for any row r_i in the support of \mathbf{p} we have

$$\mathbf{e}_i^T A \mathbf{q} \geq \max_{\ell} \mathbf{e}_{\ell}^T A \mathbf{q} - \epsilon$$

and, for any column c_j in the support of \mathbf{q} we have

$$\mathbf{p}^T B \mathbf{e}_j \geq \max_{\ell} \mathbf{p}^T B \mathbf{e}_{\ell} - \epsilon .$$

In this section we prove our main result.

Theorem 1. *For any $\epsilon < \frac{2}{3}$, there exist win-lose games for which every ϵ -well-supported Nash equilibrium has supports of cardinality $\Omega(\sqrt[3]{\log n})$.*

To prove this result, we first formulate our win-lose games graphically. This can be done in a straight-forward manner. Simply observe that we may represent a 2-player win-lose game by a directed bipartite graph $G = (R \cup C, E)$. There is a vertex for each row and a vertex for each column. There is an arc $(r_i, c_j) \in E$ if and only if $(B)_{ij} = b_{ij} = 1$; similarly there is an arc $(c_j, r_i) \in E$ if and only if $(A)_{ij} = a_{ij} = 1$.

Consequently, we are searching for a graph whose corresponding game has no high quality well-supported Nash equilibrium with small supports. We show

⁷ For Nash equilibria, Althöfer [1] and Lipton and Young [15] independently proved similar results for zero-sum games, and Lipton et al. [16] later proved similar results for general-sum bimatrix and multi-player games.

the existence of such a graph probabilistically.

The Construction.

Let $T = (V, E)$ be a random tournament on N nodes. Now create from T an auxiliary bipartite graph $G(T) = (R \cup C, A)$ corresponding to a 2-player win-lose game as follows. The auxiliary graph has a vertex-bipartition $R \cup C$ where there is a vertex of R for each node of T and there is a vertex of C for each set of k distinct nodes of T . (Observe that, for clarity we will refer to nodes in the tournament T and vertices in the bipartite graph G .) There are two types of arc in $G(T)$: those oriented from R to C and those oriented from C to R . For arcs of the former type, each vertex $X \in C$ will have in-degree exactly k . Specifically, let X correspond to the k -tuple $\{v_1, \dots, v_k\}$ where $v_i \in V(T)$, for all $1 \leq i \leq k$. Then there are arcs (v_i, X) in G for all $1 \leq i \leq k$. Next consider the latter type of arc in G . For each node $u \in R$ there is an arc (X, u) in G if and only if u dominates $X = \{v_1, \dots, v_k\}$ in the tournament T , that is if (u, v_i) are arcs in T for all $1 \leq i \leq k$. This completes the construction of the auxiliary graph (game) G .

We say that a set of vertices $W = \{w_1, \dots, w_t\}$ is *covered* if there exists a vertex y such that $(w_j, y) \in A$, for all $1 \leq j \leq t$. Furthermore, a bipartite graph is *k-covered* if every collection of k vertices that lie on the same side of the bipartition is covered. Now with positive probability the auxiliary graph $G(T)$ is *k-covered*.

Lemma 1. *For all sufficiently large n and $k \leq \sqrt[3]{\log n}$, there exists a tournament T whose auxiliary bipartite graph $G(T)$ is *k-covered*.*

Proof. Observe that the payoff matrices that correspond to $G(T)$ have $m = N$ rows and $n = \binom{N}{k}$ columns. Furthermore, by construction, any set of k vertices in R is covered. Thus, first we must verify that any set of k vertices in C is also covered.

So consider a collection $\mathcal{X} = \{X_1, \dots, X_k\}$ of k vertices in C . Since each $X_i \in C$ corresponds to a k -tuple of nodes of T , we see that \mathcal{X} corresponds to a collection of at most k^2 nodes in T . Thus, for any node $u \notin \cup_i X_i$, we have that u has an arc in T to every node in $\cup_i X_i$ with probability at least 2^{-k^2} . Thus with probability at most $(1 - \frac{1}{2^{k^2}})^{N-k^2}$ the subset \mathcal{X} of C not covered in $G(T)$. Applying the union bound we have that there exists the desired tournament if

$$\binom{n}{k} \cdot \left(1 - \frac{1}{2^{k^2}}\right)^{N-k^2} < 1 \tag{1}$$

Now set $k = \log^{\frac{1}{3}} n$. Therefore $\log n^{\frac{1}{k}} = \log^{\frac{2}{3}} n = k^2$.

In addition, because $n = \binom{N}{k}$, we have that $N \geq \frac{k}{e} \cdot n^{\frac{1}{k}}$. Hence, $N - k^2 > n^{\frac{1}{k}}$. (Note that, since $N \geq k$ this implies that $G(T)$ is defined.) Consequently,

$$\begin{aligned} \binom{n}{k} \cdot \left(1 - \frac{1}{2^{k^2}}\right)^{N-k^2} &\leq n^k \cdot \left(1 - \frac{1}{2^{k^2}}\right)^{n^{\frac{1}{k}}} \\ &\leq n^k \cdot e^{-\frac{1}{2^{k^2}} \cdot n^{\frac{1}{k}}} \\ &\leq n^k \cdot e^{-\frac{1}{e^{k^2 \cdot \log 2}} \cdot n^{\frac{1}{k}}} \end{aligned}$$

Thus, taking logarithms, we see that Inequality (1) holds if

$$e^{k^2 \cdot \log 2} \cdot k \cdot \log n < n^{\frac{1}{k}} \quad (2)$$

But $n^{\frac{1}{k}} = e^{k^2}$, so Inequality (2) clearly holds for large n . The result follows.

A property of the auxiliary graph $G(T)$ that will be very useful to us is that it contains no cycles with less than six vertices.

Lemma 2. *The auxiliary graph $G(T)$ contains no digons and no 4-cycles.*

Proof. Suppose $G(T)$ contains a digon $\{w, X\}$. The arc (w, X) implies that $X = \{x_1, \dots, x_{k-1}, w\}$. On the other-hand, the arc (X, w) implies that w dominates X in T and, thus, $w \notin X$.

Suppose $G(T)$ contains a 4-cycle $\{w, X, z, Y\}$ where w and z are in R and where $X = \{x_1, \dots, x_{k-1}, w\}$ and $Y = \{y_1, \dots, y_{k-1}, z\}$ are in C . Then z must dominate X in T and w must dominate Y in T . But then we have a digon in T as (w, z) and (z, w) must be arcs in T . This contradicts the fact that T is a tournament.

Lemmas 1 and 2 are already sufficient to prove a major distinction between approximate-Nash equilibria and well-supported Nash equilibria. Recall that there always exist $\frac{1}{2}$ -Nash equilibria with supports of cardinality at most two [8]. In sharp contrast, for supports of cardinality at most two, no constant approximation guarantee can be obtained for ϵ -well-supported Nash equilibria.

Theorem 2. *For any $\delta > 0$, there exist win-lose games for which no pair of strategy vectors with support sizes at most two is a $(1 - \delta)$ -well-supported Nash equilibrium.*

Proof. Take the auxiliary win-lose game $G(T)$ from Lemma 1 for the case $k = 2$. Now consider any pair of strategy vectors \mathbf{p}_1 and \mathbf{p}_2 with supports of cardinality

2 or less. Since $G(T)$ is 2-covered, the best responses to \mathbf{p}_1 and \mathbf{p}_2 both generate payoffs of exactly 1. Thus $\{\mathbf{p}_1, \mathbf{p}_2\}$ can be a $(1 - \delta)$ -well-supported Nash equilibrium only if each strategy in the support of \mathbf{p}_1 is a best response to at least one of the pure strategies in the support of \mathbf{p}_2 and vice versa. Therefore, in the subgraph H of $G(T)$ induced by the supports of \mathbf{p}_1 and \mathbf{p}_2 , each vertex has in-degree at least one. Thus, H contains a directed cycle. But $G(T)$ has no digons or 4-cycles, by Lemma 2. Hence, we obtain a contradiction as H contains at most four vertices.

In light of Lemma 2, we will be interested in the minimum in-degree required to ensure that a bipartite graph contains a 4-cycle. The following theorem may be of interest on its own right, as it resolves a variant of the well-known Caccetta-Häggkvist conjecture [4] for bipartite digraphs. For Eulerian graphs, a related but different result is due to Shen and Yuster [17].

Theorem 3. *Let $H = (L \cup R, A)$ be a directed $k \times k$ bipartite graph. If H has minimum in-degree $\lambda \cdot k$ then it contains a 4-cycle, whenever $\lambda > \frac{1}{3}$.*

Proof. To begin, by removing arcs we may assume that every vertex has in-degree exactly $\lambda \cdot k$. Now take a vertex v with the maximum out-degree in H , where without loss of generality $v \in L$. Let A_1 be the set of out-neighbours of v , and set $\alpha_1 \cdot k = |A_1|$. Similarly, let B_t be the set of vertices with paths to v that contain exactly t arcs, for $t \in \{1, 2\}$, and set $\beta_t \cdot k = |B_t|$. Finally, let C_1 be the vertices in R that are not adjacent to v , namely $C_1 = L - (A_1 \cup B_1)$. Set $\gamma_1 \cdot k = |C_1|$.

These definitions are illustrated in Figure 1.

Observe that we have the following constraints on α_1, β_1 and γ_1 . By assumption, $\beta_1 = \lambda$. Thus, we have $\gamma_1 = 1 - \alpha_1 - \lambda$. Moreover, by the choice of v , we have $\alpha_1 \geq \lambda$, since the maximum out-degree must be at least the average in-degree.

Note that if there is an arc from A_1 to B_2 then H contains a 4-cycle. So, let's examine the in-neighbours of B_2 . We know B_2 has exactly $\lambda \cdot k \cdot |B_2|$ incoming arcs. We may assume all these arcs emanate from $B_1 \cup C_1$. On the other-hand, there are exactly $\lambda \cdot k \cdot |B_1|$ arcs from B_2 to B_1 . Thus, there are at most $|B_1| \cdot (|B_2| - \lambda \cdot k)$ arcs from B_1 to B_2 . So the number of arcs from C_1 to B_2 is at least

$$\begin{aligned} \lambda \cdot k \cdot |B_2| - |B_1| \cdot (|B_2| - \lambda \cdot k) &= \lambda \cdot k \cdot \beta_2 \cdot k - \beta_1 \cdot k \cdot (\beta_2 \cdot k + \lambda \cdot k) \\ &= \lambda \cdot k \cdot \beta_2 \cdot k - \lambda \cdot k \cdot (\beta_2 \cdot k + \lambda \cdot k) \\ &= \lambda^2 \cdot k^2 \end{aligned}$$

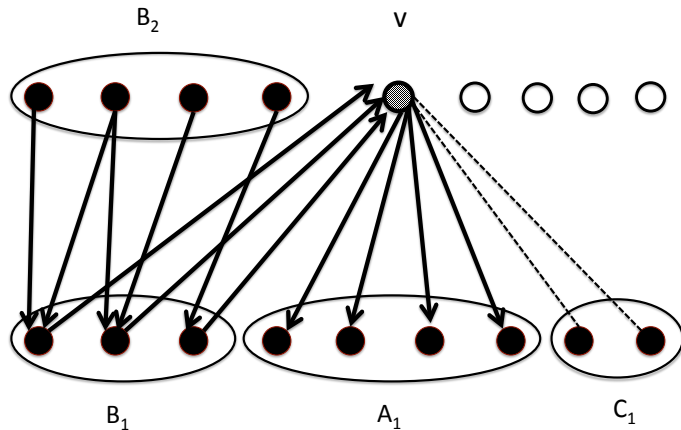


Fig. 1.

Since the maximum degree is $\alpha_1 \cdot k$, the number of arcs emanating from C_1 is at most $\gamma_1 \cdot \alpha_1 \cdot k^2$. Thus $\gamma_1 \cdot \alpha_1 \cdot (1 - \alpha_1 - \lambda) \geq \lambda^2$. Rearranging we obtain the quadratic inequality

$$\alpha_1^2 - \alpha_1(1 - \lambda) + \lambda^2 \leq 0$$

The discriminant of this quadratic is $1 - 2\lambda - 3\lambda^2$. But $1 - 2\lambda - 3\lambda^2 = (1 - 3\lambda)(1 + \lambda)$ and this is non-negative if and only if $\lambda \leq \frac{1}{3}$. This completes the proof.

We may now prove our main result: no approximation guarantee better than $\frac{2}{3}$ can be achieved unless the well-supported equilibria has supports with cardinality $\Omega(\sqrt[3]{\log n})$.

Proof of Theorem 1. Take a tournament T whose auxiliary bipartite graph is k -covered. By Lemma 1, such a tournament exists. Consider the win-lose game corresponding to the auxiliary graph $G(T)$, and take strategy vectors \mathbf{p}_1 and \mathbf{p}_2 with supports of cardinality k or less. Without loss of generality, we may assume that \mathbf{p}_1 and \mathbf{p}_2 are rational. Denote these supports as $S_1 \subseteq R$ and $S_2 \subseteq C$, respectively. As $G(T)$ is k -covered, there is a pure strategy $c^* \in C$ that covers S_1 and a pure strategy $r^* \in R$ that covers S_2 . Thus, in the win-lose game, $c^* \in C$ has an expected payoff of 1 against \mathbf{p}_1 and $r^* \in R$ has an expected payoff of 1 against \mathbf{p}_2 .

Suppose \mathbf{p}_1 and \mathbf{p}_2 form a ϵ -well-supported equilibrium for some $\epsilon < \frac{2}{3}$. Then it must be the case that each $r_i \in S_1$ has expected payoff at least $1 - \epsilon > \frac{1}{3}$ against \mathbf{p}_2 . Similarly, each $c_j \in S_2$ has expected payoff at least $1 - \epsilon > \frac{1}{3}$ against \mathbf{p}_1 . But this cannot happen. Consider the subgraph of $G(T)$ induced by $S_1 \cup S_2$ where each $r_i \in S_1$ has weight $w_i = p_1(r_i)$ and each $c_j \in S_2$ has weight $w_j = p_2(c_j)$. We convert this into an unweighted graph H by making $L \cdot w_v$ copies of each vertex v , for some large integer L . Now H is an $L \times L$ bipartite graph with minimum in-degree $(1 - \epsilon) \cdot L > \frac{1}{3} \cdot L$. Thus, by Theorem 3, H contains a 4-cycle. This is a contradiction, by Lemma 2. \square

We remark that the $\frac{2}{3}$ in Theorem 1 cannot be improved using this proof technique. Specifically the minimum in-degree requirement of $\frac{1}{3} \cdot k$ in Theorem 3 is tight. To see this, take a directed 6-cycle C and replace each vertex in C by $\frac{1}{3} \cdot k$ copies. Thus each arc in C now corresponds to a complete $\frac{k}{3} \times \frac{k}{3}$ bipartite graph with all arc orientations in the same direction. The graph H created in this fashion is bipartite with all in-degrees (and all out-degrees) equal to $\frac{1}{3} \cdot k$. Clearly the minimum length of a directed cycle in H is six.

3 Conclusion

An outstanding open problem is whether any constant approximation guarantee better than 1 is achievable with constant cardinality supports. We have shown that supports of cardinality two cannot achieve this; a positive resolution of Conjecture 1 would suffice to show that supports of cardinality three can. However, Conjecture 1 seems a hard graph problem and it is certainly conceivable that it is false.⁸ If so, that would lead to the intriguing possibility of a very major structural difference between ϵ -Nash and ϵ -WSNE; namely, that for any $\delta > 0$, there exist win-lose games for which no pair of strategies with constant cardinality supports is a $(1 - \delta)$ -well-supported Nash equilibrium.

The existence of small support ϵ -WSNE clearly implies the existence of polynomial time approximation algorithms to find such equilibria. Obtaining better approximation guarantees using more complex algorithms is also an interesting question. As discussed, the best known polynomial-time approximation algorithm for well-supported equilibria in win-lose games finds a $\frac{1}{2}$ -well supported equilibrium [13] by solving a linear program (LP). For games with

⁸ For example, the conjecture resembles a question about the existence of k -existentially complete triangle-free graphs for $k > 3$ referred to in [9], which the authors consider to be wide open.

payoffs in $[0, 1]$ that algorithm finds a $\frac{2}{3}$ -well-supported equilibrium. The algorithm has been modified in [10] to achieve a slightly better approximation of about $\frac{2}{3} - \zeta$ where $\zeta = 0.00473$. That modification solves an almost identical LP as [13] and then either transfers probability mass within the supports of a solution to the LP or returns a small support strategy profile that uses at most two pure strategies for each player. The results of this paper show that both parts of that approach are needed, and any improvement to the approximation guarantee must allow for super-constant support sizes.

Acknowledgements. We thank John Fearnley and Troels Sørensen for useful discussions. The first author is supported by a fellowship from MITACS and an NSERC grant. The second author is supported by an NSERC grant 418520. The third author is supported by an EPSRC grant EP/L011018/1. The fourth author is supported by NSERC grants 288334 and 429598.

References

1. I. Althöfer, “On sparse approximations to randomized strategies and convex combinations”, *Linear Algebra and its Applications*, **199**, pp339-355, 1994.
2. I. Bárány, S. Vempala, and A. Vetta, “Nash equilibria in random games”, *Random Struct. Algorithms*, **31(4)**, pp391-405, 2007.
3. H. Bosse, J. Byrka, and E. Markakis, “New algorithms for approximate Nash equilibria in bimatrix games”, *Theoretical Computer Science*, **411(1)**, pp164-173, 2010.
4. L. Caccetta and R. Häggkvist, “On minimal digraphs with given girth”, *Congressus Numerantium*, **21**, pp181-187, 1978.
5. X. Chen, X. Deng, and S. Teng, “Settling the complexity of computing two-player Nash equilibria”, *Journal of the ACM*, **56(3)**, pp1-57, 2009.
6. C. Daskalakis, P. Goldberg, and C. Papadimitriou, “The complexity of computing a Nash equilibrium”, *SIAM Journal on Computing*, **39(1)**, pp195-259, 2009.
7. C. Daskalakis, A. Mehta, and C. Papadimitriou “Progress in approximate Nash equilibria”, *ACM Conference on Electronic Commerce (EC)*, pp355-358, 2007.
8. C. Daskalakis, A. Mehta, and C. Papadimitriou, “A note on approximate Nash equilibria”, *Theoretical Computer Science*, **410(17)**, pp1581-1588, 2009.
9. C. Even-Zohar and N. Linial, “Triply existentially complete triangle-free graphs”, *arxiv.org/1306.5637*, 2013.
10. J. Fearnley, P. Goldberg, R. Savani, and T. Sørensen, “Approximate well-supported Nash equilibria below two-thirds”, *International Symposium on Algorithmic Game Theory (SAGT)*, pp108-119, 2012.
11. T. Feder, H. Nazerzadeh, and A. Saberi, “Approximating Nash equilibria using small-support strategies”, *ACM Conference on Electronic Commerce (EC)*, 2007.

12. P. Goldberg and C. Papadimitriou, “Reducability among equilibrium problems”, *STOC*, 2006.
13. S. Kontogiannis and P. Spirakis, “Well supported approximate equilibria in bimatrix games”, *Algorithmica*, **57**, pp653-667, 2010.
14. S. Kontogiannis and P. Spirakis, “Efficient algorithms for constant well supported approximate equilibria in bimatrix games”, *ICALP*, pp595-606, 2007.
15. R. Lipton and N. Young, “Simple strategies for zero-sum games with applications to complexity theory”, *STOC*, pp734-740, 1994.
16. R. Lipton, E. Markakis, and A. Mehta, “Playing large games using simple strategies”, *ACM Conference on Electronic Commerce (EC)*, pp36-41, 2003.
17. J. Shen and R. Yuster, “A note on the number of edges guaranteeing a C_4 in Eulerian bipartite digraphs”, *Electronic Journal of Combinatorics*, **9(1)**, Note 6, 2002.
18. H. Tsaknakis and P. Spirakis, “An optimization approach for approximate Nash equilibria”, *Internet Mathematics*, **5(4)**, pp365-382, 2008.