Computational Aspects of Multimarket Price Wars

Nithum Thain[1] and Adrian Vetta[2]

[1] Department of Mathematics and Statistics, McGill University. *
nithum.thain@mail.mcgill.ca
[2] Department of Mathematics and Statistics and School of Computer Science, McGill University. vetta@math.mcgill.ca **

Abstract. We consider the complexity of decision making with regards to predatory pricing in multimarket oligopoly models. Specifically, we present multimarket extensions of the classical single-market models of Bertrand, Cournot and Stackelberg, and introduce the War Chest Minimization Problem. This is the natural problem of deciding whether a firm has a sufficiently large war chest to win a price war. On the negative side we show that, even with complete information, it is hard to obtain any multiplicative approximation guarantee for this problem. Moreover, these hardness results hold even in the simple case of linear demand, price, and cost functions. On the other hand, we give algorithms with arbitrarily small *additive* approximation guarantees for the Bertrand and Stackelberg multimarket models with linear demand, price, and cost functions. Furthermore, in the absence of fixed costs, this problem is solvable in polynomial time in all our models.

1 Introduction

This paper concerns price wars and predatory pricing in markets. We focus on multiple markets (or a single segmentable market) as it allows us to model a broader and more realistic set of interactions between firms. A firm may initiate a price war in order to increase market share or to deter other firms from competing in particular markets. The firm suffers a short-term loss but may gain large future profits, particularly if the price war forces out the competition and allows it to price as a monopolist.

Price wars (and predatory pricing) have been studied extensively from both an economic and a legal perspective. A detailed examination of all aspects of price wars is far beyond the scope of this paper. Rather, we focus on just one important aspect: the complexity of decision making in oligopolies (e.g. duopolies). Specifically, we consider the budget required by a firm in order to successfully launch a price war. This particular question is fundamental in determining the risk and benefits arising from predatory practices. Moreover, it arises naturally in the following two scenarios:

* Supported in part by NSERC CGS grant.
** Supported in part by NSERC grant 28833.

Entry Deterrence: How much of a war chest must a monopolist or cartel have on hand so that they are able to successfully repel a new entrant?

Competition Reduction: How much money must a firm or cartel have to force another firm out of business? For example, in a duopoly how much does a firm need to save before it can defeat the other to create a monopoly?

We formulate the War Chest Minimization Problem as a generalization of both of these scenarios and study the computational complexity of and approximation algorithms for this more general problem.

1.1 Background

Price wars and predatory pricing are tools that have been long associated with monopolies and cartels. The literature on these topics is vast and we touch upon just a small sample in this short background section.

Given the possible rewards for monopolies and cartels engaging in predatory behaviour, it is not surprising that it has been a recurrent theme over time. The late 19th century saw cartels engaging in predation in a plethora of industries. Prominent examples include the use of "fighting ships" by the British Shipping Conferences ([36], [30]) to control trade routes, the setting up of phoney independents by the American Tobacco Company to undercut smaller competitors [9]. Perhaps the most infamous instance, though, of a cartel concerns Standard Oil under the leadership of John D. Rockefeller ([27], [35], [12]). More recent examples of price wars include the cigarette industry [15], the airline industry [7], and the retail industry [8]. In the computer industry, Microsoft regularly faces accusations of predatory practices ([18], [25], [26]).

Antitrust legislation has been introduced in many countries to prevent anticompetitive behaviour like predatory pricing or oligopolistic collusion[3]. In the United States, the most important such legislation is the Sherman Act of 1890. One of the Act's earliest applications came in 1911 when the Supreme Court ordered the break-up of both Standard Oil and American Tobacco; more recently, it was applied when the Court ordered the break-up of American Telephone and Telegraph (AT&T) in 1982.[4]

Given that such major repercussions may arise, there is a need for a cloak of secrecy around any act of predation. This has meant the extent of predatory pricing is unknown and has been widely debated in the literature. Indeed, early

[3] Whilst it is easy to see the negative aspect of cartels, it is interesting to note that there may even be some positive consequences. For example, it has been argued [17] that the predatory actions of cartels may *increase* consumer surplus.

[4] In 2000, a lower court also ordered the breakup of Microsoft for antitrust violations under the Sherman Act. On appeal, this punishment was removed under an agreed settlement in 2002.

economic work of McGee [27] suggested that predatory pricing was not rational. However, in Stigler's seminal work on oligopolies [39], price wars can be viewed as a break-down of a cartel, *albeit* they do not arise in equilibria because collusion can be enforced via punishment mechanisms. Moreover, recent models have shown how price wars can be recurrent in a "functioning" cartel! For example, this can happen assuming the presence of imperfect monitoring [22] or of business cycles [33]. This is particularly interesting as recurrent price wars were traditionally seen as indicators of a healthy competitive market[5].

Based primarily on the work of McGee, the US Supreme court now considers predatory pricing to be *generally implausible*[6]. As a result of this, and in an attempt to strike a balance between preventing anti-competitive behaviour and overly restricting normal competition, the Court applied the following strict definition to test for predatory practices.

(a) The predator is pricing below its short-run costs.
(b) The predator has a strong chance or recouping the losses incurred during the price-war.

The established way for the Court to test for the first requirement is the Areeda-Turner rule of 1975 [2] which established marginal cost (or, as an approximate surrogate, average variable cost) as the primary criteria for predatory pricing.[7] We will incorporate the Areeda-Turner rule as a legal element in our multimarket oligopoly models in Section 2.2. The second requirement essentially states that the "short-run loss is an investment in prospective monopoly profits" [14]. This requirement is typically simpler to test for in practice, and will be implicit in our models.

Finally, we remark that we are not aware of any other work concerning the complexity of price wars. One interesting related pricing strategy is that of loss-leaders which Balcan et al. [4] examine with respect to profit optimization. For the scale and type of problem we consider, however, using strategies that correspond to "loss-leaders" is illegal. Alternative models for oligopolistic competition and collusion in a single market setting can be found in the papers of Ericson and Pakes [16] and Weintraub et al. [43].

[5] Therefore, should such behaviour also arise in practice it would pose intriguing questions for policy makers. Specifically, when is a price war indicative of competition and when is it indicative of the presence of a cartel or a predatory practice?

[6] See the 1986 case `Matsushita Electric Industrial Company vs Zenith Radio Corporation` and the 1993 case `Brooke Group Limited vs Brown and Williamson Tobacco Corporation`.

[7] We note that the Areeda-Turner rule may be inappropriate in high-tech industries because fixed costs there are typically high. Therefore, measures of variable costs may not be reflective of the presence of a price-wars. In fact, hi-tech industries may be particularly susceptible to predatory practices as large marginal profits are required to cover the high fixed costs. Consequently, predatory pricing can be used to inflict great damage on smaller firms.

1.2 Our Results

A firm with price-making power belongs to an industry that is a monopoly or oligopoly. In Section 2, after reviewing the classical single-market models of Bertrand, Cournot and Stackelberg, we develop three multimarket models of oligopolistic competition. We then introduce the Minimum War Chest Problem to capture the essence of the Entry Deterrence and Competition Reduction scenarios outlined above.

In Section 3, we prove that this problem is NP-Hard in all three multimarket models under the legal constraints imposed by the Areeda-Turner rule. We emphasise that decision making is hard even under complete information. These hardness results utilise the fact that we have multiple markets. This assumption, however, is not essential. Decision-making can be hard in single-markets if either the number of firms is large or if the number of strategic options available to a firm is large. We give a simple example to illustrate this in Section 5.

The hardness results of Section 4.3 imply that no multiplicative approximation guarantee can be obtained for the Minimum War Chest Problem, even in the simple case of linear cost, price, and demand functions. The situation for potential predators is less bleak than this result appears to imply. To see this we present two positive results in Section 4, assuming linear cost, price, and demand functions. First, the problem can be solved in polynomial time if the predator faces no fixed costs. In addition, for the Bertrand and Stackelberg models there is a natural way to separate the markets into two types, those where player one is making a profit and those in which she is truly fighting a price war. Our second result states that in these models, we can solve the problem on the former set of markets exactly and can find a fully polynomial time approximation scheme for the problem on the latter markets. This leads to a polynomial time algorithm with an arbitrarily small additive guarantee.

2 Models

2.1 Three Classical Models of Oligopoly

Before presenting our models, in this section we review the classical Bertrand, Cournot, and Stackelberg models for competition within an oligarchy.

The Bertrand Model The Bertrand is a natural model of price competition between firms (henceforth referred to as "players") in an oligarchy [6]. We will define the model for the duopoly case, but the generalization to more firms is obvious. Suppose we have two players each producing identical goods that are not differentiated by any consumers. Player i has marginal cost c_i to produce one unit of the good and chooses a price p_i at which to sell one unit in the market. Since the goods are not differentiated, each consumer simply purchases the good from whomever charges the least. If both players charge the same price, then the market is shared evenly.

So, if $D(p)$ is the market demand function, this gives rise to the following profit function for player i:

$$\Pi_i(p_i, p_j) = (p_i - c_i)D_i(p_i, p_j)$$

where $D_i(p_i, p_j)$ is the demand for player i's good under the current prices and is defined by

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

A natural consequence of this model is that there is only one Nash equilibrium and in it the player with the lower marginal cost gets the entire market by charging the best price that is at least the other player's marginal cost. Thus, if she is player i, she makes a profit of $(c_j - c_i)D(c_j)$ and the other player makes no profit.

Our hardness results will apply even when the demand functions are linear. So, unless stated otherwise, for the remainder of this paper we will assume our demand functions are of the form $D(p) = a - bp$. In addition, we will allow for more general cost functions. Specifically, we will assume that Player i has a fixed cost f_i for competing in the market. Thus, its cost function becomes $C_i(q_i) = c_i q_i + f_i$ for $q_i > 0$.

The Cournot Model Economists have considered a number of alternative models for competition [40]. One prominent alternative is the Cournot model, formulated by Augustin Cournot in 1838 [11]. This model again assumes players selling identical, nondifferentiated goods, but studies competition in terms of quantity instead of price. Again, we will only define the model for a duopoly and leave the generalization to the reader.

In this model, each player chooses some quantity of good to produce, q_i, and pays some cost to produce it, $C_i(q_i)$. The price for the good is then set as a function of the quantities produced by both players, $P(q_i + q_j)$. Each player i makes profit:

$$\Pi_i(q_i, q_j) = q_i P(q_i + q_j) - C_i(q_i).$$

Again, we will typically assume that the price and cost functions are linear. In particular, we will only consider cost functions of the form $C_i(q_i) = c_i q_i + f_i$ for $q_i > 0$ where c_i is a constant marginal cost and f_i is a fixed cost. We will also only consider price functions of the form $P(q) = a - q$. Since our complexity results are negative ones, they still apply if we allow for more general price and cost functions.

In the absence of fixed costs, this model then has only has one equilibrium, called the *Cournot equilibrium*, where $q_i = (a - 2c_i + c_j)/3$ for every player. If both players play this equilibrium strategy, then they will each make profit $\Pi_i(q_i, q_j) = q_i^2$. If there are positive fixed costs then the model may have multiple equilibria.

The Stackelberg Model The Stackelberg model was formulated by Heinrich von Stackelberg in 1934 as an adaptation of the Cournot model of quantity competition [38]. The profit functions, price functions, and cost functions are identical to the above model. The Stackelberg model, however, separates the players into two types: leaders and followers. In the duopoly case, the model assumes that leader chooses its quantity first and commits to it, after which follower make its choice with perfect information about the leader's choice.

Being able to commit first gives the leader an enormous advantage, as it forces the follower to optimize her profit on the leader's terms. Thus, with the linear price and cost functions described above and zero fixed costs, the equilibrium is for the leader to choose quantity $q_1 = (a + c_2 - 2c_1)/2$ and for the follower to choose quantity $q_2 = (a - 3c_2 + 2c_1)/4$. Then the leader's equilibrium profit is $\Pi_1(q_1, q_2) = \frac{1}{2}q_1^2$ while the follower's is $\Pi_2(q_1, q_2) = q_2^2$. Notice that the leader makes more profit than in the Cournot model, while the follower makes less. Again, more complicated equilibria are possible if we have positive fixed costs.

2.2 Multimarket Models of Oligopoly

In this section, we formulate the multimarket Bertrand, Cournot, and Stackelberg models. These allow for the investigation of more numerous and assorted interactions between firms.

A Multimarket Bertrand Model Let us consider the following generalization of the asymmetric Bertrand model to multiple markets[8]. We will describe the model for the duopoly case, but again all of the definitions are easily generalizable. Suppose we have two players and n markets $m_1, m_2, ..., m_n$. Every player i has a budget B_i where a negative budget is thought of as the fixed cost for the firm to exist and a positive budget is thought of as a war chest available to that firm in the round. Every market m_k has a demand curve $D_k(p)$ and each player i also has a marginal cost, c_{ik}, for producing one unit of good in market m_k. In addition, each player i has a fixed cost, f_{ik}, for each market m_k that she pays if and only if she enters the market, i.e. if she sets some finite price.

We model the price war as a game between the two players. A strategy for player i is a complete specification of prices in all the markets. Both players choose their strategies simultaneously. If $p_{ik} < \infty$ then we will say that player i enters market m_k. If player i chooses not to enter market m_k, this is signified by setting $p_{ik} = \infty$. The demand for each market then all goes to the player with the lowest price. If the players set the same price, then the demand is shared equally. Thus analogously to Section 2.1, if player i participates, then she gets profit Π_{ik} in market m_k where

$$\Pi_{ik}(p_{ik}, p_{jk}) = (p_{ik} - c_{ik})D_{ik}(p_{ik}, p_{jk}) - f_{ik}$$

[8] We remark that this multimarket Bertrand model is also a generalization of the multiple market model used in the facility location game of Vetta [42].

and where D_{ik} is the demand for player i's good in market m_k and is defined as

$$D_{ik}(p_{ik}, p_{jk}) = \begin{cases} D_k(p_{ik}) & \text{if } p_{ik} < p_{jk} \\ \frac{1}{2} D_k(p_{ik}) & \text{if } p_{ik} = p_{jk} \\ 0 & \text{if } p_{ik} > p_{jk} \end{cases}$$

If player i chooses not to participate then her revenue and costs are both zero; thus, she gets 0 profit.

The sum of these profits over all markets is added to each player's budget. A player is eliminated if her budget is negative at the end of the round.

Multimarket Cournot and Stackelberg Models We now formulate a multimarket version of the Cournot model. Again we will restrict ourselves to the case of the duopoly as the generalization is obvious. In this Cornout model, there are n independent Cournot markets $m_1, ..., m_n$. Each market m_k has a price function $P_k(q) = a_k - q$. Each player also has a budget B_i, which serves the same role as in the Bertrand case. Each player also has a cost function in every market $C_{ik}(q_{ik})$.

As before we model the price war as a game. This time, a strategy for each player i is a choice of quantities q_{ik} for each market m_k. Again, both players choose a strategy simultaneously. We say that player i enters market m_k if $q_{ik} > 0$. Analogously to Section 2.1, player i then makes a profit in market m_k equal to

$$\Pi_{ik}(q_{ik}, q_{jk}) = q_{ik} P_k(q_{ik} + q_{jk}) - C_{ik}(q_{ik}).$$

Again, each player's total profit is added to their budget at the end of the round. As above, a player is eliminated if her resulting budget is negative. The multimarket Stackelberg model can then simply be adapted from the Cournot model. We define all of the quantities and functions as above. However, we now consider one player to be the leader and one to be the follower. The game is no longer simultaneous, as the leader gets to commit to a production level before the follower moves.

2.3 The War Chest Minimization Problem

We will examine the questions of entry deterrence and competition reduction in the two-firm setting. Thus, we focus on the computational problems facing (i) a monopolist fighting against a potential market entrant (entry deterrence) and (ii) a firm in a duopoly trying to force out the other firm (competition reduction). We model both these situations using the same duopolistic multimarket models of Section 2.2.

We remark that our focus on a firm rather than a cartel does not effect the fundamental computational aspects of the problem. This restriction, however, will allow us to avoid the distraction arising from the strategic complications that occur in ensuring coordination amongst members of a cartel.

Our game is then as follows. We assume that players one and two begin with budgets B_1 and B_2, respectively. They then play one of our three multimarket games. The goal of firm one is to stay/become a monopoly; if it succeeds it will subsequently be able to act monopolistically in each market. To achieve this goal the firm needs a non-negative payoff at the end of the game whilst its opponent has a negative payoff (taking into account their initial budgets). This gives us the following natural question:

War Chest Minimization Problem: *How large a budget B_1 does player one need to ensure that it can eliminate an opponent with a budget $B_2 < 0$.*

The players can play any strategy they wish *provided* it is legal, that is, they must abide by the Areeda-Turner Rule. All our results will be demonstrated under the assumptions of this rule, as it represents the current legal environment. However, similar complexity results can be obtained without assuming this rule.

Areeda-Turner Rule: *It is illegal for either player to price below their marginal cost in any market.*

Before presenting our results we make a few comments about the problem and what the legal constraints mean in our setting. First, notice that we specify a negative budget for player two but place no restriction on the budget for player one. This is natural for our models. We can view the budget as the money a firm initially has at its disposable minus the fixed costs required for it to operate; these fixed costs are additional to the separate fixed costs required to operate in any individual market. Consequently, if the second firm has a positive budget it cannot be eliminated from the game as it has sufficient resources to operate (cover its fixed costs) even without competing in any of the individual markets; thus we must constrain the second firm to have a negative budget. On the other hand, for the first firm no constraint is needed. Even if its initial budget is negative, it is plausible that it can still eliminate the second firm and end up with a positive budget at the end of the game, by making enough profit from the individual markets. Specifically, the legal constraints imposed by the Areeda-Turner rule may ensure that the second firm cannot maliciously bankrupt the first firm even if the first firm has a negative initial war-chest.

Second, since we are assuming that player one wishes to ensure success regardless of the strategy player two chooses, we will analyze the game as an asynchronous game where player two may see player one's choices before making her own. Player two will then first try to survive despite player one's choice of strategy. If she cannot do so, she will undercut player one in every market in an attempt to eliminate her also. To win the price war, player one must find strategies that keep herself safe and eliminate player two irrespective of how player two plays. Therefore, an optimal strategy for player one has maximum profit (i.e. minimum negative profit) amongst the collection of strategies that achieve these goals, assuming that player two plays maliciously.

Finally, the Areeda-Turner Rule has a straightforward interpretation in the Bertrand model of price competition, that is, neither player can set the price in any market below their marginal cost in that market! In models of quantity competition, however, the interpretation is necessarily less direct. For the Cournot model of quantity competition, we interpret the rule as saying that neither player can produce a quantity that will result in a price less than their marginal costly assuming the other player produces nothing, in other words $q_{ik} < a_k - c_{ik}$. This is the weakest interpretation possible for this simultaneous game. Finally, for the Stackelberg game, we assume that the restriction imposed by the Areeda-Turner rule is the same for player one as in the Cournot model, as she acts first and player two has not set a quantity when player one decides. Player two on the other hand, must produce a quantity so that her marginal price is greater than her marginal cost, given what player one has produced. In other words, for the Stackelberg game $q_{1k} < a_k - c_{1k}$ and $q_{2k} < a_k - q_{1k} - c_{2k}$.

3 Hardness Results

We are now in a position to show that the War Chest Minimization Problem is hard in all three models.

Theorem 1. *The War Chest Minimization Problem is NP-hard for the multi-market Bertrand model, even in the case with linear demand functions.*

Proof. We give a reduction from the knapsack problem. There we have n items, each with value v_i and weight w_i, and a bag which can hold weight at most W. In general, it is NP-hard to decide whether we can pack the items into the bag so that $\sum w_i \leq W$ and $\sum v_i > V$ for some constant V (where the sums are taken over packed items).

We will now create a multimarket Bertrand game based on the above instance. First suppose that there are n markets and each market m_k has the linear demand function

$$D_k(p) = 5\sqrt{v_k} - p.$$

Set player two's fixed costs to $f_{2k} = 0$ for all k and her marginal costs to $c_{2k} = 3\sqrt{v_k}$ for all k. Also set player one's marginal costs to $c_{1k} = 0$ for all k and her fixed costs to $f_{1k} = (25/4)v_k + w_k$ for all k. Set the budgets to be $B_1 = W$ and $B_2 = V - \sum_{k=1}^{n} v_k$.

First, we calculate the monopoly prices for player one and player two. If player i wins market m_k at price p_{ik} then their profit in that market is

$$\Pi_{ik}(p_{ik}) = (p_{ik} - c_{ik})D_k(p_{ik}) - f_{ik} = -p_{ik}^2 + (c_{ik} + 5\sqrt{v_k})p_{ik} - 5\sqrt{v_k}c_{ik} - f_{ik}.$$

Taking derivatives, we see that the monopoly price for player i in market m_k is

$$p_{ik}^* = \frac{c_{ik} + 5\sqrt{v_k}}{2} \geq c_{ik}.$$

In particular, notice that the monopoly price for player one is

$$p_{1k}^* = \frac{5}{2}\sqrt{v_k} < 3\sqrt{v_k} = c_{2k}.$$

Moreover

$$p_{2k}^* = 4\sqrt{v_k} > 3\sqrt{v_k} = c_{2k}.$$

Thus, if player one enters market m_k then she can price at her monopoly price without fear that player two will undercut her. If she does not enter, then player two could price at her monopoly price to maximize revenue, as her fixed costs are zero. In the first case, player two earns 0 profit and player one earns monopoly profit

$$\begin{aligned} \Pi_{1k}(p_{1k}^*) &= -{p_{1k}^*}^2 + (c_{1k} + 5\sqrt{v_k})p_{1k}^* - 5\sqrt{v_k}c_{1k} - f_{1k} \\ &= -{p_{1k}^*}^2 + 5\sqrt{v_k}p_{1k}^* - f_{1k} \;=\; -w_k \end{aligned}$$

In the second case, player one earns zero profit while player two earns her monopoly profit

$$\begin{aligned} \Pi_{2k}(p_{2k}^*) &= -{p_{2k}^*}^2 + (c_{2k} + 5\sqrt{v_k})p_{2k}^* - 5\sqrt{v_k}c_{2k} - f_{2k} \\ &= p_{2k}^*(8\sqrt{v_k} - p_{2k}^*) - 5\sqrt{v_k}c_{2k} \;=\; v_k \end{aligned}$$

Thus, if player one could solve the War Chest Minimization Problem then she could determine whether or not there exists a set of indices K of markets that she should enter such that both of the following equations hold simultaneously:

$$W - \sum_{k\in K} w_k \geq 0$$

$$V - \sum_{k=1}^{n} v_k + \sum_{k\notin K} v_k < 0$$

Rearranging these equations, we obtain the conditions of the knapsack equations, namely $\sum_{k\in K} w_k \leq W$ and $\sum_{k\in K} v_k > V$.

Theorem 2. *The War Chest Minimization Problem is NP-hard for the multi-market Cournot model, even in the case of linear price and cost functions.*

Proof. We again reduce from an instance of the knapsack problem. The Cournot game we create is as follows. Set $a_k = 6\sqrt{v_k}$, then for each market m_k let the price function be $P_k(q) = a_k - q$. We now set player one's marginal cost in market m_k to be $c_{1k} = 0$ and her fixed cost to be $f_{1k} = 4v_k + w_k$. Player two's marginal cost in market m_k is set to be $c_{2k} = 2a_k/3 = 4\sqrt{v_k}$ and her fixed cost is set to be $f_{2k} = 0$.

Suppose now that player one has chosen which markets to enter and has, in particular, chosen to enter market m_k by producing quantity $q_{1k} > 0$. Consider player two's response. At first, player two will try to survive and will thus try to maximize her profit, given player one's quantity. She will consequently try to choose q_{2k} that maximizes $\Pi_{2k}(q_{1k}, q_{2k})$, call this quantity q_{2k}^+. By taking derivatives, we can calculate q_{2k}^+ to be $(a_k - 3q_{1k})/6$.

If player two calculates that she can't survive by choosing q_{2k}^+ in every market, then she will try to undercut player one in every market in an attempt to also drive her out. She will therefore choose $q_{2k} = q_{2k}^-$, the quantity which minimizes $\Pi_{1k}(q_{1k}, q_{2k})$. This can be achieved by making q_{2k} as large a possible; given the constraints of the Areeda-Turner rule this implies that $q_{2k}^- = a_k - c_{2k}$. Thus, we calculate $q_{2k}^- = a_k/3 = 2\sqrt{v_k}$.

Now, if we assume that player one enters market m_k (i.e. assume $q_{1k} > 0$) then by calculating the partial derivatives of $\Pi_{2k}(q_{1k}, q_{2k}^+)$ and $\Pi_{1k}(q_{1k}, q_{2k}^-)$ with respect to q_{1k}, we see that the quantity $q_{1k}^* = a_k/3 = 2\sqrt{v_k}$ minimizes the former and maximizes the latter. Therefore if player one chooses to enter market m_k she will produce quantity q_{1k}^*. So if player one enters market m_k then she, in the worst case, makes profit

$$\begin{aligned}\Pi_{1k}(q_{1k}^*, q_{2k}^-) &= q_{1k}^*(P_k(q_{1k}^* + q_{2k}^-) - c_{1k}) - f_{1k} \\ &= q_{1k}^*((6\sqrt{v_k} - 4\sqrt{v_k}) - 0) - f_{1k} \ = \ -w_k\end{aligned}$$

Against this, player two, in her best case, plays $q_{2k}^+ = (a_k - 3q_{1k}^*)/6 = 0$. This clearly gives her a profit $\Pi_{2k}(q_{1k}^*, q_{2k}^+) = 0$. On the other hand, if player one doesn't enter market m_k then she makes profit 0 in that market and player two makes her monopoly profit, which in this case is

$$\Pi_{2k}(0, q_{2k}^*) = q_{2k}^*(P_k(q_{2k}^*) - c_{2k}) - f_{2k} \ = \ q_{2k}^*(2\sqrt{v_k} - q_{2k}^*) - 0 \ = \ v_k$$

The proof follows.

A similar proof holds for the Stackelberg case. We include the proof as it will be needed in Section 4.2.

Theorem 3. *The War Chest Minimization Problem is NP-hard for the multimarket Stackelberg model if player one is the Stackelberg leader, even in the case linear price and cost functions.*

Proof. We again reduce from the knapsack problem. Take any instance of the knapsack problem and define the quantities n, W, V, the w_is, and the v_is as in the proof of Theorem 1. We will now create a multimarket Stackelberg game based on the above instance. Set $a_k = 4\sqrt{v_k}$, and suppose that there are n markets and each market has price function $P_k(q) = a_k - q$. We now set player one's marginal cost in market m_k to be $c_{1k} = 0$ and her fixed cost to be $f_{1k} = 4v_k + w_k$. Player two's marginal cost in market m_k is set to be $c_{2k} = a_k/2 = 2\sqrt{v_k}$ and her fixed cost is set to be $f_{2k} = 0$. Finally, set the budgets to be $B_1 = W$ and $B_2 = V - \sum_{k=1}^{n} v_k$ as before.

Now consider the decision player one faces when deciding whether or not to enter market m_k. First notice that her monopoly quantity is $q_{1k}^* = a_k/2 = 2\sqrt{v_k}$ which we can calculate by maximizing $\Pi_{1k}(q_{1k}, 0)$ through simple calculus. Notice also that $a_k - q_{1k}^* - c_{2k} = 0$ and so, by the Areeda-Turner rule, player two cannot produce in any market in which player one is producing.

Thus, if player one enters any market then she will produce her monopoly quantity in that market and player two will not enter that market. In this case, player one makes profit

$$\Pi_{1k}(q^*_{1k}, 0) = q^*_{1k}(P_k(q^*_{1k}) - c_{1k}) - f_{1k} \;=\; 2\sqrt{v_k}(2\sqrt{v_k} - 0) - (4v_k + w_k) \;=\; -w_k$$

and player two makes profit $\Pi_{2k}(q^*_{1k}, 0) = 0$. On the other hand, if player one does not enter the market then player two will produce her monopoly quantity, $q^*_{2k} = a_k/4 = \sqrt{v_k}$, and will make profit

$$\Pi_{2k}(0, q^*_{2k}) = q^*_{2k}(P_k(q^*_{2k}) - c_{2k}) - f_{2k} \;=\; \sqrt{v_k}(3\sqrt{v_k} - 2\sqrt{v_k}) - 0 \;=\; v_k$$

Since player one did not enter, she will make profit 0. Thus we find ourselves back in the exact circumstances of the proof of Theorem 1. The rest of the proof follows.

4 Algorithms

In this section, we explore algorithms for solving the War Chest Minimization Problem. We highlight a case where the problem can be solved exactly and explore the approximability of the problem in general. For the entirety of this section, we assume linear cost, demand, and price functions.

4.1 A Polynomial Time Algorithm in the Absence of Fixed Costs

All of the complexity proofs in Section 3 have a similar flavor. We essentially use the fixed costs in the markets to construct weights in a knapsack problem. In this section, we demonstrate that in the absence of fixed costs, it is computationally easy for a player to determine if they can win a multimarket price war even under the restrictions of the Areeda-Turner rule. This rule adds additional complications in this Stackelberg model, so we analyse that model first here. Again, we assume player one is the Stackelberg leader. Without fixed costs, the profit functions of both players in each market m_k are particularly simple:

$$\Pi_{ik}(q_{ik}, q_{jk}) = q_{ik}(a_k - q_{1k} - q_{2k} - c_{ik})$$

As discussed, there are two strategies that player two may employ to prevent player one from winning the price war. She may play so as to survive or, if that is destined to fail, she may play so as to leave player one with a negative budget. In the former strategy, she will choose in every round and in every market the quantity, q^+_{2k}, that maximizes her own profit. In the latter strategy she will choose the quantity, q^-_{2k}, that minimizes player one's profit (while obeying the Areeda-Turner rule). By consider the partial derivatives of the players' profits, one can calculate q^+_{2k} and q^-_{2k} as functions of q_{1k}:

$$q_{2k}^{+} = \frac{a_k - q_{1k} - c_{2k}}{2}$$
$$q_{2k}^{-} = \begin{cases} a_k - q_{1k} - c_{2k} & \text{if } q_{1k} < a - c_{2k} \\ 0 & \text{otherwise} \end{cases}$$

The latter case for q_{2k}^{-} occurs if player one chooses a quantity so high that player two can choose nothing by the Areeda-Turner rule; this can only occur if $c_{1k} < c_{2k}$ as otherwise the Areeda-Turner rule itself prevents player one from choosing a sufficiently high quantity.

We now partition the markets into two sets: let $k \in A$ denote the set of markets for which $c_{1k} \leq c_{2k}$ and let $k \in B$ denote those markets where $c_{1k} > c_{2k}$. For the first subset A of markets, we will show that there is a natural choice of quantity for player one in every market. Namely, $q_{1k}^{+} = \max\{q_{1k}^{*}, a_k - c_{2k}\}$, where $q_{1k}^{*} = \frac{a_k - c_{1k}}{2}$ is player one's monopoly quantity. Clearly player one will never choose more than this as either (i) she is at her monopoly *and* player two can't enter or (ii) she is at a quantity that prevents player two from entering *and* increasing her quantity can only decrease her profit (since her profit is a concave quadratic). She will also never choose less than q_{1k}^{+} as she is either (i) at her monopoly quantity *and* preventing player two from entering or (ii) decreasing her quantity allows player two to enter the market with quantity $a_k - q_{1k} - c_{2k}$, resulting in player one selling fewer goods at a lower (or equal) price. Thus, in those markets A where player one is more competitive than player two, she will always enter at quantity q_{1k}^{+} and will always make a positive profit. Consequently, the optimal strategy for player one in these markets is clear. The problem, therefore, reduces to selecting quantities only in the subset B of markets where player one is less competitive.

So take a market $k \in B$. Then $q_{2k}^{-} = a_k - q_{1k} - c_{2k}$ always. Thus player one's profit, in the worst case is given by the linear function $q_{1k}(c_{2k} - c_{1k})$. So, again assuming that player two will first try to survive in every market and then try to undercut player one, the War Chest Minimization Problem for these markets is equivalent to the following quadratically constrained program:

$$\begin{aligned} \min\ & \textstyle\sum_{k \in B} q_{1k}(c_{1k} - c_{2k}) \\ \text{s.t.}\ & \textstyle\sum_{k \in B} (\frac{a_k - q_{1k} - c_{2k}}{2})^2 \leq B_2 \\ & 0 \leq q_{1k} \leq a_k - c_{1k} \end{aligned}$$

We can solve this convex program in polynomial time. The Bertrand case and Cournot case are similar, the former reduces to a linear program and the latter reduces to convex program, this time with a convex quadratic objective function. Thus we have shown the following.

Theorem 4. *In the absence of fixed costs and assuming linear cost, price, and demand functions, the War Chest Minimization Problem in the Cournot, Bertrand, and Stackelberg models can be solved in polynomial time.*

4.2 An Inapproximability Result

In this section, we will explore approximation algorithms for the War Chest Minimization Problem. A first inspection is disheartening for would-be predators, as demonstrated by the following theorem.

Theorem 5. *It is NP-hard to obtain any approximation algorithm for the War Chest Minimization Problem under the Bertrand, Stackelberg, and Cournot models.*

Proof. We prove this for the Stackelberg model - the other cases are similar. Let n, W, V, w_i, and v_i be an instance of the knapsack problem. Construct markets $m_1, ..., m_n$ exactly as in Theorem 3, with identical price functions, fixed costs, and marginal costs. Let W^* denote the optimal solution to the War Chest Minimization Problem in this case. Notice that $W^* > 0$ since all player one makes a negative profit in all of her markets. We now construct a new market m_{n+1} as follows. Let $P_{n+1}(q) = 2\sqrt{W^*} - q$ be the price function. Let player one's fixed and marginal costs be $c_{1,n+1} = f_{1,n+1} = 0$. Let player two's marginal cost be $c_{2,n+1} = 2\sqrt{W^*}$ and let her fixed cost be an arbitrary nonnegative value. Then player one will clearly enter the market and produce her monopoly quantity, $q_{1,n+1} = \sqrt{W^*}$, thereby forcing player two not to stay out of the market, by the Areeda-Turner rule. Thus player one will earn her monopoly quantity of W^* in this market. Consequently, the budget required for this War Chest Minimization Problem is zero. Any approximation algorithm would then have to solve this problem, and thereby the knapsack problem, exactly.

4.3 Additive Approximation Guarantees

Observe that the difficulty in obtaining a multiplicative approximation guarantee arises due to conflict between markets that generate a loss for player and markets that generate a profit. Essentially the strategic problem for player one is to partition the markets into two groups, α and β, and then conduct a price war in the markets in group α and try to gain revenue to fund this price war from markets in group β. This is still not sufficient because, in the presence of fixed costs, the optimal way to conduct a price war is not obvious even when the group α has been chosen. However, in this section we will show how to partition the markets and generate an arbitrarily small additive guarantee in the Bertrand and Stackelberg cases.

Given an optimal solution with optimal partition $\{\alpha^*, \beta^*\}$, let w_{α^*} be the absolute value of the sum of the profits of the markets with negative profit, and let w_{β^*} be the sum of the profits in positive profit markets. Then the optimal budget for player one is simply $OPT = w_{\alpha^*} - w_{\beta^*}$. For both the Bertrand and Stackelberg models, we will present algorithms that produce a budget of most $(1 + \epsilon)w_{\alpha^*} - w_{\beta^*}$, for any constant ϵ. Observe this can be expressed as $OPT + \epsilon w_{\alpha^*}$, and since w_{α^*} represents the actual cost of the price war (which takes place in the markets in α^*), our solution is then at most OPT plus epsilon times the optimal cost of fighting the price war. Let's begin with the Bertrand model.

Theorem 6. *There is an algorithm that solves the War Chest Minimization Problem for the Bertrand model within an additive bound of ϵw_{α^*}, and runs in time polynomial in the input size and $\frac{1}{\epsilon}$, assuming linear demand functions.*

Proof. We begin by proving that we can find the optimal partition $\{\alpha^*, \beta^*\}$ of the markets. Towards this goal we show that there is a optimal pricing scheme for any market, should player one choose to enter the market. Using this scheme we will be able to see which markets are revenue generating for player one and which are not. This will turn out to be sufficient to obtain $\{\alpha^*, \beta^*\}$. This is because, in the Bertrand model, player two cannot make a profit in a market if player one does and vice versa and because player one needs a strategy that maintains a non-negative budget even if player two acts maliciously (but legally).

The pricing scheme for player one should she choose to enter market m_k is $p^+_{1k} = \max\{c_{1k}, \min\{p^*_{1k}, c_{2k} - \gamma\}\}$, where γ is the minimum increment of price and p^*_{1k} is player one's monopoly price. Certainly, she should not price below p^+_{1k} as either (i) it is illegal by the Areeda-Turner rule or (ii) she cannot increase her profit by doing so (as the profit function for player one is a concave quadratic in p_{1k}). She also should not price above p^+_{1k}. If she did then either (i) she cannot increase her profit (due to concavity) or (ii) player two could undercut her or increase her own existing profits in the market. Indeed, it is certain that player two will try to undercut her if player one succeeds in keeping player two's budget negative.

Given that we have the optimal pricing scheme for player one, we may calculate the profit she could make on entering a market assuming that player two acts maliciously. Let α be the set of markets where she makes a negative profit under these conditions, and let β be the set of markets where she makes a non-negative profit. Since all markets in β give player one a non-negative profit even if player two is malicious, she will clearly always enter all of them. Consequently, as we are in the Bertrand model, player two cannot make any profit from markets in β. Thus by entering every market in β player one will earn w_β profit, and this must be the optimal for player one if the goal is to put player two out of business. So $\{\beta, \alpha\} = \{\beta^*, \alpha^*\}$ is an optimal partition.

It remains only to show that there is a fully polynomial time approximation scheme for the markets in α. We will prove this result by demonstrating an approximation preserving reduction of the War Chest Minimization Problem with only α-type Bertrand markets to the *Minimization Knapsack Problem.* Define w_k to be the negative of the profit earned by player one if she enters the market m_k and assuming player two undercuts if possible. By the above, she will price at $p_{1k} = p^+_{1k}$ and thus

$$w_k = \begin{cases} -(p^+_{1k} - c_{1k})D(p^+_{1k}) + f_{1k} & \text{if } c_{1k} < c_{2k} \\ f_{1k} & \text{otherwise.} \end{cases}$$

Recall that w_k is non-negative for markets in α. Let p^*_{2k} be player two's monopoly price in market m_k and let Π^*_{2k} be her monopoly profit in that market. We also let $v_k = \Pi^*_{2k} - \Pi_{2k}(p^+_{1k})$, where $\Pi_{2k}(p^+_{1k})$ is the maximum profit that player two can achieve in market m_k if player one enters and prices at p^+_{1k}. The

War Chest Minimization Problem is that of maximizing player one's profit (i.e. minimizing the negative of her profit) even if player two acts maliciously, while ensuring that player two's budget is always negative. So it can be expressed as

$$\begin{array}{ll} \min & \sum_k w_k y_k \\ \text{s.t.} & B_2 + \sum_k (\Pi^*_{2k}(1-y_k) + \Pi_{2k}(p^+_{1k}) \cdot y_k) \leq 0 \\ & y_k \in \{0,1\} \end{array}$$

Setting the constant C to be the sum of player two's budget and her monopoly profit in all of the markets, that is $C = B_2 + \sum_k \Pi^*_{2k}$, the problem can be rewritten as

$$\begin{array}{ll} \min & \sum_k w_k y_k \\ \text{s.t.} & \sum_k v_k y_k \geq C \\ & y_k \in \{0,1\} \end{array}$$

Finally, since the w_k are non-negative, this formulation is exactly the minimization knapsack problem. The reduction is approximation preserving and so we are done as there is a fully polynomial time approximation scheme for the minimization knapsack problem [20].

We now turn to the Stackelberg problem.

Theorem 7. *There is an algorithm that solves the War Chest Minimization Problem for the Stackelberg model within an additive bound of ϵw_{α^*}, and runs in time polynomial in the input size and $\frac{1}{\epsilon}$, assuming linear cost and price functions.*

Proof. As we have seen, there are two strategies that player two may employ to prevent player one from winning the price war. She may play so as to survive or, if that is destined to fail, she may play so as to leave player one with a negative budget. As in Section 4.1, define the quantity q^+_{2k} to be the quantity that maximizes player two's own profit in every market and q^-_{2k} to be the quantity that minimizes player one's profit (while obeying the Areeda-Turner rule). As before, though now adjusting for fixed costs, we get:

$$q^+_{2k} = \begin{cases} \frac{a_k - q_{1k} - c_{2k}}{2} & \text{if } (\frac{a_k - q_{1k} - c_{2k}}{2})^2 \geq f_{2k} \\ 0 & \text{otherwise} \end{cases}$$

$$q^-_{2k} = \begin{cases} a_k - q_{1k} - c_{2k} & \text{if } q_{1k} < a - c_{2k} \\ 0 & \text{otherwise.} \end{cases}$$

We initially split the markets of the Stackelberg case into two sets: let $k \in A$ denote the set of markets for which $c_{1k} \leq c_{2k}$ and let $k \in B$ denote those markets where $c_{1k} > c_{2k}$. In the former case, if player one enters the market m_k then she will necessarily produce quantity $q^+_{1k} = \max\{q^*_{1k}, a_k - c_{2k}\}$, where $q^*_{1k} = \frac{a_k - c_{1k}}{2}$ is player one's monopoly quantity. The argument for this is identical to that in Section 4.1, as the fixed costs here do not change anything. Let β be the set of all markets for which player one's worst case profit, $\Pi_{1k}(q^+_{1k}, q^-_{2k})$, is now nonnegative. Clearly she will enter all of these markets, and again $\beta = \beta^*$. Let

$\alpha = \alpha^*$ be the set of markets for which her worst case profit is negative. These include some of the markets in A and all of the markets in B, since player two as the follower can always force a price that is less than player one's marginal cost in these latter markets.

Again, player two will clearly enter each market in β^* and produce quantity q^+_{1k}, earning a positive profit of w_{β^*}. Thus, we need only find a fully polynomial time approximation scheme for the markets in α^*. So for the remainder of the proof, we will deal solely with the markets of α^*. By scaling, we may also assume that all of the variables and constants are integral.

As discussed above, the markets in $\alpha^* \cap A$ have a canonical choice of quantity for player one, q^+_{1k}. The worst case profit for player one in these markets will always be negative, by definition. Now let Π^*_{2k} be player two's monopoly profit in market m_k. Define V to be the sum of player two's monopoly profits in every market. Also define $v_k(q_{1k})$ to be the difference between player two's monopoly profit in market m_k and her maximum profit if player one enters the market with quantity q_{1k}. So, $v_k(q_{1k}) = \Pi^*_{2k} - \Pi_{2k}(q_{1k}, q^+_{2k})$. Notice that $v_k(q_{1k})$ is monotonically nondecreasing in q_{1k}.

Define $w_k(q_{1k})$ to be player one's worst case cost (negative profit) if she chooses to produce quantity q_{1k} in market m_k. For those markets where $c_{1k} \leq c_{2k}$, there is a natural strategy for player one and so $w_k(q_{1k}) = \Pi_{1k}(q^+_{1k}, q^-_{2k}) > 0$. For markets with $c_{2k} \leq c_{1k}$, we have $q^-_{2k} = a_k - q_{1k} - c_{2k}$ and so

$$w_k(q_{1k}) = \begin{cases} 0 & \text{if } q_{1k} = 0 \\ q_{1k}(c_{1k} - c_{2k}) + f_{1k} & \text{otherwise} \end{cases}$$

All of these weights are also non-negative.

The War Chest Minimization Problem requires player one minimize the cost of the markets she enters while keeping the sum of player two's budget and profits below zero. Since player two may "win" either by reducing player one's budget below zero or keeping her final budget nonnegative, player one needs to work with both her worst case costs and player two's best case profits. Thus the War Chest Minimization Problem, after some simple algebra, may be formulated as the problem of finding the integer vector $(q_{11}, q_{12}, ..., q_{1n})$ that solves

$$\begin{array}{ll} \min \sum_k w_k(q_{1k}) & \\ \text{s.t} \quad \sum_k v_k(q_{1k}) \geq V & \\ \qquad q_{1k} \in \{0, q^+_{1k}\} & \text{if } c_{1k} < c_{2k} \\ \qquad 0 \leq q_{1k} \leq a_k - c_{1k} & \text{if } c_{1k} \geq c_{2k}. \end{array}$$

The last constraint comes from the Areeda-Turner rule. We will refer to this problem as Stackelberg War Chest Minimization (SWCM). The weight of the vector $(q_{11}, q_{12}, ..., q_{1n})$ will mean $\sum_k w_k(q_{1k})$ and the value of the vector will mean $\sum_k v_k(q_{1k})$.

The remainder of this proof will be broken into parts. We first show that there is a pseudo-polynomial time dynamic program for SWCM. We then show how to using rounding techniques to obtain a polynomial time approximation scheme.

So let's describe the dynamic program. Let $\bar{W}$ be the maximum attainable weight. For each market m_i with $i \in \{1, ..., n\}$ and for each weight $w \in \{0, 1, ..., \bar{W}\}$, let $U_{i,w}$ denote the vector $(q_{11}, ..., q_{1n})$ such that $q_{1j} = 0$ for all $j > i$ which has total weight w and with the maximum value amongst all such vectors. Let $f(i, w)$ denote the value of $U_{i,w}$; if no such vector exists, then we set $f(i, w) = -\infty$. It is easy to calculate the base cases $f(1, w)$ for every w. We then get the recurrence:

$$f(i+1, w) = \max_{q_{1,i+1}} f(i, w - w_{i+1}(q_{1,i+1})) + v_i(q_{1,i+1})$$

where the maximum is taken over the feasible values of $q_{1,i+1}$, where we understand that $f(n, w) = -\infty$ for all $w < 0$. Thus we get a dynamic program that solves SWCM exactly and whose running time is polynomial in $n, \bar{W}$, and $a_k - c_{1k}$ for those markets k with $c_{1k} \geq c_{2k}$.

This dynamic program is pseudo polynomial. We can make it polynomial by a suitable scheme to round the quantities and to round the weights. Rounding the quantities, we shall try to make the running time depend on $\log(a_k - c_{1k})$ instead of $a_k - c_{1k}$, for those markets k with $c_{1k} \geq c_{2k}$. To do this, we will restrict the possible feasible choices of quantity, in each of these market, in the following manner. First fix some $\delta_0 > 0$. For each interval $I = [0, a_k - c_{1k}]$, partition it into the subintervals $I_0 = \{0\}, I_1 = \{1\}, I_2 = (1, 2], ..., I_i = (2^{i-2}, 2^{i-1}], ..., I_{\lceil \log(a_k - c_{1k}) + 1 \rceil} = (2^{\lceil \log(a_k - c_{1k}) - 1 \rceil}, a_k - c_{1k}]$. Each subinterval I_i, $i > 1$, is further partitioned into the minimum number of subintervals J_{ij} whose lengths are at most $\delta_0 2^{i-2}$. For each i, there are at most $\lceil \frac{1}{\delta_0} \rceil$ subintervals. For each quantity q_{1k} let $h_k(q_{1k})$ be the maximum value of the J_{ij} subinterval that contains q_{1k} (we define $h_k(0) = 0$ and $h_k(1) = 1$). Thus h_k maps the integer values of the interval $[0, a_k - c_{1k}]$ into a set of $O(\frac{1}{\delta_0} \log(a_k - c_{1k}))$ integers.

Now let $q = (q_{11}, ..., q_{1n})$ be any solution to SWCM. Since the objective function is linear, by replacing each q_{1k} with $h_k(q_{1k})$ we change the weight of the resulting vector by at most $\delta_0 w(q)$. By standard arguments, using these rounded quantities gives a $(1 + \delta_0)$ approximate algorithm whose running time is polynomial in $n, \frac{1}{\delta_0}, \bar{W}$ and $\log(a_k - c_{1k})$ for those markets k with $c_{1k} \geq c_{2k}$.

We can round the weights using a similar trick to obtain a $(1 + \epsilon)$ approximation algorithm for SWCM whose running time is polynomial in $n, \log(a_1 - c_{11}), ..., \log(a_n - c_{1n}), \frac{1}{\epsilon}$ and $\log(\bar{W})$. This completes the proof.

The approach taken here does not apply directly to the Cournot model. In particular, a more subtle rounding scheme is required there when player one is more competitive than player two. We conjecture, however, that a similar type of additive approximation guarantee is possible in the Cournot model.

5 Single Market Case

Clearly, our hardness results require that there be a large number of markets (or submarkets). Whilst the multimarket problem is the most interesting one in our opinion, we remark that hardness results can be obtained even in the single-market case, provided that each firm has a sufficient number of strategic choices available to it. For example, in the appendix, we introduce a very simple modified single market model, where firms are able to invest in themselves by increasing their fixed cost to decrease their marginal cost. Despite the simplicity of this model, the War Chest Minimization Problem is trivially hard, indicating that more complex and realistic single market models will typically also be hard.

References

1. D. Abreu, D. Pearce and E. Stacchetti, "Optimal cartel equilibria with imperfect monitoring", *Journal of Economic Theory*, **39**, pp251-69, 1986.
2. P. Areeda and D. Turner, "Predatory pricing and related issues under Section 2 of the Sherman Act", *Harvard Law Review*, **88**, pp997-733, 1975.
3. C. Bayer, "The other side of limited liability: predatory behavior and investment timing", *working paper*, 2004.
4. M. Balcan, A. Blum, H. Chan, M. Hajiaghayi, "A theory of loss-leaders: making money by pricing below cost", *Proceedings of 3rd International Workshop on Internet and Network Economics (WINE)*, **LNCS 4858**, pp293-299, 2007.
5. J. Benoit, "Financially constrained entry into a game with incomplete information", *RAND Journal of Economics*, **15**, pp490-499, 1984.
6. J. Bertrand, "Théorie mathématique de la richesse sociale", *Journal des Savants*, pp499-508, 1883.
7. R. Blair and J. Harrison, "Airline price wars: competition or predation", *Antitrust Bulletin*, **44(2)**, pp489-518, 1999.
8. D. Boudreaux, "Predatory pricing in the retail trade: the Wal-Mart case", in M. Coate and A. Kleit (eds.), *The Economics of the Antitrust Process*, Klewer, pp195-215, 1989.
9. M. Burns, "Predatory pricing and the acquisition cost of competitors", *Journal of Political Economy*, **94**, pp266-96, 1986.
10. M. Busse, "Firm financial condition and airline price wars", *RAND Journal of Economics*, **33(2)**, pp298-318, 2002.
11. A. Cournot, *Recherces sur les Principes Mathématiques de la Théorie des Richesse*, Paris, 1838.
12. J. Dalton and L. Esposito, "Predatory price cutting and Standard Oil: a re-examination of the trial record", *Research in Law and Economics*, **22**, pp155-205, 2007.
13. G. Ellison, "Theories of cartel stability and the joint executive committee", *RAND Journal of Economics*, **25**, pp37-57, 1994.
14. K. Elzinga and D. Mills, "Testing for predation: Is recoupment feasible?", *Antitrust Bulletin*, **34**, pp869-893, 1989.
15. K. Elzinga and D. Mills, "Price wars triggered by entry", *International Journal of Industrial Organization*, **17**, pp 179-198, 1999.
16. R. Ericson and A. Pakes, "Markov-perfect industry dynamics: A framework for empirical work", *The Review of Economic Studies*, **62(1)**, pp53-82, 1995.
17. C. Fershtman and A. Pakes, "A dynamic oligopoly with collusion and price wars", *RAND Journal of Economics*, **31(2)**, pp207-236, 2000.
18. F. Fisher, "The IBM and Microsoft cases: what's the difference?", *American Economic Review*, **90(2)**, pp180-183, 2000.

19. D. Fudenberg and J. Tirole, "A 'signal-jamming' theory of predation", *RAND Journal of Economics*, **17**, pp366-376, 1986.
20. G.V. Gens and E.V. Levner, "Computational complexity of approximation algorithms for combinatorial problems", in *Mathematical Foundations of Computer Science* (*Lecture Notes in Computer Science*, **74**), Springer, Berlin, pp292-300, 1979.
21. D. Genesove and W. Mullin, "Predation and its rate of return: the sugar industry 1887-1914", *RAND Journal of Economics*, **37(1)**, pp47-69, 2006.
22. E. Green and R. Porter, "Noncooperative collusion under imperfect price competition", *Econometrica*, **52**, pp87-100, 1984.
23. M. Güntzer and D. Jungnickel, "Approximate minimization algorithms for the 0/1 Knapsack and Subset-Sum Problem", *Operations Research Letters*, **26(2)**, pp55-66, 2000.
24. P. Klemperer, "Price wars caused by shifting costs", *Review of Economic Studies*, **56**, pp405-420, 1989.
25. B. Klein, "The Microsoft case: what can a dominant firm do to defend its market position", *Journal of Economic Perspectives*, **15(2)**, pp45-62, 2001.
26. B. Klein, "Did Microsoft engage in anticompetitive exclusionary behavior" *Antitrust Bulletin*, **46(1)**, pp71-113, 2001.
27. J. McGee, "Predatory price cutting: the Standard Oil (NJ) case", *Journal of Law and Economics*, **1**, pp137-169, October, 1958.
28. P. Milgrom and J. Roberts, "Predation, reputation, and entry deterrence", *Journal of Economic Theory*, **27**, pp280-312, 1982.
29. J. Ordover and G. Saloner, "Predation, monopolization, and anti-trust", in R. Schmalensee and R. Willig (eds.), *The Handbook of Industrial Organization*, North-Holland, pp537-596, 1989.
30. J. Podolny and F. Scott Morton, "Social status, entry and predation: the case of British shipping cartels 1879-1929", *Journal of Industrial Economics*, **47(1)**, pp41-67, 1999.
31. R. Porter, "On the incidence and duration of price wars", *The Journal of Industrial Economics*, **33(4)**, pp415-426, 1985.
32. A. Rao, M. Bergen and S. Davis, "How to fight a price war", *Harvard Business Review*, pp107-120, March/April, 2000.
33. J. Rotemberg and G. Soloner, "A supergame-theoretic model of price wars during booms", *American Economic Review*, **76(3)**, pp390-407, 1986.
34. R. Selten, "The chain store paradox", *Theory and Decision*, **9**, pp127-159, 1978.
35. F. Scherer, *Industrial Market Structure and Economic Performance*, Houghton Miffin, 1990.
36. F. Scott Morton, "Entry and predation: British shipping cartels 1879-1929", *Journal of Economics and Management Strategy*, **6**, pp679-724, 1997.
37. M. Slade, "Strategic pricing models and interpretation of price war data", *European Economic Review*, **34**, pp524-537, 1990.
38. H. von Stackelberg, *Marktform und Gleichgewicht*, Julius Springer, Vienna, 1934.
39. G. Stigler, "A theory of oligopoly", *Journal of Political Economy*, **72(1)**, pp44-61, 1964.
40. J. Tirole, *The Theory of Industrial Organization*, MIT Press, 1988.
41. L. Telser, "Cutthroat competition and the long purse", *Journal of Law and Economics*, **9**, pp259-277, 1966.
42. A. Vetta, "Nash equilibria in competitive societies with applications to facility location, traffic routing and auctions", *Proceedings of the 43rd Symposium on the Foundations of Computer Science (FOCS)*, pp416-425, 2002.

43. G. Weintraub and C.L. Benkard and B.V. Roy, "Markov perfect industry dynamics with many firms", *Econometrica*, **76(6)**, pp1375-1411, 2008.

A Single Market Case

Suppose player one and player two are competing in a single Bertrand market. Player two has a certain marginal cost c_2. Player one begins with a marginal cost c_1. However, she may choose to invest in any subset of n technologies each of which will cost her a fixed cost f_i but will reduce her marginal cost by λ_i. Suppose player one begins with a budget B_1 and may not spend more than this budget in technology investment. Player one wins the market from player two if she can reduce her marginal cost c_1 to below player two's c_2 within her budget constraints. This produces the problem:

Single Market War Chest Minimization Problem: *If the initial c_1 and c_2 are fixed, what is the minimum budget B_1 that player one needs so that she can win the market from player two?*

Theorem 8. *This problem is NP-hard but has a fully polynomial time approximation scheme.*

Proof. We prove the theorem by showing that this problem is completely equivalent to the minimization knapsack problem in an approximization preserving way. Notice that the problem can be formulated as

$$\begin{aligned} &\min \textstyle\sum_i f_i x_i \\ &\text{s.t. } c_1 - \textstyle\sum_i \lambda_i x_i < c_2 \\ &\quad\; x_i \in \{0,1\} \end{aligned}$$

But then if we write $v_i = f_i, w_i = \lambda_i, C = c_1 - c_2$ then we have reduced the problem to the minimization knapsack problem as seen in Section 4.3. This reduction clearly preserves approximation.