Math 340: Discrete Structures II

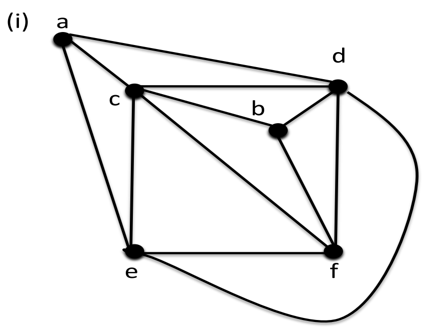
Practice Exam 2: Solutions

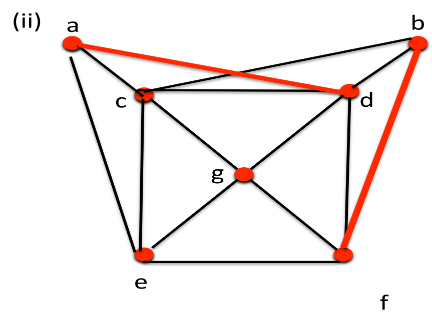
1. **Graph Theory.**

   (a) *State Kuratowski's theorem.*

   A graph is planar if and only if it contains no K_3 nor $K_{5,5}$ minor.

   (b) *Explain whether or not each of the following two graphs is planar.*





The first graph is planar, the second is not. A planar drawing of the first graph is shown, as is a $K_{5,5}$ minor of the second graph.

2. **Graph Theory.**

(a) *State Hall's Theorem.*

A bipartite graph with $|X| = |Y|$ is planar if and only if $|\Gamma(A)| \geq |A|$ for all $A \subseteq X$.

(b) *Let $\pi_1\pi_2 \cdots \pi_n$ be a permutation of the numbers $\{0, 1, \ldots, n-1\}$. Suppose that the degree of x_i is π_i for each $1 \leq i \leq n$. Does G contain a perfect matching?*

No, because some vertex has degree zero!

(c) *Let $\pi_1\pi_2 \cdots \pi_n$ be a permutation of the numbers $\{1, 2, \ldots, n\}$. Suppose that the degree of x_i is π_i for each $1 \leq i \leq n$. Does G contain a perfect matching?*

Yes. Take any $A \subseteq X$, wlog $A = \{x_1, x_2, \ldots, x_r\}$. Each vertex has a distinct degree in $\{1, 2, \ldots, n\}$, so at least one vertex $x_j \in A$ has degree at least r. So

$$|\Gamma(A)| \geq |\Gamma(\{x_j\})| \geq r = |A|$$

So Halls condition in satisfied.

(d) *Let $\pi_1\pi_2 \cdots \pi_n$ be a permutation of the numbers $\{n+1, n+2, \ldots, 2n\}$. Suppose that the degree of x_i is $(\pi_i - i)$ for each $1 \leq i \leq n$. Does G contain a perfect matching?*

Yes. Observe that the maximum degree of any node is n. But $2n - i > n$ unless $i = n$. So it must be the case that $\pi_n = 2n$ and $deg(x_n) = n$. But then it must be the case that $\pi_{n-1} = 2n - 1$ and so $deg(x_{n-1}) = (2n-1) - (n-1) = n$, etc. So the graph is n-regular and contains a perfect matching (in fact it can be decomposed into n disjoint perfect matchings).

3. **Probability.**

(a) i. *State Bayes Theorem.*

For any two events A and B:

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

ii. *A large company gives a new employee a drug test. The False-Positive rate is 1% and the False-Negative rate is 1%. In addition, 1% of the population use the drug. The employee tests positive for the drug. What is the probability the employee uses the drug?*

Let T be the event an employee tests positive, and let U be the event they are a drug-user. So we want to calculate $P(U|T)$. Then $P(U) = 0.01$, $P(T|\bar{U}) = 0.01$. In addition, $P(\bar{T}|U) = 0.01$ so $P(T|U) = 0.99$. Applying Bayes rule,

$$\begin{aligned} P(U|T) &= \frac{P(T|U) \cdot P(U)}{P(T)} \\ &= \frac{P(T|U) \cdot P(U)}{P(T|U) \cdot P(U) + P(T|\bar{U}) \cdot P(\bar{U})} \\ &= \frac{\frac{99}{100} \cdot \frac{1}{100}}{\frac{99}{100} \cdot \frac{1}{100} + \frac{1}{100} \cdot \frac{99}{100}} \\ &= \frac{1}{2} \end{aligned}$$

(b) *Suppose I roll an n-sided die once. Now you repeatedly roll the die until you roll a number* at least *as large as I rolled. What is the expected number of rolls you have to make?*

With probability $\frac{1}{n}$ I roll an i. If I roll i the the probability that you get at least that much in any roll is $\frac{n-i+1}{n}$. Thus, the expected number of rolls you require in that instance $\frac{n}{n-i+1}$. So your expected number of rolls is

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{n}{n-i+1} &= \sum_{i=1}^{n} \frac{1}{n-i+1} \\ &= \sum_{i=1}^{n} \frac{1}{i} \\ &= H_n \end{aligned}$$

4. **Probability.**

(a) i. *State Boole's Inequality.*

ii. *State the Chernoff bound.*

Let $X_1, X_2, \dots X_n$ be independent poisson trials with $P(X_i = 1) = p_i$. If $X = \sum_i X_i$ and $\mu = E(X)$ then

$$P(X > (1+\delta)\mu) \;\le\; \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu} \qquad \forall \delta > 0$$

There are several other variations. The following is useful for part (b)

$$P(X < (1-\delta)\mu) \;\le\; e^{-\frac{1}{2}\mu\delta^2} \qquad \forall 1 > \delta > 0$$

(b) *In a* random graph*, for each pair of vertices i and j, we independently include the edge (i,j) in the graph with probability $\frac{1}{2}$.*

i. *Prove that, with high probability, every vertex in a random graph has degree at least $\frac{1}{2}n - 3\sqrt{n\ln n}$, where $n+1$ is the number of vertices.*

Here $\mu = \frac{1}{2}n$ as a vertex has $(n+1)-1$ potential neighbours. By Chernoff

$$\begin{aligned} P(X < \frac{1}{2}n - 3\sqrt{n\ln n}) &= P(X < \mu - 6\mu\frac{\sqrt{\ln n}}{\sqrt{n}}) \\ &= P(X < (1 - 6\frac{\sqrt{\ln n}}{\sqrt{n}})\mu) \\ &\le e^{-\frac{1}{2}\mu 36\frac{\ln n}{n}} \\ &= e^{-9\ln n} \\ &= n^{-9} \end{aligned}$$

By Boole's inequality every vertex has at least this degree with probability $(n+1)\frac{1}{n^9} \approx n^{-8}$.

ii. *The distance between a pair of vertices i and j is the length of the shortest path between them. The diameter of a graph is the maximum distance between any pair of vertices. Prove that, with high probability, a random graph has diameter 2.*

Take any pair i and j. There are $n-1$ other vertices $v_1, v_2, \dots, v_{n-1}$. These induce $n-1$ disjoint paths $P_1, P_2, \dots, P_{n-1}$ of length 2, that is paths of the form $P_k = \{i, v_k, j\}$. Each independently appears in the random graph with probability $\frac{1}{4}$. The lenth one path (i,j) isSo with probability at most $\frac{1}{2}\cdot(\frac{3}{4})^{n-1} \le (\frac{3}{4})^n$ we have i and j more than distance two apart. There are less than n^2 vertex pairs so by Booles inequality the diameter is at most 2 with probability at most a tiny

$$n^2 \cdot (\frac{3}{4})^n$$

5. **Combinatorics.**

(a) i. *Use the Binomial Theorem to prove the following identity:*

$$3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

The Binomial Theorem is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Plugging in $x = 2$ and $y = 1$ gives the result.

ii. *Give a combinatorial proof.*

The LHS counts the number of ways n elements can be coloured using three colours red/green/blue.

So does the RHS: $\binom{n}{k}$ is the number of ways to pick the k elements that will **not** be coloured blue. 2^k then is the number of ways to colour these red or green. Summing over all k thus also enumerates all possible 3-colourings.

(b) i. *Give an algebraic proof of the following identity:*

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

This follows because

$$\begin{aligned} k\binom{n}{k} &= k\frac{n!}{k!(n-k)!} \\ &= \frac{n \cdot (n-1)!}{(k-1)!(n-k)!} \\ &= n\binom{n-1}{k-1} \end{aligned}$$

ii. *Give a combinatorial proof.*

The LHS counts the number of ways to pick k elements from an n-set and to colour one of them red.

So does the RHS. First pick one of the n elements to be in the k-set and colour it red. Next pick another $(k-1)$ elements from amongst the remaining $(n-1)$ elements.

6. **Combinatorics.**

Consider strings of length n that use the digits $\{0, 1, 2\}$. Let $f(n)$ be the number of such strings that contain an even *number of* 0*s.*

(a) *Prove that $f(n)$ satisfies the recurrence relation $f(n) = f(n-1) + 3^{n-1}$.*

Let $g(n)$ be the number of such sequences with an odd number of 0s. Then

$$\begin{aligned} f(n) &= 2 \cdot f(n-1) + 1 \cdot g(n-1) \\ &= 2 \cdot f(n-1) + 1 \cdot (3^{n-1} - f(n-1)) \\ &= f(n-1) + 3^{n-1} \end{aligned}$$

To see this, either 1 or 2 can be the nth digit if the first $n-1$ digits contain an even number of zeros, but only 0 can be the nth digit if the first $n-1$ digits contain an odd number of zeros.

(b) *Use the recurrence to find the ordinary generating function $F(x)$.*

The base cases are $f(0) = 1$ and $f(1) = 2$. Then

$$\begin{aligned} F(x) &= \sum_{n\geq 0} f(n)x^n \\ &= 1 + \sum_{n\geq 1} f(n)x^n \\ &= 1 + \sum_{n\geq 1} (f(n-1) + 3^{n-1})x^n \\ &= 1 + \sum_{n\geq 1} f(n-1)x^n + \sum_{n\geq 1} 3^{n-1}x^n \\ &= 1 + x\sum_{n\geq 0} f(n)x^n + x\sum_{n\geq 0} 3^n x^n \\ &= 1 + xF(x) + \frac{x}{1-3x} \end{aligned}$$

Rearranging gives

$$F(x) = \frac{1-2x}{(1-x)(1-3x)}$$

(c) *Use the generating function to obtain a closed formula for $f(n)$.*

By partial fractions

$$\begin{aligned} F(x) &= \frac{1-2x}{(1-x)(1-3x)} \\ &= \frac{A}{1-x} + \frac{B}{1-3x} \\ &= \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1-3x} \end{aligned}$$

This follows as $A(1-3x)+B(1-x) = 1-2x$ and so $A+B = 1$ and $3A+B = 2$. Thus

$$F(x) \quad = \quad \frac{1}{2}(1+x+x^2+\cdots)+\frac{1}{2}(1+3x+(3x)^2+\cdots)$$

So

$$f(n) = \frac{1}{2}(1+3^n)$$