Math 340: Discrete Structures II

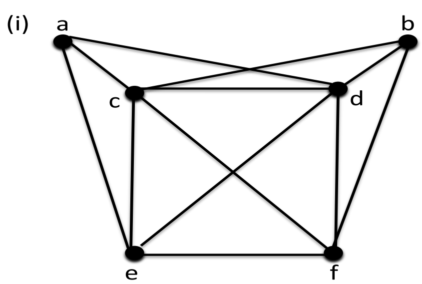
Practice Exam 2

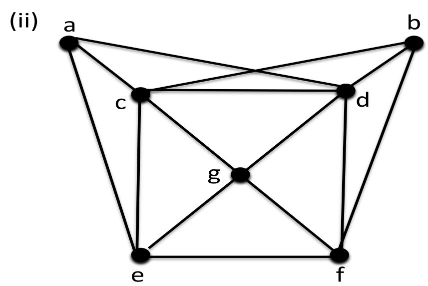
Instructions. The exam is 3 hours long and contains 6 questions. Write your answers <u>*clearly*</u> *in the notebook provided. You may quote any result/theorem seen in the lectures or in the assignments without proving it (unless, of course, that is what you are asked to prove).*

1. **Graph Theory.**

   (a) State Kuratowski's theorem. [2 marks]

   (b) Explain whether or not each of the following two graphs is planar. [8 marks]





2. **Graph Theory.**

   Take a bipartite graph $G = (V, E)$ where the two parts of V in the bipartition are $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

   (a) State Hall's Theorem. [1 mark]

   (b) Let $\pi_1\pi_2 \cdots \pi_n$ be a permutation of the numbers $\{0, 1, \ldots, n-1\}$. Suppose that the degree of x_i is π_i for each $1 \le i \le n$. Does G contain a perfect matching? [3 marks]

   (c) Let $\pi_1\pi_2 \cdots \pi_n$ be a permutation of the numbers $\{1, 2, \ldots, n\}$. Suppose that the degree of x_i is π_i for each $1 \le i \le n$. Does G contain a perfect matching? [3 marks]

   (d) Let $\pi_1\pi_2 \cdots \pi_n$ be a permutation of the numbers $\{n+1, n+2, \ldots, 2n\}$. Suppose that the degree of x_i is $(\pi_i - i)$ for each $1 \le i \le n$. Does G contain a perfect matching? [3 marks]

3. **Probability.**

   (a) i. State Bayes Theorem. [1 mark]

   ii. A large company gives a new employee a drug test. The False-Positive rate is 1% and the False-Negative rate is 1%. In addition, 1% of the population use the drug. The employee tests positive for the drug. What is the probability the employee uses the drug? [4 marks]

   (b) Suppose I roll an n-sided die[1] once. Now you repeatedly roll the die until you roll a number *at least* as large as I rolled. What is the expected number of rolls you have to make? [5 marks]

4. **Probability.**

   (a) i. State Boole's Inequality. [1 mark]

   ii. State the Chernoff bound. [2 marks]

   (b) In a *random graph*, for each pair of vertices i and j, we independently include the edge (i, j) in the graph with probability $\frac{1}{2}$.

   i. Prove that, with high probability, every vertex in a random graph has degree at least $\frac{1}{2}n - 3\sqrt{n \ln n}$, where n is the number of vertices. [3 marks]

   ii. The *distance* between a pair of vertices i and j is the length of the shortest path between them. The *diameter* of a graph is the maximum distance between any pair of vertices. Prove that, with high probability, a random graph has diameter 2. [4 marks]

[1] That is, its faces have the values $\{1, 2, \ldots, n\}$.

5. **Combinatorics.**

   (a) i. Use the Binomial Theorem to prove the following identity: [2 marks]

$$3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

   ii. Give a combinatorial proof. [3 marks]

   (b) i. Give an algebraic proof of the following identity: [2 marks]

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

   ii. Give a combinatorial proof. [3 marks]

6. **Combinatorics.**
   Consider strings of length n that use the digits $\{0, 1, 2\}$. Let $f(n)$ be the number of such strings that contain an *even* number of 0s.

   (a) Prove that $f(n)$ satisfies the recurrence relation [4 marks]

$$f(n) = f(n-1) + 3^{n-1}$$

   (b) Use the recurrence to find the ordinary generating function $F(x)$. [3 marks]

   (c) Use the generating function to obtain a closed formula for $f(n)$. [3 marks]