Math 340: Discrete Structures II

Practice Exam 1: Solutions

1. *Matchings.*

   (a) *What is a perfect matching?*

   A matching M is a set of vertex-disjoint edges. The matching is perfect if every vertex in the graph in incident to some edge in the matching.

   (b) *True or false: a non-bipartite graph cannot contain a perfect matching.*

   False. For example K_n clearly has a perfect matching if n is even.

   (c) *Take a bipartite graph $G = (V, E)$ where the two parts of V in the bipartition are X and Y, where $|X| = |Y|$.*

   i. *State Hall's Theorem.*

   The bipartite graph has a perfect matching if and only if $\Gamma(A) \geq |A|$ for all $A \subseteq X$, where $\Gamma(A)$ is the neighbourhood of A.

   ii. *Prove Hall's Theorem.*

   If there is a perfect matching then clearly $\Gamma(A) \geq |A|$ for all $A \subseteq X$.

   So suppose Hall's condition is satisfied and take a maximum matching M. Take an unmatched vertex $x_1 \in X$. If By Hall's condition it has at least one neighbour $y_1 \in Y$. Now y_1 is matched to some vertex x_2 otherwise M is not maximum. Now iterating, we have sets $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_{k-1}\}$, where $(y_i, x_{i+1}) \in M$ for all $1 \leq i \leq k-1$, and where there is an alternating path from x_1 to x_j, for all $1 \leq j \leq k$, By Hall's condition, $\{x_1, \ldots, x_k\}$ has at least one neighbour not in $\{y_1, \ldots, y_{k-1}\}$. Call this y_k. Now y_k is matched to some vertex x_{k+1} otherwise we have an augmenting path from x_1 to y_k and M is not maximum. Thus we have an alternating path from x_1 to x_{k+1}. But the graph is finite so this process cannot go on forever. Therefore there could be no unmatched vertex x_1 and M is a perfect matching.

2. *Planar Graphs.*

(a) i. *State Kuratowski's theorem.*

A graph is planar if and only if it contains no K_5 nor $K_{3,3}$ minor.

ii. *State Euler's formula.*

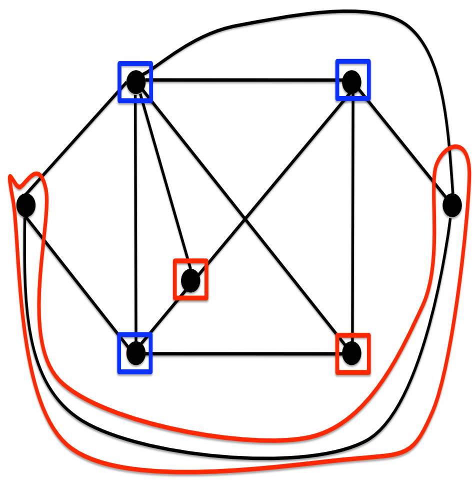
A connected planar graph satisfies $m + 2 = n + f$, where f is the number of faces.

(b) *Prove that any graph with at most eight edges is planar.*

A graph with a K_5 minor has at least 10 edges and a graph with a $K_{3,3}$ minor has at least 9 edges.

(c) *Explain whether or not the following graph is planar.*

It is not planar. A $K_{3,3}$ minor is shown below.



3. *Probability.*

(a) *Two dice are rolled. At least one of the dice is a* 6*. What is the probability that the sum of the dice is* 8*?*

Let A be the event that the sum of the dice is 8 and let be B the event that at least one of the die is 6. Then

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{2}{36}}{\frac{11}{36}} \\ &= \frac{2}{11} \end{aligned}$$

(b) *We repeatedly roll a fair die until any number appears twice in a row. What is the expected number of rolls until we stop?*

After the first roll, we stop with probability $\frac{1}{6}$ at each roll. This is the **geometric distribution** so we stop after $1 + 6 = 7$ rolls in expectation.

Formally

$$\begin{aligned} E(R) &= \sum_{i \geq 2} i \cdot P(R = i) \\ &= \sum_{i \geq 2} i \cdot (\frac{5}{6})^{i-2} \cdot \frac{1}{6} \\ &= \frac{1}{6}\frac{6}{5} \sum_{i \geq 2} i \cdot (\frac{5}{6})^{i-1} \\ &= \frac{1}{5} \cdot \left(\sum_{i \geq 1} i \cdot (\frac{5}{6})^{i-1} - 1 \right) \\ &= \frac{1}{5} \cdot \left(\sum_{i \geq 1} i \cdot (\frac{5}{6})^{i-1} - 1 \right) \\ &= \frac{1}{5} \cdot \left(\frac{1}{\frac{1}{6}^2} - 1 \right) \\ &= 7 \end{aligned}$$

Recall here that $\sum_{n \geq 1} n \cdot x^{n-1} = \frac{1}{(1-x)^2}$.

• A simpler proof comes from the fact that for a non-negative integral random variable R we have (from class)

$$\begin{aligned} E(R) &= \sum_{i \geq 1} P(R \geq i) \\ &= 1 + \sum_{i \geq 2} P(R \geq i) \end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{i \geq 2} (\frac{5}{6})^{i-2} \\
&= 1 + \sum_{i \geq 0} (\frac{5}{6})^{i} \\
&= 1 + \frac{1}{1 - \frac{5}{6}} \\
&= 7
\end{aligned}
$$

(c) *A coin is tossed until the first head appears. You win 2^n dollars if the first head appears on the nth toss.*

i. *What are the expected winnings if you play the game?*

$$
\begin{aligned}
E(W) &= \sum_{i \geq 1} 2^i \cdot P(First\ head\ at\ time\ i) \\
&= \sum_{i \geq 1} 2^i \cdot \frac{1}{2^{i-1}} \cdot \frac{1}{2} \\
&= \sum_{i \geq 1} 1 \\
&= \infty
\end{aligned}
$$

ii. *Are you willing to pay this amount to play the game?*
No!

4. *Probability.*

   (a) *State the Chernoff bound.*

   Let $X_1, X_2, \ldots X_n$ be independent poisson trials with $P(X_i = 1) = p_i$. If $X = \sum_i X_i$ and $\mu = E(X)$ then

$$P(X > (1+\delta)\mu) \;\leq\; \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu \qquad \forall \delta > 0$$

   (b) *Suppose we have n boxes and we start randomly and independently throwing balls into the boxes.*

   i. *If we throw exactly n balls, give an upper bound on the probability that Box #1 contains more than $2 \ln n$ balls.*

   We also have a simple Chernoff bound of

$$P(X > b) \;\leq\; 2^{-b} \qquad \forall b \geq 6\mu$$

   Clearly as $\mu = 1$ we have $2 \ln n \geq 6\mu$ we can use this. Thus $P(X > 2 \ln n) \leq 2^{-2 \ln n} = 4^{-\ln n} \approx n^{-1.39}$. [Stronger bounds can be obtained with the original Chernoff bound.]

   ii. *What is the expected number of balls we need to throw before* every *bin contains at least one ball?*

   This classical problem is known as the *Coupon Collector's Problem.* We will solve it using geometric distributions. Denote by N_i the number of balls taken to go from having $i-1$ non-empty bins to i non-empty bins. That is, if it took k balls to hit $i-1$ bins, it took $k + N_i$ balls to hit one more bin. Let N denote the number of balls thrown before every bin is non-empty. Then $N = N_1 + N_2 + ... + N_n$ throws have hit every bin. But how are our N_i distributed? If we have $i-1$ bins hit, then the probability of hitting an empty bin is $1-(i-1)/n$. So N_i is geometrically distributed with parameter $p = 1-(i-1)/n$. Then we have $E(N_i) = 1/(1-(i-1)/n)$ and by linearity of expectation

$$E(N) = \sum_{i=1}^{n} \frac{1}{1-(i-1)/n} = n \sum_{i=1}^{n} \frac{1}{n-(i-1)} = n \sum_{i=1}^{n} \frac{1}{i} = nH_n$$

   where we reversed the order of summation in the second to last equality and we denote the nth harmonic number, $\sum_{i=1}^{n} 1/i$, by H_n. Note $H_n = \Theta(\ln(n))$.

5. *Combinatorics.*

(a) *State the binomial theorem.*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

or

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(b) *Consider the equality*

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

i. *Prove this equality using the binomial theorem.*
Just plug in $x = y = 1$.

ii. *Prove it combinatorially.*
There are 2^n subsets of an n-set. This is the LHS. There are $\binom{n}{k}$ subsets of cardinality k in an n-set. Summing over k gives the RHS.

(c) *Prove combinatorially that, for $n \geq 1$,*

$$\sum_{0 \leq k \leq n:\ k \,\mathrm{odd}} \binom{n}{k} = \sum_{0 \leq k \leq n:\ k \,\mathrm{even}} \binom{n}{k}$$

Take an n-set and pick an arbitrary element x. We create a bijection between odd and even subsets as follows. Let S be a subset. If $x \in S$ then $f(S) = S - x$; if $x \notin S$ then $f(S) = S + x$.

6. *Combinatorics.*

(a) *How many different ways are there to make up 22 cents using coins of denomination 1, 5 and 10 cents?*

There are nine ways:

- Two 10s and two 1s.
- One 10, and twelve 1s.
- One 10, one 5, and seven 1s.
- One 10, two 5s, and two 1s.
- One 5 and seventeen 1s.
- Two 5s and twelve 1s.
- Three 5s and seven 1s.
- Four 5s, and two 1s.
- Twenty two 1s.

(b) *Let $f(n)$ be the number of ways to make up n cents using coins of denomination 1, 5 and 10 cents if we can use at most four 1 cent coins. Give the ordinary generating function $F(x)$.*

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} f(n)x^n \\
&= (1+x+x^2+x^3+x^4)(1+(x^5)+(x^5)^2+(x^5)^3+\cdots)(1+(x^{10})+(x^{10})^2+(x^{10})^3+\cdots) \\
&= (1+x+x^2+x^3+x^4)\cdot\frac{1}{(1-x^5)}\cdot\frac{1}{(1-x^{10})} \\
&= \frac{(1-x^5)}{1-x}\cdot\frac{1}{(1-x^5)}\cdot\frac{1}{(1-x^{10})} \\
&= \frac{1}{1-x}\cdot\frac{1}{1-x^{10}}
\end{aligned}$$

(c) *Using b) or otherwise, obtain a simple expression for $f(n)$.*

So

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} f(n)x^n \\
&= \frac{1}{1-x}\cdot\frac{1}{1-x^{10}} \\
&= (1+x+x^2+x^3\cdots)\cdot(1+x^{10}+x^{20}+x^{30}\cdots)
\end{aligned}$$

But this is the same as the number of ways to get n cents using coins of denomination 1 and 10 cents. And this is just the number ways to pick 10 cents so that we have a total value at most n cents (as the number of 1 cent coins is then determined for us). Thus $f(n) = 1 + \lfloor\frac{n}{10}\rfloor$.