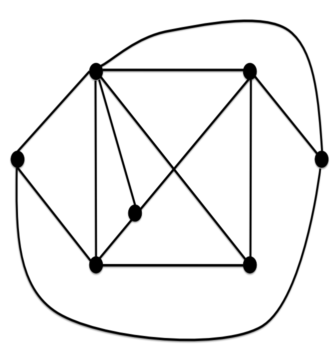
Math 340: Discrete Structures II

Practice Exam 1

Instructions. The exam is 3 hours long and contains 6 questions. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures or in the assignments without proving it (unless, of course, that is what you are asked to prove).

1. *Matchings.*
   (a) What is a perfect matching?
   (b) True or false: a non-bipartite graph cannot contain a perfect matching.
   (c) Take a bipartite graph $G = (V, E)$ where the two parts of V in the bipartition are X and Y, where $|X| = |Y|$.
      i. State Hall's Theorem.
      ii. Prove Hall's Theorem.

2. *Planar Graphs.*
   (a) i. State Kuratowski's theorem.
      ii. State Euler's formula.
   (b) Prove that any graph with at most eight edges is planar.
   (c) Explain whether or not the following graph is planar.



3. *Probability.*

   (a) Two dice are rolled. At least one of the dice is a 6. What is the probability that the sum of the dice is 8?

   (b) We repeatedly roll a fair die until any number appears twice in a row. What is the expected number of rolls until we stop?

   (c) A coin is tossed until the first head appears. You win 2^n dollars if the first head appears on the nth toss.

      i. What are the expected winnings if you play the game?

      ii. Are you willing to pay this amount to play the game?

4. *Probability.*

   (a) State the Chernoff bound.

   (b) Suppose we have n boxes and we start randomly and independently throwing balls into the boxes.

      i. If we throw exactly n balls, give an upper bound on the probability that Box #1 contains more than $2 \ln n$ balls.

      ii. What is the expected number of balls we need to throw before *every* bin contains at least one ball?

5. *Combinatorics.*

   (a) State the binomial theorem.

   (b) Consider the equality

   $$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

      i. Prove this equality using the binomial theorem.

      ii. Prove it combinatorially.

   (c) Prove combinatorially that, for $n \geq 1$,

   $$\sum_{0 \leq k \leq n:\ k\,\mathrm{odd}} \binom{n}{k} = \sum_{0 \leq k \leq n:\ k\,\mathrm{even}} \binom{n}{k}$$

6. *Combinatorics.*

(a) How many different ways are there to make up 22 cents using coins of denomination $1, 5$ and 10 cents?

(b) Let $f(n)$ be the number of ways to make up n cents using coins of denomination $1, 5$ and 10 cents if we can use at most four 1 cent coins. Give the ordinary generating function $F(x)$.

(c) Using b) or otherwise, obtain a simple expression for $f(n)$.