Math 340: Discrete Structures II

Midterm Exam : Solutions

1. *Matchings. Take a bipartite graph $G = (V, E)$ where the two parts of V in the bipartition are X and Y, where $|X| = |Y| = n$.*

   (a) *State Hall's Theorem.*

   A bipartite graph $G = (V, E)$ with $|X| = |Y|$ has a perfect matching **if and only if** $|\Gamma(A)| \geq |A|$ for all $A \subseteq X$, where $\Gamma(A)$ is the neighbour set of A.

   (b) *Let the bipartite graph G be connected and have maximum degree 2. Explain why (without using Hall's Theorem) G must contain a perfect matching.*

   As seen in class, any graph with maximum degree degree two consists just of a disjoint collection of paths and cycles. But G here is **connected** so has only one component. Thus G is a single cycle/path containing all the vertices (i.e. either a Hamiltonian Cycle or a Hamiltonian Path). Now G has an even number of vertices, $2n$. So if G is a cycle it consists of two disjoint perfect matchings. If G is a path, then taking alternating edges along the path will give a perfect matching. ■

   (c) *Now prove the result in (b) using Hall's Theorem.*

   We need to show that Hall's condition holds. So take any $A \subseteq X$. Again, the graph induced by $A \cup \Gamma(A)$ consists of a collection of disjoint paths $P_1, P_2, \ldots, P_k$. It cannot contain any cycle C otherwise the cycle C is disconnected from the rest of the graph, contradicting the connectivity of G. We claim that each path P_i has at least one endpoint in Y. If not, because A has all its neighbours in $\Gamma(A)$ the path P_i is disconnected from the rest of the graph. Thus P_i has an endpoint $y_i \in Y \cap \Gamma(A)$. Thus P_i contains at least as many vertices in Y as in X. It follows that $|\Gamma(A)) \geq |A|$. ■

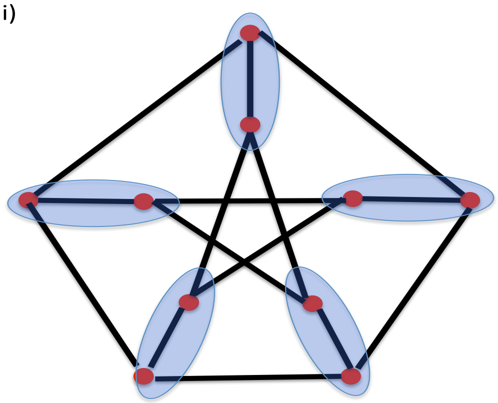
2. *Planar Graphs.*

   (a) *State Kuratowski's theorem.*

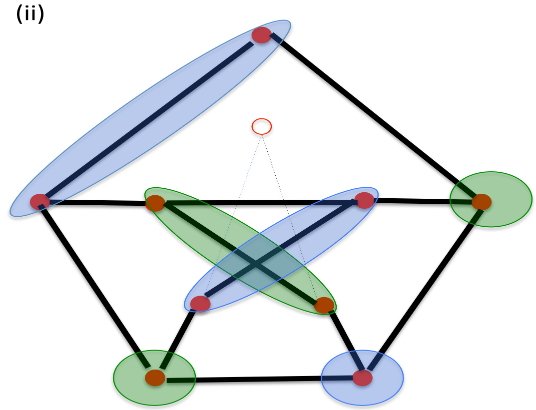
   A graph is planar if and only if it contains no K_3 nor $K_{5,5}$ minor.

   (b) *Explain whether or not each of the following two graphs is planar.*

(i)



(ii)



Neither graph is planar. A K_5 minor is shown in the first graph and $K_{3,3}$ minor is shown in the second graph.

3. *Planar Graphs.*

(a) *State Euler's Formula for planar graphs.* Let $G = (V, E)$ be a **connected** planar graph, where $n = |V|$, $m = |E|$, and f be the number of faces on the graph. Then $n + f = m + 2$.

(b) *Use Euler's Formula to give an upper bound on the number of edges (in terms of the number of vertices) in G.*

We repeat the argument from class. We may assume the graph is connected and that $n \geq 5$. So every face has at least 3 edges. Furthermore, every edge touches at most two faces. Thus, $2m \geq 3f$ and so $f \leq \frac{2}{3}m$. Plugging this into Euler's Formula we get $n - 2 = m - f \geq \frac{1}{3}m$. This gives $m \leq 3n - 6$. ■

(c) *Prove that at least one of G or $\bar{G}$ is not planar if $|V| = n \geq 11$.*

Observe that G and $\bar{G}$ are edge-disjoint and that their union is the complete graph K_n. Thus, the total number of edges in G plus the number of edges in $\bar{G}$ is exactly $\binom{n}{2} = \frac{1}{2}n \cdot (n-1)$. By (b), we know that if both G and $\bar{G}$ are planar they have at most $3n - 6$ edges each. So if they are both planar the total number of edges between them is at most $6n - 12$. Thus it must be that

$$\begin{aligned} \frac{1}{2}n \cdot (n-1) &\leq 6n - 12 \\ n^2 - n &\leq 12n - 24 \\ n^2 &\leq 13n - 24 \\ n^2 - 13n + 24 &\leq 0 \end{aligned}$$

Clearly this is not true if $n \geq 13$. It is also easy to check this not true for any $n \geq 11$. (For example, find the roots of the quadratic.) Thus at least one of them is not planar. ■