Revenue Equivalence Theorem (RET)

Definition

Consider an auction mechanism in which, for n risk-neutral bidders, each has a privately know value drawn independently from a common, strictly increasing distribution.

Then, any such mechanism, in which

- the object always goes to the bidder with highest value or bid
- any bidder with lowest value expects zero utility

yields same expected revenue.

Proof

For simplicity, lets take the when there are *n risk-neutral* bidders, competing for a single item.

Lets say, each bidder *i* values the item at v_i , which is private

And each vi is chosen independently from the same continuous distribution function F(v) on $[v_l, v_h]$ with density function f(v)

Now consider any auction mechanism, for which, the expected utility for each bidder i is $U_i(v_i)$ that a bidder i obtains in equilibrium by participating in the auction mechanism.

Let $P_i(v_i)$ be the probability of bidder i to win the item and E_i is the payment made bidder i for the value v_i.

So, Expected Utility is given by

$$U_i(v_i) = v_i P_i(v_i) - E_i$$

If bidder i having value v deviates from the equilibrium behavior and follow another strategy with value $\tilde{v}$, then

$$U_i(v_i) \geq U_i(\tilde{v}_i) + (v_i - \tilde{v}_i) P_i(\tilde{v}_i)$$

- Since value v_i must not mimic v+dv, so

$$U_i(v_i) \geq U_i(v_i+dv) + (-dv)\, P_i(v_i+dv) \text{ ------------- } (1)$$

Also, v_i + dv must not mimic v_i

$$U_i(v_i + dv) \geq U_i(v_i) + (dv)\, P_i(v_i) \text{ ------------------ } (2)$$

Combining (1) and (2)

$$P_i(v_i+dv) \geq \frac{Ui\,(vi+dv) - U_i\,(vi)}{dv} \geq P_i(v_i)$$

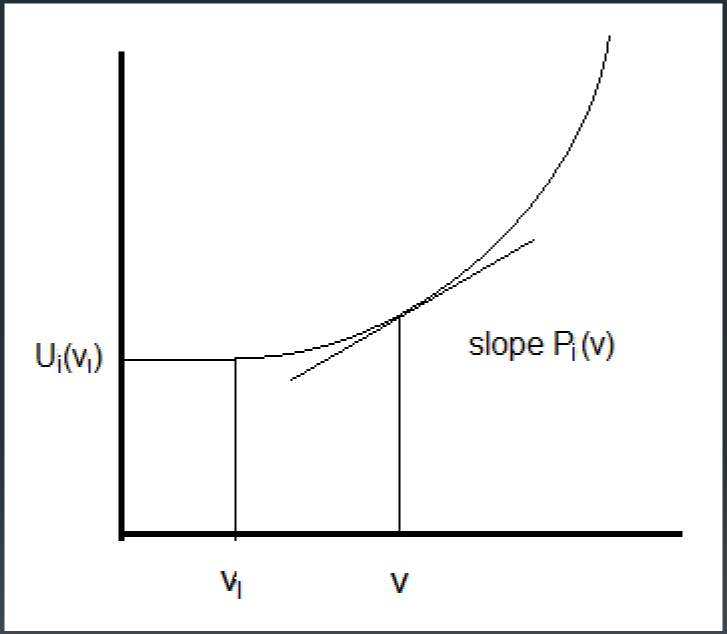
Taking limit dv → 0

$$\frac{dU_i}{dv} = P_i(v_i)$$

Integrating, we get

$$U_i\,(v_i) = U_i\,(v_l) + \int_{x=vl}^{v} P_i x dx$$

It gives this graph.



It means if we know $U_i\,(v_l)$ and
Then we can calculate utility
For any value v

Now, consider two mechanisms with same U_i (v_I) and same P_i (v_i) functions for all v_i for every bidder i.

Therefore, for every v_i for bidder i, will have same utility in both the mechanisms.

So, the bidder i will have same expected payment E_i for both.

($U_i (v_i) = v_i P_i (v_i) - E_i$)

This means her average expected payment for all the values v is also same for both the mechanisms.

As, this is true for all the bidders, this means that both the mechanisms will give same expected revenue to the seller.

This is Revenue Equivalence Theorem.

Simple Cases for RET

- MODEL:
  - Seller sells only one item
  - Two risk-neutral bidders
  - Bid value is drawn from the uniform distribution [0,1]

- STANDARD AUCTION:
  - bidders submit their bids
  - Bidder with highest bid wins the item
  - Bidder is asked to pay $T(s_i, s_j)$

Revenue Equivalence Theorem

- If there are two bidders with values drawn from uniform distribution U[0,1], then, any standard auction has an expected revenue 1/3 and gives bidder (with value v) an expected profit of $v^2/2$ same as second price auction.

Proof:

- We’ll show that Second Price Auction has expected profit is $v^2/2$, and so for standard auctions.

- Then, we’ll show that Expected Revenue in standard auctions is 1/3

- Later, we’ll show that for Expected Profit U(v) of a bidder with value v

  U’(v) = v (i.e. P(v))

Expected Profit in Second Price Auction

- Bidder has value v
- Equilibrium strategy $b(v) = v$
- She wins with the probability $P(v) = v$
- If she wins, she expects to pay $v/2$
- So the expected profit

$$U(v) = v\,(v - v/2) = v^2/2$$

Expected Profit in First Price Auction

- Bidder has value v
- Equilibrium strategy $b(v) = v/2$
- Probability to win the item , $P(v) = v$
- If she wins, she expects to pay $v/2$
- So the expected profit

$$U(v) = v\,(v - v/2) = v^2/2$$

Expected Revenue is 1/3

By def. , Expected Profit for lowest value, U(0)=0

Therefore,

$U(v) = U(0) + \int_0^v U'(x)dx = 0 + \int_0^v x dx = v^2/2$

Average Profit for each bidder

$E_v[U(v)] = \int_0^1 U(v)dv = \int_0^1 v^2/2\, dv = 1/6$

So, total avg Profit for both the bidders

E[Total Bidder Profit] = 1/3

Expected Surplus of the auction

$E[Surplus] = E[\max\{v_i, v_j\}] = 2/3$

Surplus is given by

E[surplus] = E[Revenue] + E[Total Bidder profit]

Therefore

E[Revenue] = 2/3 – 1/3 = 1/3

Proved!

$U'(v) = v$

- Second Price Auction
  - $b(v) = v$
  - $P(v) = v$
  - Expected Payment $E = v/2$ if you win
  - Total Expected Payment, $E[Total] = v^2/2$

$U(v) = P(v).v – E[Total] = v^2 - v^2/2 = v^2/2$

So, $U'(v) = v$

- First Price Auction
  - $b(v) = v/2$
  - $P(v) = v$
  - Expected Payment $E = v/2$ if you win
  - Total Expected Payment, $E[Total] = v^2/2$

$U(v) = P(v).v – E[Total] = v^2 - v^2/2 = v^2/2$

So, $U'(v) = v$

- Other Standard Auctions
  - Equilibrium Strategy, b(v)
  - P(v) = v
  - Total Expected Payment, E[Total] = ?

$U(v) = P(v).v – E[Total] = v^2 - E[Total]$

How would we get, $U'(v) = v$?
Help from Envelope Theorem.

E[Total] for Standard Auctions

$U(v) = v^2/2$

$P(v) = v$

$U(v) = P(v).v – E[Total]$

Therefore, $E[Total] = v^2 - v^2/2 = v^2/2$

Revenue Equivalence Theorem
(General Case)

If there are N bidders with values drawn uniformly from a continuous distribution, then any standard auction generates same expected revenue and same expected profit as a second price auction

References

- Auctions: Theory and Practice (Paul Klemperer)
- http://www.stanford.edu/~jdlevin/Econ%20136/RET%20Notes.pdf
- R. Myerson, Optimal auction design, *Mathematics of Operations Research*, 6(1), 58-73, 1981.

Questions?