The Simplex Algorithm: Technicalities

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\[1\] This presentation is based upon the book *Linear Programming* by Vasek Chvatal
Two Issues

- Here we discuss two potential problems with the simplex method and how to avoid them.

Termination.
How can we ensure that the algorithm terminates?

Initialisation.
How can we ensure that the algorithm starts?
Part I

Termination: Cycling
Choosing Pivots

**Entering Variable.**
We may choose *any* non-basic variable with a positive coefficient in the top row of the dictionary to enter the basis.

- If there is more than one choice to enter, *which do we pick?*

**Leaving Variable.**
We may choose *any* basic variable whose non-negativity imposes the most stringent constraint on the entering variable to leave the basis.

- If there is more than one choice to leave, *which do we pick?*
Degeneracy

Leaving Variable.

We may choose any basic variable whose non-negativity imposes the most stringent constraint on the entering variable to **leave** the basis.

- If two basic variables $x_i, x_j$ can be selected to leave the basis then their values are both forced to zero by the entering variable.
- Since one of them, say $x_i$, must stay in the basis, we now have a basic variable whose value is zero. [Its constant term in the dictionary is 0.]
- Basic solutions with some basic variables of value zero are called **degenerate**.
- For a pivot at a degenerate solution, the **objective value will not increase**, nor will any of the variable values change.
Cycling

- So pivots at **degenerate** solutions do not lead to better solutions.
- Is it possible that the algorithm can get **stuck** (before it finds an optimal solution) in a sequence of degenerate solutions?
- **YES.** - If we are **not careful** in how we choose our pivots.
An Example of Cycling

- Assume we use the following rules for selecting pivots:

**Entering Variable.**
Add the non-basic variable with the highest positive coefficient.

**Leaving Variable.**
In case of a tie, remove the basic variable with the smallest subscript.

- Apply these rules to the following example.

\[
\begin{align*}
\text{maximise} & \quad 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\text{subject to} & \quad \frac{1}{2}x_1 - \frac{11}{2}x_2 - \frac{5}{2}x_3 + 9x_4 \leq 0 \\
& \quad \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + x_4 \leq 0 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]
Dictionary Form

maximise \[ 10x_1 - 57x_2 - 9x_3 - 24x_4 \]
subject to
\[
\begin{align*}
\frac{1}{2}x_1 - \frac{11}{2}x_2 - \frac{5}{2}x_3 + 9x_4 & \leq 0 \\
\frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + x_4 & \leq 0 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

Write in **dictionary form**.

\[
\begin{align*}
    z &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
    x_5 &= -\frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\
    x_6 &= -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
    x_7 &= 1 - x_1
\end{align*}
\]
## Pivot 1

\[
\begin{align*}
z &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= -\frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\
x_6 &= -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
x_7 &= 1 - x_1 \\
\end{align*}
\]

- **Pivot:** Add \(x_1\) to basis. (We had no choice.)
- Remove \(x_5\) from basis. (We had the choice of \(x_5\) or \(x_6\)).

\[
\begin{align*}
z &= 10(11x_2 + 5x_3 - 18x_4 - 2x_5) - 57x_2 - 9x_3 - 24x_4 \\
x_1 &= \frac{11x_2}{2} + \frac{5x_3}{2} - 18x_4 - 2x_5 \\
x_6 &= -\frac{1}{2}(11x_2 + 5x_3 - 18x_4 - 2x_5) + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
x_7 &= 1 - (11x_2 + 5x_3 - 18x_4 - 2x_5) \\
\end{align*}
\]

\[
\begin{align*}
z &= 53x_2 + 41x_3 - 204x_3 - 20x_5 \\
x_1 &= 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= -4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\
\end{align*}
\]
Pivot 2

\[
\begin{align*}
  z &= 53x_2 + 41x_3 - 204x_3 - 20x_5 \\
  x_1 &= 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
  x_6 &= -4x_2 - 2x_3 + 8x_4 + x_5 \\
  x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{align*}
\]

- **Pivot:** Add $x_2$ to basis. (It has a higher coefficient than $x_3$.)
- **Remove** $x_6$ from basis. (We had no choice; $x_1$ is unconstrained by $x_2$).

\[
\begin{align*}
  z &= 53\left(-\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6\right) + 41x_3 - 204x_3 - 20x_5 \\
  x_2 &= -\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\
  x_1 &= 11\left(-\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6\right) + 5x_3 - 18x_4 - 2x_5 \\
  x_7 &= 1 - 11\left(-\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6\right) - 5x_3 + 18x_4 + 2x_5
\end{align*}
\]

\[
\begin{align*}
  z &= \frac{29}{2}x_3 - 98x_4 - \frac{27}{4}x_5 - \frac{53}{4}x_6 \\
  x_2 &= -\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\
  x_1 &= -\frac{1}{2}x_3 + 4x_4 + \frac{3}{4}x_5 - \frac{11}{4}x_6 \\
  x_7 &= 1 + \frac{1}{2}x_3 - 4x_4 - \frac{3}{4}x_5 + \frac{11}{4}x_6
\end{align*}
\]
Pivot 3

\[
\begin{array}{ccccccc}
    z = & & & \frac{29}{2}x_3 & - & 98x_4 & - & \frac{27}{4}x_5 & - & \frac{53}{4}x_6 \\
    x_2 = & - \frac{1}{2}x_3 & + & 2x_4 & + & \frac{1}{4}x_5 & - & \frac{1}{4}x_6 \\
    x_1 = & - \frac{1}{2}x_3 & + & 4x_4 & + & \frac{3}{4}x_5 & - & \frac{11}{4}x_6 \\
    x_7 = & 1 & + & \frac{1}{2}x_3 & - & 4x_4 & - & \frac{3}{4}x_5 & + & \frac{11}{4}x_6 \\
\end{array}
\]

- **Pivot**: Add \( x_3 \) to basis. (No choice.)
- **Remove** \( x_1 \) from basis. (We had the choice of choice of \( x_1 \) or \( x_2 \).)

\[
\begin{array}{ccccccc}
    z = & \frac{29}{2} (8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6 - 2x_1) & - & 98x_4 & - & \frac{27}{4}x_5 & - & \frac{53}{4}x_6 \\
    x_3 = & & & 8x_4 & + & \frac{3}{2}x_5 & - & \frac{11}{2}x_6 & - & 2x_1 \\
    x_2 = & - \frac{1}{2} (8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6 - 2x_1) & + & 2x_4 & + & \frac{1}{4}x_5 & - & \frac{1}{4}x_6 \\
    x_7 = & 1 & + & \frac{1}{2} (8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6 - 2x_1) & - & 4x_4 & - & \frac{3}{4}x_5 & + & \frac{11}{4}x_6 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
    z = & 18x_4 & + & 15x_5 & - & 93x_6 & - & 29x_1 \\
    x_3 = & & & 8x_4 & + & \frac{3}{2}x_5 & - & \frac{11}{2}x_6 & - & 2x_1 \\
    x_2 = & - & 2x_4 & - & \frac{1}{2}x_5 & + & \frac{5}{2}x_6 & + & x_1 \\
    x_7 = & 1 & & & & & & - & x_1 \\
\end{array}
\]
Pivot 4

\[
\begin{align*}
    z &= 18x_4 + 15x_5 - 93x_6 - 29x_1 \\
    x_3 &= 8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6 - 2x_1 \\
    x_2 &= -2x_4 - \frac{1}{2}x_5 + \frac{5}{2}x_6 + x_1 \\
    x_7 &= 1
\end{align*}
\]

- **Pivot**: Add \(x_4\) to basis. (It has a higher coefficient than \(x_5\).)
- Remove \(x_2\) from basis. (No choice as \(x_3\) unconstrained by \(x_4\).)

\[
\begin{align*}
    z &= 18\left(-\frac{1}{4}x_5 + \frac{5}{4}x_6 + \frac{1}{2}x_1 - \frac{1}{2}x_2\right) + 15x_5 - 93x_6 - 29x_1 \\
    x_4 &= -\frac{1}{4}x_5 + \frac{5}{4}x_6 + \frac{1}{2}x_1 - \frac{1}{2}x_2 \\
    x_3 &= 8\left(-\frac{1}{4}x_5 + \frac{5}{4}x_6 + \frac{1}{2}x_1 - \frac{1}{2}x_2\right) + \frac{3}{2}x_5 - \frac{11}{2}x_6 - 2x_1 \\
    x_7 &= 1
\end{align*}
\]

\[
\begin{align*}
    z &= \frac{21}{2}x_5 - \frac{141}{2}x_6 - 20x_1 - 9x_2 \\
    x_4 &= -\frac{1}{4}x_5 + \frac{5}{4}x_6 + \frac{1}{2}x_1 - \frac{1}{2}x_2 \\
    x_3 &= -\frac{1}{2}x_5 + \frac{9}{2}x_6 + 2x_1 - 4x_2 \\
    x_7 &= 1
\end{align*}
\]
### Pivot 5

<table>
<thead>
<tr>
<th>Variable</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>[ \frac{21}{2} x_5 - \frac{141}{2} x_6 - 20 x_1 - 9 x_2 ]</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>[ - \frac{1}{4} x_5 + \frac{5}{2} x_6 + \frac{1}{2} x_1 - \frac{1}{2} x_2 ]</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>[ - \frac{1}{2} x_5 + \frac{9}{2} x_6 + 2 x_1 - 4 x_2 ]</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>[ 1 ]</td>
</tr>
</tbody>
</table>

- **Pivot:** Add \( x_5 \) to basis. (No choice.)
- **Remove** \( x_3 \) from basis. (We had the choice of \( x_3 \) or \( x_4 \).)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>[ \frac{21}{2} \left( 9 x_6 + 4 x_1 - 8 x_2 - 2 x_3 \right) - \frac{141}{2} x_6 - 20 x_1 - 9 x_2 ]</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>[ 9 x_6 ]</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>[ - \frac{1}{4} \left( 9 x_6 + 4 x_1 - 8 x_2 - 2 x_3 \right) + \frac{5}{4} x_6 + \frac{1}{2} x_1 - \frac{1}{2} x_2 ]</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>[ 1 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>[ 24 x_6 + 22 x_1 - 93 x_2 - 21 x_3 ]</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>[ 9 x_6 + 4 x_1 - 8 x_2 - 2 x_3 ]</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>[ - x_6 - \frac{1}{2} x_1 + \frac{3}{2} x_2 + \frac{1}{2} x_3 ]</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>[ 1 ]</td>
</tr>
</tbody>
</table>
**Pivot 6**

\[
\begin{align*}
  z &= 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
  x_5 &= 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
  x_4 &= -x_6 - \frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 \\
  x_7 &= 1 - x_1
\end{align*}
\]

- **Pivot:** Add $x_6$ to basis. (It has a higher coefficient than $x_1$.)
- **Remove** $x_4$ from basis. (No choice as $x_5$ unconstrained by $x_6$.)

\[
\begin{align*}
  z &= 24(-\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4) + 22x_1 - 93x_2 - 21x_3 \\
  x_6 &= -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
  x_5 &= 9(-\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4) + 4x_1 - 8x_2 - 2x_3 \\
  x_7 &= 1 - x_1
\end{align*}
\]

\[
\begin{align*}
  z &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
  x_6 &= -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
  x_5 &= -\frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\
  x_7 &= 1 - x_1
\end{align*}
\]
A Cycle is Obtained

\[
\begin{align*}
  z & = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
  x_6 & = - \frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
  x_5 & = - \frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\
  x_7 & = 1 - x_1 \\
\end{align*}
\]

- But this is exactly the same as the original dictionary!

\[
\begin{align*}
  z & = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
  x_5 & = - \frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\
  x_6 & = - \frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
  x_7 & = 1 - x_1 \\
\end{align*}
\]

- We are stuck in a cycle (with top row coefficients not all negative).
- So the simplex algorithm did not terminate!
Part II

Bland’s Rule
Bland’s Rule

- Can we stop the algorithm from cycling?
  - **YES.** If we are more *careful* in how we choose the entering and leaving variables.

**Bland’s Rule.**

Given a choice, *always* choose the variable with the *smallest subscript* (for both entering and leaving variables).
Let’s see how this works with the previous example.

\[
\begin{align*}
    z &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
    x_5 &= -\frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\
    x_6 &= -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\
    x_7 &= 1 - x_1
\end{align*}
\]

- **Pivot:** Add \( x_1 \) to basis. (We had no choice.)
- **Remove** \( x_5 \) from basis. (It has a lower subscript than \( x_6 \).)

\[
\begin{align*}
    z &= 53x_2 + 41x_3 - 204x_3 - 20x_5 \\
    x_1 &= 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
    x_6 &= 4x_2 - 2x_3 + 8x_4 + x_5 \\
    x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{align*}
\]
Bland’s Rule: Pivot 2

<table>
<thead>
<tr>
<th></th>
<th>53x₂ +</th>
<th>41x₃ -</th>
<th>204x₃ -</th>
<th>20x₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>53x₂</td>
<td>41x₃</td>
<td>204x₃</td>
<td>20x₅</td>
</tr>
<tr>
<td>x₁</td>
<td>11x₂</td>
<td>5x₃</td>
<td>18x₄</td>
<td>2x₅</td>
</tr>
<tr>
<td>x₆</td>
<td>-4x₂</td>
<td>-2x₃</td>
<td>8x₄</td>
<td>x₅</td>
</tr>
<tr>
<td>x₇</td>
<td>11x₂</td>
<td>5x₃</td>
<td>18x₄</td>
<td>2x₅</td>
</tr>
</tbody>
</table>

- **Pivot**: Add $x₂$ to basis. (It has a lower subscript than $x₃$.)
- **Remove $x₆$ from basis.** (We had no choice; $x₁$ is unconstrained by $x₂$.)

<table>
<thead>
<tr>
<th></th>
<th>29/2x₃ -</th>
<th>98x₄ -</th>
<th>27/4x₅ -</th>
<th>53/4x₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>29/2x₃</td>
<td>98x₄</td>
<td>27/4x₅</td>
<td>53/4x₆</td>
</tr>
<tr>
<td>x₂</td>
<td>-1/2x₃</td>
<td>2x₄</td>
<td>1/4x₅</td>
<td>-1/4x₆</td>
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<tr>
<td>x₁</td>
<td>-1/2x₃</td>
<td>4x₄</td>
<td>3/4x₅</td>
<td>11/4x₆</td>
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<tr>
<td>x₇</td>
<td>1/2x₃</td>
<td>-4x₄</td>
<td>3/4x₅</td>
<td>11/4x₆</td>
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</table>
### Bland’s Rule: Pivot 3

<table>
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<tr>
<th></th>
<th>29/2 (x_3)</th>
<th>98(x_4)</th>
<th>27/4 (x_5)</th>
<th>53/4 (x_6)</th>
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</thead>
<tbody>
<tr>
<td>(z)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
<td>(-)</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
<td>(-)</td>
</tr>
<tr>
<td>(x_7)</td>
<td>(+)</td>
<td>(-)</td>
<td>(-)</td>
<td>(-)</td>
</tr>
</tbody>
</table>

- **Pivot**: Add \(x_3\) to basis. (No choice.)
- **Remove** \(x_1\) from basis. (It has a lower subscript \(x_2\).)

<table>
<thead>
<tr>
<th>(z)</th>
<th>18(x_4)</th>
<th>15(x_5)</th>
<th>93(x_6)</th>
<th>29(x_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_3)</td>
<td>(+)</td>
<td>(-)</td>
<td>(-)</td>
<td>(+)</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>(x_7)</td>
<td>(+)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Bland’s Rule: Pivot 4

\[
\begin{align*}
  z &= 18x_4 + 15x_5 - 93x_6 - 29x_1 \\
  x_3 &= 8x_4 + \frac{3}{2}x_5 + \frac{11}{2}x_6 - 9x_1 \\
  x_2 &= -2x_4 - \frac{9}{2}x_5 + \frac{5}{2}x_6 + x_1 \\
  x_7 &= 1
\end{align*}
\]

- **Pivot**: Add \(x_4\) to basis. (It has a lower subscript than \(x_5\).)
- Remove \(x_2\) from basis. (No choice as \(x_3\) unconstrained by \(x_4\).)

\[
\begin{align*}
  z &= \frac{21}{2}x_5 - \frac{141}{2}x_6 - 20x_1 - \frac{9}{2}x_2 \\
  x_4 &= -\frac{1}{4}x_5 + \frac{5}{4}x_6 + \frac{1}{2}x_1 - \frac{1}{2}x_2 \\
  x_3 &= -\frac{1}{2}x_5 + \frac{9}{2}x_6 + 2x_1 - 4x_2 \\
  x_7 &= 1
\end{align*}
\]
Bland’s Rule: Pivot 5

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>=</td>
<td>$\frac{21}{2} x_5$</td>
<td>$-\frac{141}{2} x_6$</td>
<td>$-20 x_1$</td>
<td>$-9 x_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_4$</td>
<td>=</td>
<td>$-\frac{1}{4} x_5$</td>
<td>$+\frac{1}{2} x_6$</td>
<td>$+\frac{1}{2} x_1$</td>
<td>$-\frac{1}{2} x_2$</td>
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</tr>
<tr>
<td>$x_3$</td>
<td>=</td>
<td>$-\frac{1}{2} x_5$</td>
<td>$+\frac{1}{2} x_6$</td>
<td>$+2 x_1$</td>
<td>$-4 x_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_7$</td>
<td>=</td>
<td>$1$</td>
<td>$- x_1$</td>
<td></td>
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</table>

- **Pivot:** Add $x_5$ to basis. (No choice.)
- **Remove** $x_3$ from basis. (It has a lower subscript than $x_4$.)

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<tbody>
<tr>
<td>$z$</td>
<td>=</td>
<td>$24 x_6$</td>
<td>$+22 x_1$</td>
<td>$-93 x_2$</td>
<td>$-21 x_3$</td>
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<tr>
<td>$x_5$</td>
<td>=</td>
<td>$9 x_6$</td>
<td>$+4 x_1$</td>
<td>$-8 x_2$</td>
<td>$-2 x_3$</td>
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<tr>
<td>$x_4$</td>
<td>=</td>
<td>$-x_6$</td>
<td>$-\frac{1}{2} x_1$</td>
<td>$+\frac{3}{2} x_2$</td>
<td>$+\frac{1}{2} x_3$</td>
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<tr>
<td>$x_7$</td>
<td>=</td>
<td>$1$</td>
<td>$- x_1$</td>
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</table>
Bland’s Rule: Pivot 6

\[
\begin{align*}
  z &= 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
  x_5 &= 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
  x_4 &= -x_6 - \frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 \\
  x_7 &= 1 - x_1
\end{align*}
\]

- So far all our pivots has been the same as before, but now....
- **Pivot**: Add \(x_1\) to basis. (It has a lower subscript than \(x_6\).)
- Remove \(x_4\) from basis. (No choice as \(x_5\) unconstrained by \(x_6\).)

\[
\begin{align*}
  z &= 24x_6 + 22(-2x_6 + 3x_2 + x_3 - 2x_4) - 93x_2 - 21x_3 \\
  x_1 &= -2x_6 + 3x_2 + x_3 - 2x_4 \\
  x_5 &= 9x_6 + 4(-2x_6 + 3x_2 + x_3 - 2x_4) - 8x_2 - 2x_3 \\
  x_7 &= 1 - (-2x_6 + 3x_2 + x_3 - 2x_4)
\end{align*}
\]

\[
\begin{align*}
  z &= -20x_6 - 27x_2 + x_3 - 44x_4 \\
  x_1 &= -2x_6 + 3x_2 + x_3 - 2x_4 \\
  x_5 &= -x_6 + 4x_2 + 2x_3 - 8x_4 \\
  x_7 &= 1 + 2x_6 - 3x_2 - x_3 - 2x_4
\end{align*}
\]
Bland’s Rule: Pivot 7

\[
\begin{align*}
  z &= -20x_6 - 27x_2 + x_3 - 44x_4 \\
  x_1 &= -2x_6 + 3x_2 + x_3 - 2x_4 \\
  x_5 &= -x_6 + 4x_2 + 2x_3 - 8x_4 \\
  x_7 &= 1 + 2x_6 - 3x_2 - x_3 - 2x_4 \\
\end{align*}
\]

- **Pivot**: Add \( x_3 \) to basis. (No choice.)
- **Remove** \( x_7 \) from basis. (No choice as \( x_1, x_5 \) unconstrained by \( x_3 \).)

\[
\begin{align*}
  z &= -20x_6 - 27x_2 + (1 + 2x_6 - 3x_2 - 2x_4 - x_7) - 44x_4 \\
  x_3 &= 1 + 2x_6 - 3x_2 - 2x_4 - x_7 \\
  x_1 &= -2x_6 + 3x_2 + (1 + 2x_6 - 3x_2 - 2x_4 - x_7) - 2x_4 \\
  x_5 &= -x_6 + 4x_2 + 2(1 + 2x_6 - 3x_2 - 2x_4 - x_7) - 8x_4 \\
\end{align*}
\]

\[
\begin{align*}
  z &= 1 - 18x_6 - 30x_2 - 46x_4 - x_7 \\
  x_3 &= 1 + 2x_6 - 3x_2 - 2x_4 - x_7 \\
  x_1 &= 1 - 4x_4 - x_7 \\
  x_5 &= 2 + 3x_6 - 2x_2 - 12x_4 - 2x_7 \\
\end{align*}
\]
Bland’s Rule: Breaking the Cycle

\[
\begin{align*}
    z &= 1 - 18x_6 - 30x_2 - 46x_4 - x_7 \\
    x_3 &= 1 + 2x_6 - 3x_2 - 2x_4 - x_7 \\
    x_1 &= 1 - 4x_4 - x_7 \\
    x_5 &= 2 + 3x_6 - 2x_2 - 12x_4 - 2x_7 \\
\end{align*}
\]

- All the coefficients in the top row are now negative.
- Thus, we break out of the degeneracy and obtain an optimal solution \((x_1, x_2, x_3, x_4) = (1, 0, 1, 0)\) and \(z = 1\).
Remarks

- Bland’s Rule *always works*: use it and you obtain an optimal solution.
- There are other methods, e.g. **Perturbation Methods**, that also work.
- For proofs and discussions on these issues *see Chvatal’s book*. 
Part III

The Starting Point?
Is the All-Zero Solution Feasible?

- Suppose we want to solve the linear program

\[
\begin{align*}
\text{maximise} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_j \quad \forall i \in [m] \\
& \quad x_j \geq 0 \quad \forall j \in [n]
\end{align*}
\]

- To run the simplex algorithm we need to start with a feasible solution.
- So far we have used the all-zero solution.
- But if some \( b_j \) are negative then the all-zero solution is not feasible!

E.g.

\[
\begin{align*}
\text{maximise} & \quad x_1 - x_2 + x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + 2x_3 \leq 4 \\
& \quad 2x_1 - 3x_2 + x_3 \leq -5 \\
& \quad -x_1 + x_2 - 2x_3 \leq -1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
An Auxiliary Problem

\begin{align*}
\text{maximise} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_j \quad \forall i \in [m] \\
& \quad x_j \geq 0 \quad \forall j \in [n]
\end{align*}

- But our LP has a feasible solution \textit{if and only if} the following auxiliary linear program has an optimal solution of \textit{value zero}.

\begin{align*}
\text{minimise} & \quad x_0 \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_j \quad \forall i \in [m] \\
& \quad x_j \geq 0 \quad \forall j \in \{0, 1, \ldots, n\}
\end{align*}

- So can we use an optimal solution to this auxiliary problem to find a feasible solution (and starting dictionary) for our LP?
- \textbf{YES.}
Finding a Starting Feasible Solution

Let's see how this works on the problem

maximise \( z = x_1 - x_2 + x_3 \)
subject to
\[
\begin{align*}
2x_1 - x_2 + 2x_3 & \leq 4 \\
2x_1 - 3x_2 + x_3 & \leq -5 \\
-x_1 + x_2 - 2x_3 & \leq -1 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

The auxiliary problem is then

maximise \( w = \) 
subject to
\[
\begin{align*}
2x_1 - x_2 + 2x_3 - x_0 & \leq 4 \\
2x_1 - 3x_2 + x_3 - x_0 & \leq -5 \\
-x_1 + x_2 - 2x_3 - x_0 & \leq -1 \\
x_0, x_1, x_2, x_3 & \geq 0
\end{align*}
\]
Solving the Auxiliary Problem: a Dictionary

\[
\begin{align*}
\text{maximise} \quad w &= -x_0 \\
\text{subject to} \quad 2x_1 - x_2 + 2x_3 - x_0 &\leq 4 \\
2x_1 - 3x_2 + x_3 - x_0 &\leq -5 \\
-x_1 + x_2 - 2x_3 - x_0 &\leq -1 \\
x_0, \quad x_1, \quad x_2, \quad x_3 &\geq 0
\end{align*}
\]

- Let’s apply the simplex method to this auxiliary problem.
- First put it in \textit{dictionary form}.

\[
\begin{align*}
w &= -x_0 \\
x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\
x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\
x_6 &= -1 + x_1 - x_2 + 2x_3 + x_0
\end{align*}
\]
Solving the Auxiliary Problem: an Infeasible Solution

\[
\begin{align*}
\text{w} & = -x_0 \\
x_4 & = 4 - 2x_1 + x_2 - 2x_3 + x_0 \\
x_5 & = -5 - 2x_1 + 3x_2 - x_3 + x_0 \\
x_6 & = -1 + x_1 - x_2 + 2x_3 + x_0
\end{align*}
\]

- The only term, i.e. \( x_0 \), in the top row has a **negative** coefficient.
- But this solution is **not feasible** for the auxiliary problem.
- To see this observe that \( x_5, x_6 < 0 \).
- But if we increase \( x_0 \) these variables will become **non-negative** (at the expense of **decreasing** the objective value).
- Let’s make this pivot and see what happens...
A Strange Pivot: Pivot 0

\[
\begin{align*}
\text{w} &= \quad -x_0 \\
\text{x}_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\
\text{x}_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\
\text{x}_6 &= -1 + x_1 - x_2 + 2x_3 + x_0
\end{align*}
\]

- **Pivot:** Add \( x_0 \) to basis.
- Remove \( x_5 \) from basis. (As \( x_5 \) was the most negative basic variable.)

\[
\begin{align*}
\text{w} &= \quad -(5 + 2x_1 - 3x_2 + x_3 + x_5) \\
\text{x}_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
\text{x}_4 &= 4 - 2x_1 + x_2 - 2x_3 + (5 + 2x_1 - 3x_2 + x_3 + x_5) \\
\text{x}_6 &= -1 + x_1 - x_2 + 2x_3 + (5 + 2x_1 - 3x_2 + x_3 + x_5)
\end{align*}
\]

\[
\begin{align*}
\text{w} &= \quad -5 - 2x_1 + 3x_2 - x_3 - x_5 \\
\text{x}_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
\text{x}_4 &= 9 - 2x_2 - x_3 + x_5 \\
\text{x}_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5
\end{align*}
\]
Solving the Auxiliary Problem: Pivot 1

\[
\begin{align*}
w &= -5 - 2x_1 + 3x_2 - x_3 - x_5 \\
x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_4 &= 9 - 2x_2 - x_3 + x_5 \\
x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5
\end{align*}
\]

- **Hooray:** this is **feasible** for the auxiliary problem!

- **Pivot:** Add \( x_2 \) to basis.

- Remove \( x_6 \) from basis.

\[
\begin{align*}
w &= -5 - 2x_1 + 3(1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6) - x_3 - x_5 \\
x_2 &= 1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\
x_0 &= 5 + 2x_1 - 3(1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6) + x_3 + x_5 \\
x_4 &= 9 - 2(1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6) - x_3 + x_5 \\
w &= -2 + \frac{1}{4}x_1 + \frac{5}{4}x_3 - \frac{1}{4}x_5 - \frac{3}{4}x_6 \\
x_2 &= 1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\
x_0 &= 2 - \frac{1}{4}x_1 - \frac{5}{4}x_3 + \frac{1}{4}x_5 + \frac{3}{4}x_6 \\
x_4 &= 7 - \frac{3}{2}x_1 - \frac{5}{2}x_3 + \frac{1}{2}x_5 + \frac{1}{2}x_6
\end{align*}
\]
Solving the Auxiliary Problem: Pivot 2

\[ w = -2 + \frac{1}{4}x_1 + \frac{5}{4}x_3 - \frac{1}{4}x_5 - \frac{3}{4}x_6 \]

| \( x_2 \) | 1 | + | \( \frac{3}{4}x_1 \) | + | \( \frac{3}{4}x_3 \) | + | \( \frac{1}{4}x_5 \) | - | \( \frac{1}{4}x_6 \) |
| \( x_0 \) | 2 | - | \( \frac{1}{4}x_1 \) | - | \( \frac{5}{4}x_3 \) | + | \( \frac{1}{4}x_5 \) | + | \( \frac{3}{4}x_6 \) |
| \( x_4 \) | 7 | - | \( \frac{3}{2}x_1 \) | - | \( \frac{5}{2}x_3 \) | + | \( \frac{1}{2}x_5 \) | + | \( \frac{1}{2}x_6 \) |

- **Pivot:** Add \( x_3 \) to basis. [**Note:** here it is quicker not to use Bland’s rule.]
- Remove \( x_0 \) from basis.

\[ w = -2 + \frac{1}{4}x_1 + \frac{5}{4}\left(\frac{8}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0\right) - \frac{1}{4}x_5 - \frac{3}{4}x_6 \]

\[ x_2 = \frac{1}{4} + \frac{3}{4}x_1 + \frac{3}{4}\left(\frac{8}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0\right) + \frac{1}{4}x_5 - \frac{1}{4}x_6 \]
\[ x_3 = \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \]
\[ x_4 = 7 - \frac{3}{2}x_1 - \frac{5}{2}\left(\frac{8}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0\right) + \frac{1}{2}x_5 + \frac{1}{2}x_6 \]

\[ w = \frac{8}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \]
\[ x_2 = \frac{11}{5} + \frac{3}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \]
\[ x_4 = 3 - \frac{1}{5}x_1 - x_6 + 2x_0 \]
The Auxiliary Problem: an Optimal Solution

\[
\begin{align*}
  w &= -x_0 \\
  x_3 &= \frac{8}{5} - \frac{1}{5} x_1 - \frac{1}{5} x_5 + \frac{3}{5} x_6 - \frac{4}{5} x_0 \\
  x_2 &= \frac{11}{5} + \frac{3}{5} x_1 + \frac{2}{5} x_5 + \frac{1}{5} x_6 - \frac{3}{5} x_0 \\
  x_4 &= 3 - x_1 - x_6 + 2x_0
\end{align*}
\]

- All coefficients in the top row are now \textbf{negative}.
- So we have an \textbf{optimal solution} for the auxiliary problem.
- \((x_0, x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, \frac{11}{5}, \frac{8}{5}, 3, 0, 0)\).
- The \textbf{objective value} is \(w = -x_0 = 0\).
- So this corresponds to a \textbf{feasible solution} to the original problem.
maximise \[ x_1 - x_2 + x_3 \]
subject to \[
\begin{align*}
2x_1 - x_2 + 2x_3 & \leq 4 \\
2x_1 - 3x_2 + x_3 & \leq -5 \\
-x_1 + x_2 - 2x_3 & \leq -1 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

- So we have an **optimal** solution for the auxiliary problem.
  \((x_0, x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, \frac{11}{5}, \frac{8}{5}, 3, 0, 0)\).
- This gives the solution \((x_1, x_2, x_3) = (0, \frac{11}{5}, \frac{8}{5})\) to the original problem.
- This is **feasible**. [Check it!]
A Feasible Dictionary for the Original Problem

maximise \quad x_1 - x_2 + x_3

subject to
\begin{align*}
2x_1 - x_2 + 2x_3 & \leq 4 \\
2x_1 - 3x_2 + x_3 & \leq -5 \\
-x_1 + x_2 - 2x_3 & \leq -1 \\
x_1, x_2, x_3 & \geq 0
\end{align*}

So we have a feasible solution for the original LP.

How do we find a feasible starting dictionary?

Consider the final dictionary for the auxiliary LP.

\[
\begin{align*}
W &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 - \frac{3}{5}x_0 \\
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 - \frac{3}{5}x_0 \\
x_3 &= \frac{1}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \\
x_4 &= 3 - x_1 - x_6 + 2x_0
\end{align*}
\]

Remove the column for \( x_0 \) and add back the original objective...
maximise \( x_1 - x_2 + x_3 \)
subject to
\[
\begin{align*}
2x_1 - x_2 + 2x_3 & \leq 4 \\
2x_1 - 3x_2 + x_3 & \leq -5 \\
-x_1 + x_2 - 2x_3 & \leq -1
\end{align*}
\]
\( x_1, x_2, x_3 \geq 0 \)

- Remove the column for \( x_0 \) and add back the original objective.

\[
\begin{align*}
z &= x_1 - x_2 + x_3 \\
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 \\
x_3 &= \frac{5}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{1}{5}x_6 \\
x_4 &= 3 - x_1 - x_6
\end{align*}
\]

- This gives a feasible dictionary for the original LP.

\[
\begin{align*}
z &= -\frac{3}{5} + \frac{1}{5}x_1 - \frac{3}{2}x_5 + \frac{2}{5}x_6 \\
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 \\
x_3 &= \frac{5}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{3}{5}x_6 \\
x_4 &= 3 - x_1 - x_6
\end{align*}
\]
The Two Phase Simplex Method

- So we use a two phase approach to solve the linear program.
- **First** we solve the auxiliary problem:

  **Case I.**
  If the optimal value *is zero* we have a feasible solution for the original. We then run the simplex algorithm for the **second** time.

  **Case II.**
  If the optimal value *is less than zero* the original problem is infeasible.
Part IV

Efficiency
Termination

- With Bland’s Rule the algorithm will terminate.
- To see this, observe that there are \( \binom{m+n}{m} \) bases.
- The dictionary corresponding to each basis can appear only once.
- So there are at most \( \binom{m+n}{m} \) iterations.
- But \( \binom{m+n}{m} \gg 2^{m+n} \).
- Does the algorithm always make an exponential \# pivots?
The simplex method typically makes $O(m)$ pivots, where $m$ is the number of constraints.

Each pivot takes time $O(mn)$ with the dictionary method.
[But there are actually faster ways to implement pivots.]

So typically the algorithm is fast.

On the other hand...
Exponential Running Times

- ....all known pivot rules have pathological examples on which they make an exponential number of pivots.
- So the simple algorithm is a fast practical algorithm but has exponential worst case running time.
- In some sense the simplex algorithm is the exception that proves the rule: “Polynomial algorithms good; exponential algorithms bad”.
- In fact, recent theoretical results show the simplex algorithm isn’t really an exception to this rule after all.
  [But that research is beyond the scope of this course...]
Polynomial Time Algorithms

- There are polynomial time algorithms for solving linear programs.
- The ellipsoid method and the interior point method are polytime algorithms.

Conjecture.

The simplex algorithm is polynomial time if, given a choice, it chooses the pivot variables at random.