BOUNDED STABILITY FOR STRONGLY COUPLED CRITICAL ELLIPTIC SYSTEMS BELOW THE GEOMETRIC THRESHOLD OF THE CONFORMAL LAPLACIAN

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To the memory of T. Aubin

Abstract. We prove bounded stability for strongly coupled critical elliptic systems in the inhomogeneous context of a compact Riemannian manifold when the potential of the operator is less, in the sense of bilinear forms, than the geometric threshold potential of the conformal Laplacian.

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. For $p \geq 1$ an integer, let also $M_p^s(\mathbb{R})$ denote the vector space of symmetrical $p \times p$ real matrices, and A be a C^1 map from M to $M_p^s(\mathbb{R})$. We write that $A = (A_{ij})_{i,j}$, where the A_{ij}'s are C^1 real-valued functions in M. Let $\Delta_g = -\mathrm{div}_g \nabla$ be the Laplace-Beltrami operator on M, and $H^1(M)$ be the Sobolev space of functions in $L^2(M)$ with one derivative in $L^2(M)$. The Hartree-Fock coupled systems of nonlinear Schrödinger equations we consider in this paper are written as

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}(x) u_j = |\mathcal{U}|^{2^\star - 2} u_i \tag{0.1}$$

in M for all i, where $|\mathcal{U}|^2 = \sum_{i=1}^p u_i^2$, and $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embeddings of the Sobolev space $H^1(M)$ into Lebesgue's spaces. The systems (0.1) are weakly coupled by the linear matrix A, and strongly coupled by the Gross-Pitaevskii type nonlinearity in the right-hand side of (0.1). As is easily seen, (0.1) is critical for Sobolev embeddings.

Coupled systems of nonlinear Schrödinger equations like (0.1) are now parts of several important branches of mathematical physics. They appear in the Hartree-Fock theory for Bose-Einstein double condensates, in fiber-optic theory, in the theory of Langmuir waves in plasma physics, and in the behavior of deep water waves and freak waves in the ocean. A general reference in book form on such systems and their role in physics is by Ablowitz, Prinari, and Trubatch [1]. The systems (0.1) we investigate in this paper involve coupled Gross-Pitaevskii type equations. Such equations are strongly related to two branches of mathematical physics. They arise, see Burke, Bohn, Esry, and Greene [9], in the Hartree-Fock theory for double condensates, a binary mixture of Bose-Einstein condensates in two different hyperfine states. They also arise in the study of incoherent solitons in nonlinear optics,

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as described in Akhmediev and Ankiewicz [2], Christodoulides, Coskum, Mitchell and Segev [13], Hioe [24], Hioe and Salter [25], and Kanna and Lakshmanan [26].

A strong solution $\mathcal{U}$ of (0.1) is a p-map with components in H^1 satisfying (0.1). By elliptic regularity strong solutions are of class $C^{2,\theta}$, $\theta \in (0,1)$. In the sequel a p-map $\mathcal{U} = (u_1, \dots, u_p)$ from M to $\mathbb{R}^p$ is said to be nonnegative if $u_i \geq 0$ in M for all i. We aim in this paper in discussing bounded stability for our systems (0.1). With respect to the notion of analytic stability, as defined and investigated in Druet and Hebey [19], no bound on the energy of the solution is required in the stronger notion of bounded stability. This prevents, see Section 2, the existence of standing waves with arbitrarily large amplitude for the corresponding critical vector-valued Klein-Gordon and Schrödinger equations. Let $\mathcal{S}_A$ be the set consisting of the nonnegative strong solutions of (0.1). Bounded stability is defined as follows.

Definition. *The system* (0.1) *is bounded and stable if there exist $C > 0$ and $\delta > 0$ such that for any $A' \in C^1\left(M, M_p^s(\mathbb{R})\right)$ satisfying $\|A' - A\|_{C^1} < \delta$, and for any $\mathcal{U} \in \mathcal{S}_{A'}$, there holds that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for $\theta \in (0,1)$.*

An equivalent definition is that for any sequence $(A_\alpha)_\alpha$ of C^1-maps from M to $M_p^s(\mathbb{R})$, $\alpha \in \mathbb{N}$, and for any sequence of nonnegative nontrivial strong solutions $\mathcal{U}_\alpha$ of the associated systems, if $A_\alpha \to A$ in C^1 as $\alpha \to +\infty$, then, up to a subsequence, $\mathcal{U}_\alpha \to \mathcal{U}$ in C^2 as $\alpha \to +\infty$ for some nonnegative solution $\mathcal{U}$ of (0.1). Moreover, see Druet and Hebey [19], we can assert that $\mathcal{U}$ is automatically nontrivial if $\Delta_g + A$ is coercive, or, more generally, if $\Delta_g + A$ does not possess nonnegative nontrivial maps in its kernel.

The question we address in this paper is to find conditions on the vector-valued operator $\Delta_g + A$ which guarantee the bounded stability of (0.1). We answer the question in the theorem below when the potential of the operator is less, in the sense of bilinear forms, than the geometric threshold potential of the conformal Laplacian. As one can check, there is a slight difference between the case $n = 3$, where the Green's matrix of $\Delta_g + A$ and the positive mass theorem come into play, and the case $n \geq 4$. Following standard terminology we say that $\Delta_g + A$ is coercive if the energy of the operator controls the H^1-norm, and we say that $-A$ is cooperative if the nondiagonal components A_{ij} of A, $i \neq j$, are nonpositive in M. When $-A$ is cooperative, see Hebey [23], the existence of $\mathcal{U} = (u_1, \dots, u_p)$ such that $\mathcal{U}$ solves (0.1) and $u_i > 0$ in M for all i, implies the coercivity of $\Delta_g + A$. In the sequel we let S_g be the scalar curvature of g and let Id_p be the identity matrix in $M_p^s(\mathbb{R})$. The theorem we prove is stated as follows.

Theorem. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map satisfying that*

$$A < \frac{n-2}{4(n-1)} S_g Id_p \tag{0.2}$$

in M in the sense of bilinear forms. When $n = 3$ assume also that $\Delta_g + A$ is coercive and that $-A$ is cooperative. Then the associated system (0.1) *is bounded and stable.*

A closely related notion to stability, which has been intensively investigated, is that of compactness. Among possible references we refer to Brendle [6, 7], Brendle and Marques [8], Druet [14, 15], Druet and Hebey [17], Gidas and Spruck [21], Khuri, Marques and Schoen [27], Li and Zhang [29, 30], Li and Zhu [32], Marques [33], Schoen [40, 41], and Vétois [42]. A system like (0.1) is said to be compact if sequences of nonnegative solutions of (0.1) converge, up to a subsequence, in the C^2-topology. A direct consequence of our theorem is as follows.

Corollary. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map satisfying* (0.2). *When $n = 3$ assume also that $\Delta_g + A$ is coercive and that $-A$ is cooperative. Then* (0.1) *is compact.*

Another consequence of our theorem is in terms of standing waves and phase stability for vector-valued Schrödinger and Klein-Gordon equations. Roughly speaking, we refer to Section 2 for more details, it follows from our result that fast oscillating standing waves for Schrödinger and Klein-Gordon equations cannot have arbitrarily large amplitude. The same phenomenon holds true for slow oscillating standing waves if the potential matrix A is sufficiently small. Instability comes in the intermediate regime.

Condition (0.2) in the theorem is the global vector-valued extension of the seminal condition introduced by Aubin [3]. Aubin proved in [3] that (0.2), when satisfied at one point in the manifold, and when A and $\mathcal{U}$ are functions, implies the existence of a minimizing solution of (0.1). Our theorem establishes that (0.2) does not only provide the existence of minimal energy solution to the equations, but also provides the stability of the equations in all dimensions. The condition turns out to be sharp. Assuming that (0.2) is an equality, then, see Druet and Hebey [16, 19], we can construct various examples of unstable systems like (0.1) in any dimension $n \geq 6$. These include the existence of clusters (multi peaks solutions with fewer geometrical blow-up points) and the existence of sequences $(\mathcal{U}_\alpha)_\alpha$ of solutions with unbounded energy (namely such that $\|\mathcal{U}_\alpha\|_{H^1} \to +\infty$ as $\alpha \to +\infty$). By the analysis in Brendle [6] and Brendle and Marques [8] we even get examples of noncompact systems in any dimension $n \geq 25$. Of course, the sphere, because of the noncompactness of its conformal group, is another example where noncompactness holds true (however, in this case, in all dimensions). Conversely, when we avoid large dimensions, avoid the sphere, and restrict the discussion to compactness, it follows from the analysis developed in this paper that for any smooth compact Riemannian 3-manifold (M, g), assumed not to be conformally diffeomorphic to the unit 3-sphere, for any $p \geq 1$, and any C^1-map $A : M \to M_p^s(\mathbb{R})$, if the inequality in (0.2) is large, $\Delta_g + A$ is coercive, and $-A$ is cooperative, then the associated system (0.1) is compact.

Our paper is organized as follows. In Section 1 we provide a complete classification of nonnegative solutions of the strongly coupled critical Euclidean limit system associated with (0.1) and thus obtain the shape of the blow-up singularities associated to our problem. We briefly discuss the dynamical notion of phase stability in Section 2. In Section 3 we prove strong pointwise control estimates for blowing-up sequences of solutions of perturbed equations. These estimates hold true without assuming (0.2). In Section 4 we prove sharp asymptotic estimates for sequences of solutions of perturbed equations when we assume (0.2) and get that rescalings of such sequences locally converge to the Green's function plus a globally well-defined harmonic function with no mass. We construct parametrix for vector-valued Schrödinger operators when $n = 3$ in Section 5 and get an extension of the positive mass theorem of Schoen and Yau [37] to the vector-valued case we consider here. This is the only place in the paper where we use the 3-dimensional assumptions that $\Delta_g + A$ is coercive and that $-A$ is cooperative. We prove the theorem in Section 6 by showing that there should be a mass in the rescaled expansions of blowing-up sequences of solutions of perturbed equations.

1. Nonnegative solutions of the limit system

Of importance in blow-up theory, when discussing critical equations, is the classification of the solutions of the critical limit Euclidean system we get after blowing up the equations. In our case, we need to classify the nonnegative solutions of the limit system

$$\Delta u_i = |\mathcal{U}|^{2^\star-2} u_i \ , \tag{1.1}$$

where $|\mathcal{U}|^2 = \sum_{i=1}^p u_i^2$, and $\Delta = -\sum_{i=1}^n \partial^2/\partial x_i^2$ is the Euclidean Laplace-Beltrami operator. The result we prove here provides full classification of nonnegative solutions of (1.1). It is stated as follows.

Proposition 1.1. *Let $p \geq 1$ and $\mathcal{U}$ be a nonnegative C^2-solution of* (1.1)*. Then there exist $a \in \mathbb{R}^n$, $\lambda > 0$, and $\Lambda \in S_+^{p-1}$, such that*

$$\mathcal{U}(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-a|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \Lambda \tag{1.2}$$

for all $x \in \mathbb{R}^n$, where S_+^{p-1} consists of the elements $(\Lambda_1, \ldots, \Lambda_p)$ in S^{p-1}, the unit sphere in $\mathbb{R}^p$, which are such that $\Lambda_i \geq 0$ for all i.

We prove Proposition 1.1 by using the moving sphere method and the result in Druet and Hebey [19] where the classification of nonnegative H^1-solutions of (1.1) is achieved by variational arguments. The method of moving sphere, a variant of the method of moving planes, has been intensively investigated in recent years. Among possible references we refer to Chen and Li [11], Chou and Chu [12], Li and Zhang [28], Li and Zhu [31], and Padilla [34]. Proposition 1.1 in the special case $p = 1$ was known for long time and goes back to Caffarelli, Gidas and Spruck [10]. The novelty in Proposition 1.1 is that p is arbitrary.

For any $a \in \mathbb{R}^n$, and any $\lambda > 0$, we define the Kelvin transform $\mathcal{U}_{a,\lambda} = K_{a,\lambda}(\mathcal{U})$ of a map $\mathcal{U} : \mathbb{R}^n \to \mathbb{R}^p$ as the p-map defined in $\mathbb{R}^n \backslash \{a\}$ by

$$\mathcal{U}_{a,\lambda}(x) = K_{a,\lambda}(x)^{n-2} \mathcal{U}\left(a + K_{a,\lambda}(x)^2(x-a)\right)$$

for all $x \in \mathbb{R}^n \backslash \{a\}$, where $K_{a,\lambda}$ is given by $K_{a,\lambda}(x) = \frac{\lambda}{|x-a|}$. As one can check, for any $u \in C^2(\mathbb{R}^n, \mathbb{R})$, for any $a \in \mathbb{R}^n$, for any $\lambda > 0$, and for any $x \in \mathbb{R}^n \backslash \{a\}$,

$$\Delta u_{a,\lambda}(x) = K_{a,\lambda}(x)^{n+2} \Delta u\left(a + K_{a,\lambda}(x)^2(x-a)\right) \ . \tag{1.3}$$

In particular, if $\mathcal{U}$ is a nonnegative solution of (1.1), so is $\mathcal{U}_{a,\lambda}$ in $\mathbb{R}^n \backslash \{a\}$ for all $a \in \mathbb{R}^n$ and all $\lambda > 0$. Writing that $\mathcal{U}_{a,\lambda} = ((u_1)_{a,\lambda}, \ldots, (u_p)_{a,\lambda})$, it follows that

$$\Delta (u_i)_{a,\lambda} = |\mathcal{U}_{a,\lambda}|^{2^\star-2} (u_i)_{a,\lambda} \tag{1.4}$$

in $\mathbb{R}^n \backslash \{a\}$ for all $a \in \mathbb{R}^n$, all $\lambda > 0$, and all $i = 1, \ldots, p$. Before proving Proposition 1.1 we establish three lemmas. Our approach is based on the analysis developed in Li and Zhang [28].

Lemma 1.1. *Let $\mathcal{U}$ be a nonnegative C^2-solution of* (1.1)*. For any point a in $\mathbb{R}^n$, there exists a positive real number $\lambda_0(a)$ such that for any λ in $(0, \lambda_0(a))$, there holds $(u_i)_{a,\lambda} \le u_i$ in $\mathbb{R}^n \backslash B_a(\lambda)$ for $i = 1, \ldots, p$.*

Proof of Lemma 1.1. Without loss of generality, we may take $a = 0$. We denote $(u_i)_{0,\lambda} = (u_i)_\lambda$ for $i = 1, \ldots, p$. By the superharmonicity of the function u_i and by the strong maximum principle, for $i = 1, \ldots, p$, there holds either $u_i \equiv 0$ or $u_i > 0$ in $\mathbb{R}^n$. In case $u_i > 0$, as is easily seen, there exists a positive real number r_0 such that for any $r \in (0, r_0)$ and for any point $\theta \in S^{n-1}$, there holds

$$\frac{d}{dr}\left(r^{\frac{n-2}{2}} u_i(r\theta)\right) > 0 \ ,$$

for $i = 1, \ldots, p$. It follows that for any $\lambda \in (0, r_0]$, there holds

$$(u_i)_\lambda \le u_i \tag{1.5}$$

in $\overline{B_0(r_0)} \backslash B_0(\lambda)$. On the other hand, by the superharmonicity of the function u_i and by the Hadamard Three-Sphere theorem as stated, for instance, in Protter and Weinberger [35], for any real number $r > r_0$ and for any point $x \in B_0(r) \backslash B_0(r_0)$, we get

$$\begin{aligned}\left(r_0^{2-n} - r^{2-n}\right) u_i(x) \quad &\ge \quad \left(|x|^{2-n} - r^{2-n}\right) \min_{\partial B_0(r_0)} u_i + \left(r_0^{2-n} - |x|^{2-n}\right) \min_{\partial B_0(r)} u_i \\ &\ge \quad \left(|x|^{2-n} - r^{2-n}\right) \min_{\partial B_0(r_0)} u_i\end{aligned}$$

for $i = 1, \ldots, p$. Letting $r \to +\infty$ gives

$$u_i(x) \ge \left(\frac{r_0}{|x|}\right)^{n-2} \min_{\partial B_0(r_0)} u_i \tag{1.6}$$

for $i = 1, \ldots, p$. We take

$$\lambda_0 = r_0 \min_{i \in I_0} \left(\frac{\displaystyle\min_{\partial B_0(r_0)} u_i}{\displaystyle\max_{\overline{B_0(r_0)}} u_i}\right)^{\frac{1}{n-2}} ,$$

where

$$I_0 = \left\{ i \in \{1, \ldots, p\} \text{ s.t. } u_i \not\equiv 0 \text{ in } \mathbb{R}^n \right\} .$$

For any real number $\lambda \in (0, \lambda_0)$ and for any point $x \in \mathbb{R}^n \backslash B_0(r_0)$, there holds

$$(u_i)_\lambda(x) \le \left(\frac{\lambda_0}{|x|}\right)^{n-2} \max_{\overline{B_0(r_0)}} u_i \le \left(\frac{r_0}{|x|}\right)^{n-2} \min_{\partial B_0(r_0)} u_i \tag{1.7}$$

for $i = 1, \ldots, p$. It follows from (1.5)–(1.7) that for any λ in $(0, \lambda_0)$, there holds $(u_i)_\lambda \le u_i$ in $\mathbb{R}^n \backslash B_0(\lambda)$ for $i = 1, \ldots, p$. This ends the proof of Lemma 1.1. $\square$

By Lemma 1.1, for any point a in $\mathbb{R}^n$, we can now define

$$\overline{\lambda}(a) = \sup\left\{\lambda > 0 \text{ s.t. } (u_i)_{a,\lambda} \le u_i \text{ in } \mathbb{R}^n \backslash B_a(\lambda) \text{ for } i = 1, \ldots, p\right\} .$$

The next lemma in the proof of Proposition 1.1 is as follows.

Lemma 1.2. *Let $\mathcal{U}$ be a nonnegative C^2-solution of* (1.1)*. If there holds that $\overline{\lambda}(a) < +\infty$ for some point a in $\mathbb{R}^n$, then there holds $\left|\mathcal{U}_{a,\overline{\lambda}(a)}\right| \equiv |\mathcal{U}|$ in $\mathbb{R}^n \backslash \{a\}$.*

Proof of Lemma 1.2. Without loss of generality, we may take $a = 0$. We denote $\overline{\lambda}(0) = \overline{\lambda}$ and $(u_i)_{0,\lambda} = (u_i)_\lambda$ for $i = 1, \dots, p$. By definition of $\overline{\lambda}$, in case $\overline{\lambda} < +\infty$, we get that for any $\lambda \in (0, \overline{\lambda}]$, there holds

$$(u_i)_\lambda \le u_i \tag{1.8}$$

in $\mathbb{R}^n \backslash B_0(\lambda)$ for $i = 1, \dots, p$, and that there exist an index i_0 and a sequence of real numbers $(\lambda_\alpha)_\alpha$ in $(\overline{\lambda}, +\infty)$ converging to $\overline{\lambda}$ such that property (1.8) does not hold true for $i = i_0$ and $\lambda = \lambda_\alpha$. For any positive real number λ, we let v_λ be the function defined on $\mathbb{R}^n \backslash \{0\}$ by $v_\lambda = u_{i_0} - (u_{i_0})_\lambda$. By (1.1), (1.4), and (1.8), we get

$$-\Delta v_{\overline{\lambda}} = |\mathcal{U}|^{2^*-2} u_{i_0} - \left|\mathcal{U}_{\overline{\lambda}}\right|^{2^*-2} (u_{i_0})_{\overline{\lambda}} \ge 0 \tag{1.9}$$

in $\mathbb{R}^n \backslash B_0(\overline{\lambda})$. We clearly have that

$$\min_{\mathbb{R}^n \backslash B_0(\overline{\lambda})} v_{\overline{\lambda}} = \min_{\partial B_0(\overline{\lambda})} v_{\overline{\lambda}} = 0 . \tag{1.10}$$

We claim that there holds $v_{\overline{\lambda}} \equiv 0$ in $\mathbb{R}^n \backslash B_0(\overline{\lambda})$. In order to prove this claim, we proceed by contradiction and assume that $v_{\overline{\lambda}} \not\equiv 0$ in $\mathbb{R}^n \backslash B_0(\overline{\lambda})$. By (1.10) and by the Hopf lemma, it follows that the outward normal derivative of the function $v_{\overline{\lambda}}$ on $\partial B_0(\overline{\lambda})$ is positive. By the continuity of ∇u_{i_0}, we then get that there exists a real number $r_0 > \overline{\lambda}$ such that for any $\lambda \in [\overline{\lambda}, r_0)$, there holds

$$v_\lambda > 0 \tag{1.11}$$

in $\overline{B_0(r_0)} \backslash B_0(\lambda)$. Using the Hadamard Three-Sphere theorem as in Lemma 1.1, we also get that for any point $x \in \mathbb{R}^n \backslash B_0(r_0)$, there holds

$$v_{\overline{\lambda}}(x) \ge \left(\frac{r_0}{|x|}\right)^{n-2} \min_{\partial B_0(r_0)} v_{\overline{\lambda}} . \tag{1.12}$$

On the other hand, by the uniform continuity of the function u_{i_0} on $\overline{B_0(r_0)}$, there exists a positive real number ε such that for any $\lambda \in \left[\overline{\lambda}, \overline{\lambda} + \varepsilon\right]$ and for any point $x \in \mathbb{R}^n \backslash B_0(r_0)$, there holds

$$\left|v_{\overline{\lambda}}(x) - v_\lambda(x)\right| = \left|(u_{i_0})_\lambda(x) - (u_{i_0})_{\overline{\lambda}}(x)\right| \le \frac{1}{2}\left(\frac{r_0}{|x|}\right)^{n-2} \min_{\partial B_0(r_0)} v_{\overline{\lambda}} . \tag{1.13}$$

It follows from (1.11)–(1.13) that for any $\lambda \in \left[\overline{\lambda}, \overline{\lambda} + \varepsilon\right]$, there holds $v_\lambda \ge 0$ in $\mathbb{R}^n \backslash B_0(\lambda)$. This contradicts the definition of $\overline{\lambda}$, and this ends the proof of our claim, namely that there holds $v_{\overline{\lambda}} \equiv 0$ in $\mathbb{R}^n \backslash B_0(\overline{\lambda})$. Taking into account that

$$v_{\overline{\lambda}}(x) = -\left(\frac{\overline{\lambda}}{|x|}\right)^{n-2} v_{\overline{\lambda}}\left(\left(\frac{\overline{\lambda}}{|x|}\right)^2 x\right)$$

for all points x in $\mathbb{R}^n \backslash \{0\}$, we even get that there holds $v_{\overline{\lambda}} \equiv 0$ in $\mathbb{R}^n \backslash \{0\}$. Moreover, the function u_{i_0} cannot be identically zero without contradicting the definition of $\overline{\lambda}$, and thus, by the maximum principle, u_{i_0} is nowhere vanishing. By (1.9), it follows that there holds $\left|\mathcal{U}_{\overline{\lambda}}\right| \equiv |\mathcal{U}|$ in $\mathbb{R}^n \backslash \{0\}$. This ends the proof of Lemma 1.2. □

The third and last lemma in the proof of Proposition 1.1 states as follows.

Lemma 1.3. *Let $\mathcal{U}$ be a nonnegative C^2-solution of* (1.1). *If there holds that $\overline{\lambda}(a) = +\infty$ for some point a in $\mathbb{R}^n$, then the p-map $\mathcal{U}$ is identically zero.*

Proof of Lemma 1.3. By definition of $\overline{\lambda}(a)$, in case $\overline{\lambda}(a) = +\infty$, we get that for any positive real number λ, there holds

$$(u_i)_{a,\lambda} \le u_i$$

in $\mathbb{R}^n \backslash B_a(\lambda)$ for $i = 1, \dots, p$. Without loss of generality we may here again assume that $a = 0$. In particular, we get

$$\lambda^{n-2} u_i(0) \le \liminf_{|x| \to +\infty} \left(|x|^{n-2} u_i(x) \right) .$$

Letting $\lambda \to +\infty$, it follows that for $i = 1, \dots, p$, either $u_i(0) = 0$ or

$$|x|^{n-2} u_i(x) \to +\infty$$

as $|x| \to +\infty$. If there holds $u_i(0) = 0$ for some $i = 1, \dots, p$, then by the superharmonicity of the function u_i and by the strong maximum principle, u_i is identically zero. Therefore, we may now assume that there holds $|x|^{n-2} u_i(x) \to +\infty$ as $|x| \to +\infty$ for all $i = 1, \dots, p$ such that $u_i \not\equiv 0$. We then claim that there holds $\overline{\lambda}(y) = +\infty$ for all points y in $\mathbb{R}^n$. Indeed, if not the case, namely if there holds $\overline{\lambda}(y) < +\infty$ for some point y in $\mathbb{R}^n$, then by Lemma 1.2, we get

$$|x|^{n-2} |\mathcal{U}(x)| = |x|^{n-2} \left| \mathcal{U}_{y,\overline{\lambda}(y)}(x) \right| \longrightarrow \overline{\lambda}(y)^{n-2} |\mathcal{U}(y)|$$

as $|x| \to +\infty$, which is a contradiction. By Lemma 11.2 in Li and Zhang [28] if there holds $\overline{\lambda}(y) = +\infty$ for all points y in $\mathbb{R}^n$, then we get that the p-map $\mathcal{U}$ is constant. Taking into account that $\mathcal{U}$ satisfies (1.1), it follows that $\mathcal{U}$ is identically zero. □

We are now in position to end the proof of Proposition 1.1.

Proof of Proposition 1.1. By Lemma 1.3, we may assume that for any point $y \in \mathbb{R}^n$, there holds $\overline{\lambda}(y) < +\infty$. By Lemma 1.2, it follows that for any point y in $\mathbb{R}^n$, there holds $\left| \mathcal{U}_{y,\overline{\lambda}(y)} \right| \equiv |\mathcal{U}|$ in $\mathbb{R}^n \backslash \{y\}$. By Lemma 11.1 in Li and Zhang [28], we then get that there exist a point $a \in \mathbb{R}^n$ and two positive real numbers λ and λ' such that

$$|\mathcal{U}(x)| = \left(\frac{\lambda'}{\lambda + |x-a|^2} \right)^{\frac{n-2}{2}} \tag{1.14}$$

for all points x in $\mathbb{R}^n$. For any positive real number R, we define the function η_R in $\mathbb{R}^+$ by $\eta_R(x) = \eta(x/R)$, where η is a smooth cutoff function in $\mathbb{R}^+$ satisfying $\eta \equiv 1$ in $[0, 1]$, $0 \le \eta \le 1$ in $[1, 2]$, and $\eta \equiv 0$ in $[2, +\infty)$. For any positive real number R, multiplying (1.1) by $\eta_R u_i$, summing over i and integrating by parts in $\mathbb{R}^n$ gives

$$\int_{\mathbb{R}^n} |\nabla \mathcal{U}|^2 \eta_R dx + \frac{1}{2} \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta \eta_R dx = \int_{\mathbb{R}^n} |\mathcal{U}|^{2^\star} \eta_R dx . \tag{1.15}$$

By (1.14), we get

$$\left| \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta \eta_R dx \right| \le \frac{\|\Delta \eta\|_{C^0(\mathbb{R}^n)}}{R^2} \int_{B_0(2R) \backslash B_0(R)} |\mathcal{U}|^2 dx = O\left(R^{2-n}\right) \tag{1.16}$$

as $R \to +\infty$. Passing to the limit into (1.15) as $R \to +\infty$, it follows from (1.16) that

$$\int_{\mathbb{R}^n} |\nabla \mathcal{U}|^2 dx = \int_{\mathbb{R}^n} |\mathcal{U}|^{2^\star} dx < +\infty .$$

By Proposition 3.1 in Druet and Hebey [19] we then get that the p-map $\mathcal{U}$ is of the form (1.2). This ends the proof of Proposition 1.1. □

2. Phase stability

We very briefly discuss the implications that the stationary notion of bounded stability introduced in the introduction has in terms of dynamics. For this we define a notion of phase stability, see below, and discuss standing waves of critical nonlinear Klein-Gordon and Schrödinger equations associated with (0.1). The critical nonlinear vector-valued Schrödinger equations we consider in this section are written as

$$i\frac{\partial u_i}{\partial t} - \Delta_g u_i - \sum_{j=1}^{p} A_{ij}(x)u_j + |\mathcal{U}|^{2^\star-2} u_i = 0 \tag{2.1}$$

in M for all i. The critical nonlinear vector-valued Klein-Gordon equations we consider are written as

$$\frac{\partial^2 u_i}{\partial t^2} + \Delta_g u_i + \sum_{j=1}^{p} A_{ij}(x)u_j - |\mathcal{U}|^{2^\star-2} u_i = 0 \tag{2.2}$$

in M for all i. In the above equations $A \in C^1\left(M, M_p^s(\mathbb{R})\right)$. The vector-valued Schrödinger equations traditionally arise as a limiting case of the Zakharov system associated with plasma physics. In this framework equation (2.1) is a special case of the traditional vector nonlinear Schrödinger equation corresponding to the addition of a matrix potential in the linear part of the equation, and to the choice $\alpha = 1$ of the thermal velocity parameter in the original equations. Let $\mathcal{U}e^{-i\omega t}$ be the standing waves model for (2.1) and (2.2), where the amplitude $\mathcal{U} : M \to \mathbb{R}^p$ is assumed to be nonnegative. It is easily checked that $\mathcal{U}e^{-i\omega t}$ is a standing wave for (2.1) if and only if $\mathcal{U}$ solves

$$\Delta_g u_i + \sum_{j=1}^{p} \left(A_{ij}(x) - \tilde{\omega}\delta_{ij}\right) u_j = |\mathcal{U}|^{2^\star-2} u_i \tag{2.3}$$

in M for all i, where $\tilde{\omega} = \omega$, and that it is a standing wave for (2.2) if and only if $\mathcal{U}$ solves (2.3) with $\tilde{\omega} = \omega^2$. In other words, $\mathcal{U}e^{-i\omega t}$ is a standing wave for (2.1) and (2.2) if and only if $\mathcal{U}$ solves (0.1) with the phase translated matrix $A - \omega\mathrm{Id}_p$ and $A - \omega^2\mathrm{Id}_p$.

In what follows, we define phase stability by the property that a convergence of the phase implies a convergence of the amplitude. When phase stability holds true, the corresponding standing wave sequence converges to another standing wave and phase stability clearly prevents the existence of standing waves with arbitrarily large amplitude in L^∞-norm.

Definition. *A phase ω is stable if for any sequence of standing waves with amplitudes $\mathcal{U}_\alpha$ and phases ω_α, the convergence $\omega_\alpha \to \omega$ in $\mathbb{R}$ as $\alpha \to +\infty$ implies that, up to a subsequence, $\mathcal{U}_\alpha \to \mathcal{U}$ in C^2 as $\alpha \to +\infty$.*

An easy consequence of our theorem and of (2.3) is that large phases are always stable (with extra assumptions on A when $n = 3$). In particular, fast oscillating standing waves ($|\omega| \gg 1$) for the critical nonlinear vector-valued Klein-Gordon and Schrödinger equations cannot have arbitrarily large amplitude. We also get that small phases are stable, and thus that slow oscillating standing waves ($|\omega| \ll 1$)

cannot have arbitrarily large amplitude as well, if the potential A is sufficiently small. Recall that standing waves here are like $\mathcal{U}e^{-i\omega t}$, where $\mathcal{U} \geq 0$.

Corollary. *Large phases, required to be positive for* (2.1)*, are generically stable. In particular, fast oscillating standing waves cannot have arbitrarily large amplitude. Small phases are also stable, and slow oscillating standing waves cannot have arbitrarily large amplitude as well, if the potential A is sufficiently small.*

To be more precise, assume that $-A$ is cooperative, that $\tilde{\omega} = \omega$ (resp. $\tilde{\omega} = \omega^2$) is such that $\Delta_g + (A - \tilde{\omega}\mathrm{Id}_p)$ is coercive, and that

$$A < \left(\frac{n-2}{4(n-1)}S_g + \tilde{\omega}\right)\mathrm{Id}_p \ . \tag{2.4}$$

Classical minimization arguments give that standing waves with nonnegative amplitude and phase ω exist for the critical nonlinear vector-valued Klein-Gordon and Schrödinger equations. Our theorem provides the stability of such standing waves with respect to ω. As is easily checked, (2.4) is satisfied by large phases. Let $(a_{ij})_{i,j}$ be a symmetrical matrix of C^1 functions $a_{ij} : M \to \mathbb{R}$ such that $\sum_{j=1}^p a_{ij}(x) = 1$ for all $i = 1, \dots, p$ and all $x \in M$, and let $A(g)$ be the C^1 maps from M to $M_p^s(\mathbb{R})$ given by $A(g)_{ij} = \frac{n-2}{4(n-1)}S_g a_{ij}$ for all $i, j = 1, \dots, p$. By Druet and Hebey [16, 19], the system (0.1) associated with $A(g)$ is unstable when posed on spherical space forms in any dimension $n \geq 6$. By the noncompactness of the conformal group on the sphere the system is noncompact when posed on the sphere in any dimension $n \geq 3$, and by the constructions in Brendle [6] and Brendle and Marques [8], there are examples of nonconformally flat manifolds for any $n \geq 25$ such that the system (0.1) associated with $A(g)$ is noncompact, and thus also unstable. If $A - \omega\mathrm{Id}_p = A(g)$, or $A - \omega^2\mathrm{Id}_p = A(g)$, we then get instability of the phase ω for (2.1) and (2.2). However, if A is sufficiently small such that (0.2) is satisfied, then (2.4) is still satisfied with $|\omega| \ll 1$ sufficiently small, and our theorem provides the stability of such ω's. In particular, small phases are also stable, and thus slow oscillating standing waves cannot have arbitrarily large amplitude as well, if the potential A is sufficiently small. Instability comes in the intermediate regime.

3. Pointwise controls in blow-up theory

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M, M_p^s(\mathbb{R})\right)$, $\alpha \in \mathbb{N}$. We consider the sequence of approximated equations

$$\Delta_g u_i + \sum_{j=1}^p A_{ij}^\alpha(x) u_j = |\mathcal{U}|^{2^\star - 2} u_i \ , \tag{3.1}$$

where $A_\alpha = (A_{ij}^\alpha)_{i,j}$, and we assume that

$$A_\alpha \to A \tag{3.2}$$

in $C^1\left(M, M_p^s(\mathbb{R})\right)$ as $\alpha \to +\infty$ for some $A \in C^1\left(M, M_p^s(\mathbb{R})\right)$. We let $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of (3.1) and we assume that

$$\max_M |\mathcal{U}_\alpha| \to +\infty \tag{3.3}$$

as $\alpha \to +\infty$. For $\mathcal{U} \in C^1(M, \mathbb{R}^p)$ we define $|\mathcal{U}|_\Sigma$ by

$$|\mathcal{U}|_\Sigma = \sum_{i=1}^p u_i \ , \tag{3.4}$$

where $\mathcal{U} = (u_1, \dots, u_p)$. If $\mathcal{U} \geq 0$ solves an equation like (0.1), summing the equations in (0.1), we get that

$$\Delta_g |\mathcal{U}|_\Sigma + \Lambda |\mathcal{U}|_\Sigma \geq 0 \ ,$$

where, for example, $\Lambda = p\|A\|_\infty$ and $\|A\|_\infty = \max_M \max_{ij} |A_{ij}(x)|$. In particular, $|\mathcal{U}|_\Sigma$ satisfies the maximum principle and we get that either $|\mathcal{U}|_\Sigma \equiv 0$ or $|\mathcal{U}|_\Sigma > 0$ everywhere in M. As a consequence, either $\mathcal{U} \equiv 0$ or $|\mathcal{U}| > 0$ everywhere in M, and we get that $|\mathcal{U}|$ is of class $C^{2,\theta}$, $\theta \in (0,1)$, exactly like $\mathcal{U}$ is. In what follows we let $(x_\alpha)_\alpha$ be a sequence of points in M and $(\rho_\alpha)_\alpha$, $0 < \rho_\alpha < i_g/7$, be a sequence of positive real numbers, where i_g is the injectivity radius of g. We assume that the x_α's and ρ_α's are such that

$$\begin{aligned} &\nabla |\mathcal{U}_\alpha| (x_\alpha) = 0 \ \text{ and} \\ &d_g(x_\alpha, x)^{\frac{n-2}{2}} |\mathcal{U}_\alpha(x)| \leq C \end{aligned} \tag{3.5}$$

for all α, all $x \in B_{x_\alpha}(7\rho_\alpha)$, and some $C > 0$ independent of α and x. We define

$$\mu_\alpha = \frac{1}{|\mathcal{U}_\alpha(x_\alpha)|^{\frac{2}{n-2}}} \tag{3.6}$$

for all α, and aim in getting pointwise control estimates on the $\mathcal{U}_\alpha$'s around the x_α's. We start with a general Harnack type inequality in Lemma 3.1 and then get our control estimates in Lemmas 3.2, 3.4, and 3.5 under the additional assumption that

$$\lim_{\alpha \to +\infty} \rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} |\mathcal{U}_\alpha| = +\infty \ . \tag{3.7}$$

Lemma 3.3 is used as an intermediate state between the asymptotic description in Lemma 3.2 and the sharp pointwise control in Lemma 3.4.

Lemma 3.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M, M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of* (3.1) *such that* (3.2) *and* (3.3) *hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that* (3.5) *holds true, and let $R \geq 6$ be given. There exists $C > 1$ such that for any sequence $(s_\alpha)_\alpha$ of positive real numbers satisfying that $s_\alpha > 0$ and $Rs_\alpha \leq 6\rho_\alpha$ for all α, there holds*

$$s_\alpha \|\nabla \mathcal{U}_\alpha\|_{L^\infty(\Omega_\alpha)} \leq C \sup_{\Omega_\alpha} |\mathcal{U}_\alpha| \leq C^2 \inf_{\Omega_\alpha} |\mathcal{U}_\alpha| \ ,$$

where Ω_α is given by $\Omega_\alpha = B_{x_\alpha}(Rs_\alpha) \setminus B_{x_\alpha}\left(\frac{1}{R}s_\alpha\right)$ and, for $\mathcal{U} = (u_1, \dots, u_p)$, $\|\nabla \mathcal{U}\|_{L^\infty} = \max_i \|\nabla u_i\|_{L^\infty}$.

Proof of Lemma 3.1. Let $R \geq 6$ be given and $(s_\alpha)_\alpha$ be a sequence of positive real numbers such that $s_\alpha > 0$ and $Rs_\alpha \leq 6\rho_\alpha$ for all α. We set for $x \in B_0(\frac{7R}{6})$,

$$\begin{aligned} \hat{\mathcal{U}}_\alpha(x) &= s_\alpha^{\frac{n-2}{2}} \mathcal{U}_\alpha\left(\exp_{x_\alpha}(s_\alpha x)\right) \ , \\ \hat{A}_\alpha(x) &= A_\alpha\left(\exp_{x_\alpha}(s_\alpha x)\right) \ , \text{ and} \\ \hat{g}_\alpha(x) &= \left(\exp_{x_\alpha}^\star g\right)(s_\alpha x) \ . \end{aligned}$$

Up to a subsequence, $\hat{g}_\alpha \to \hat{g}$ in $C^2_{loc}\left(B_0(\frac{7R}{6})\right)$ as $\alpha \to +\infty$, where $\hat{g}$ is some Riemannian metric in $B_0(\frac{7R}{6})$, and $\hat{g} = \xi$ as soon as $s_\alpha \to 0$, where ξ is the Euclidean metric. We know thanks to (3.5) that

$$|\hat{\mathcal{U}}_\alpha(x)| \le C|x|^{1-\frac{n}{2}} \tag{3.8}$$

in $B_0(\frac{7R}{6}) \setminus \{0\}$. Thanks to equation (3.1), we also get that

$$\Delta_{\hat{g}_\alpha}(\hat{u}_\alpha)_i + s_\alpha^2 \sum_{j=1}^p \hat{A}_{ij}^\alpha(x)(\hat{u}_\alpha)_j = |\hat{\mathcal{U}}_\alpha|^{2^\star-2}(\hat{u}_\alpha)_i \tag{3.9}$$

in $B_0(\frac{7R}{6})$ for all i, where $\hat{\mathcal{U}}_\alpha = ((\hat{u}_\alpha)_1, \dots, (\hat{u}_\alpha)_p)$. It follows from (3.8) and (3.9) that

$$|\Delta_{\hat{g}_\alpha}(\hat{u}_\alpha)_i| \le \left(C^{2^\star-2}\,|x|^{-2} + ps_\alpha^2\,\|A_\alpha\|_\infty\right) \sup_{B_0\left(\frac{13R}{12}\right)\setminus B_0\left(\frac{12}{13R}\right)} |\hat{\mathcal{U}}_\alpha|$$

in $B_0\left(\frac{13R}{12}\right) \setminus B_0\left(\frac{12}{13R}\right)$ for all $i = 1, \dots, p$. Sobolev embeddings lead then to the existence of some $D > 0$ such that

$$\sup_{B_0(R)\setminus B_0\left(\frac{1}{R}\right)} |\nabla(\hat{u}_\alpha)_i| \le D \sup_{B_0\left(\frac{13R}{12}\right)\setminus B_0\left(\frac{12}{13R}\right)} |\hat{\mathcal{U}}_\alpha| \tag{3.10}$$

for all $i = 1, \dots, p$. Let $i \in \{1, \dots, p\}$ be given and let $\hat{u}_\alpha = |\hat{\mathcal{U}}_\alpha|_\Sigma$, where $|\cdot|_\Sigma$ is as in (3.4). By the maximum principle, $\hat{u}_\alpha > 0$. Summing the equations in (3.9) we have that

$$\Delta_{\hat{g}_\alpha}\hat{u}_\alpha = F_\alpha \hat{u}_\alpha \tag{3.11}$$

in $B_0(\frac{7R}{6})$, where

$$F_\alpha = |\hat{\mathcal{U}}_\alpha|^{2^\star-2} - s_\alpha^2 \frac{\sum_{i,j=1}^p \hat{A}_{ij}^\alpha(\hat{u}_\alpha)_j}{\hat{u}_\alpha}\,. \tag{3.12}$$

Combining (3.8) and (3.12) we get that

$$|F_\alpha| \le \left(\frac{7R}{6}\right)^2 C^{2^\star-2} + s_\alpha^2\,\|A_\alpha\|_\infty \tag{3.13}$$

in $B_0(\frac{7R}{6}) \setminus B_0\left(\frac{6}{7R}\right)$. Thanks to the Harnack inequality that we apply to the solutions $\hat{u}_\alpha$ of (3.11), see for instance Theorem 4.17 of [22], we get the existence of some $D > 0$ independent of α, K and x such that

$$\sup_{B_x(2K)} \hat{u}_\alpha \le D\left(\inf_{B_x(K)} \hat{u}_\alpha + K\,\|F_\alpha\|_{L^n(B_x(2K))} \sup_{B_x(2K)} \hat{u}_\alpha\right)$$

for all α and all balls $B_x(2K) \subset B_0(\frac{7R}{6})$. Using (3.13) and choosing K small enough clearly leads to the existence of some $D > 0$ such that

$$\begin{aligned} \sup_{B_0(R)\setminus B_0(\frac{1}{R})} \hat{u}_\alpha &\le \sup_{B_0(\frac{13R}{12})\setminus B_0(\frac{12}{13R})} \hat{u}_\alpha \\ &\le D \inf_{B_0(R)\setminus B(\frac{1}{R})} \hat{u}_\alpha \le D \sup_{B_0(R)\setminus B_0(\frac{1}{R})} \hat{u}_\alpha \end{aligned} \tag{3.14}$$

for all α. It remains to note that $\frac{1}{p}\hat{u}_\alpha^2 \le |\mathcal{U}_\alpha|^2 \le \hat{u}_\alpha^2$ to conclude the lemma with (3.10) and (3.14). □

Lemmas 3.2 to 3.5 below are involved with getting pointwise control estimates on the $\mathcal{U}_\alpha$'s.

Lemma 3.2. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge 3$, $p\ge 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M,M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of (3.1) such that (3.2) and (3.3) hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that (3.5) and (3.7) hold true. After passing to a subsequence,*

$$\mu_\alpha^{\frac{n-2}{2}}\mathcal{U}_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)\to\left(\frac{1}{1+\frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}\Lambda \tag{3.15}$$

in $C^1_{loc}(\mathbb{R}^n)$ as $\alpha\to+\infty$, where μ_α is as in (3.6), $\Lambda\in S_+^{p-1}$, and S_+^{p-1} is the set of vectors in $\mathbb{R}^p$ with nonnegative components and such that $|\Lambda|=1$. Moreover, $\frac{\rho_\alpha}{\mu_\alpha}\to+\infty$ as $\alpha\to+\infty$. In particular, $\mu_\alpha\to 0$ as $\alpha\to+\infty$.

Proof of Lemma 3.2. Let $y_\alpha\in B_{x_\alpha}(6\rho_\alpha)$ and $\nu_\alpha>0$ be such that

$$|\mathcal{U}_\alpha(y_\alpha)|=\sup_{B_{x_\alpha}(6\rho_\alpha)}|\mathcal{U}_\alpha|\quad\text{and}\quad|\mathcal{U}_\alpha(y_\alpha)|=\nu_\alpha^{1-\frac{n}{2}}$$

for all α. By (3.7), $\nu_\alpha\to 0$ and $\rho_\alpha\nu_\alpha^{-1}\to+\infty$ as $\alpha\to+\infty$. By (3.5),

$$d_g\left(x_\alpha,y_\alpha\right)\le C\nu_\alpha \tag{3.16}$$

for all α. Let $\Omega_\alpha=B_0\left(\rho_\alpha\nu_\alpha^{-1}\right)$, $\Omega_\alpha\subset\mathbb{R}^n$. For $x\in\Omega_\alpha$ we set

$$\tilde{\mathcal{U}}_\alpha(x)=\nu_\alpha^{\frac{n-2}{2}}\mathcal{U}_\alpha\left(\exp_{x_\alpha}(\nu_\alpha x)\right)$$

and $g_\alpha(x)=\left(\exp_{x_\alpha}^\star g\right)(\nu_\alpha x)$. Since $\nu_\alpha\to 0$ we get that $g_\alpha\to\delta$ in $C^2_{loc}\left(\mathbb{R}^n\right)$ as $\alpha\to+\infty$, where ξ is the Euclidean metric. As is easily checked,

$$\Delta_{g_\alpha}(\tilde{u}_\alpha)_i+\nu_\alpha^2\sum_{j=1}^p\tilde{A}_{ij}^\alpha(x)(\tilde{u}_\alpha)_j=|\tilde{\mathcal{U}}_\alpha|^{2^\star-2}(\tilde{u}_\alpha)_i \tag{3.17}$$

for all i, where $\tilde{\mathcal{U}}_\alpha=((\tilde{u}_\alpha)_1,\dots,(\tilde{u}_\alpha)_p)$ and

$$\tilde{A}_{ij}^\alpha(x)=A_{ij}^\alpha\left(\exp_{x_\alpha}(\nu_\alpha x)\right)$$

for all α and all i,j. Since $|\tilde{\mathcal{U}}_\alpha|\le 1$ in Ω_α, and since $\rho_\alpha\nu_\alpha^{-1}\to+\infty$ so that $\Omega_\alpha\to\mathbb{R}^n$, we get from (3.17) and standard elliptic theory that $\tilde{\mathcal{U}}_\alpha\to\tilde{\mathcal{U}}$ in $C^1_{loc}(\mathbb{R}^n)$ as $\alpha\to+\infty$, where $\tilde{\mathcal{U}}\ge 0$ solves (1.1). Let $\tilde{y}_\alpha$ be given by

$$\tilde{y}_\alpha=\frac{1}{\nu_\alpha}\exp_{x_\alpha}^{-1}(y_\alpha)\ .$$

By (3.16) we have that $|\tilde{y}_\alpha|\le C$ for all α and we may thus assume that, up to a subsequence, $\tilde{y}_\alpha\to\tilde{y}_0$ as $\alpha\to+\infty$. Since $|\tilde{\mathcal{U}}_\alpha(\tilde{y}_\alpha)|=1$, we get that $|\tilde{\mathcal{U}}(\tilde{y}_0)|=1$ and $\tilde{y}_0$ is a point where $|\tilde{\mathcal{U}}|$ attains its maximum. Also we have that 0 is a critical point of $|\tilde{\mathcal{U}}|$ since x_α is a critical point of $|\tilde{\mathcal{U}}_\alpha|$, and we have that

$$|\tilde{\mathcal{U}}(0)|=\lim_{\alpha\to+\infty}\left(\frac{\nu_\alpha}{\mu_\alpha}\right)^{\frac{n-2}{2}}\ . \tag{3.18}$$

By Proposition 1.1, since $|\tilde{\mathcal{U}}|$ attains its maximum 1 at $\tilde{y}_0$, we get that

$$\tilde{\mathcal{U}}(x)=\left(\frac{1}{1+\frac{|x-\tilde{y}_0|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}\Lambda\ ,$$

for all $x \in \mathbb{R}^n$, where $\Lambda \in S_+^{p-1}$. Since 0 is a critical point of $|\tilde{\mathcal{U}}|$, we get that $\tilde{y}_0 = 0$, and by (3.18) we get that $\nu_\alpha = \mu_\alpha (1 + o(1))$. This proves Lemma 3.2. □

At this point we define $\varphi_\alpha : (0, \rho_\alpha) \mapsto \mathbb{R}^+$ by

$$\varphi_\alpha(r) = \frac{1}{|\partial B_{x_\alpha}(r)|_g} \int_{\partial B_{x_\alpha}(r)} |\mathcal{U}_\alpha|_\Sigma \, d\sigma_g \,, \tag{3.19}$$

where $|\partial B_{x_\alpha}(r)|_g$ is the volume of the sphere of center x_α and radius r for the induced metric and $|\cdot|_\Sigma$ is as in (3.4). As a consequence of Lemma 3.2 we have that

$$(\mu_\alpha r)^{\frac{n-2}{2}} \varphi_\alpha(\mu_\alpha r) \to \left(\frac{r}{1 + \frac{r^2}{n(n-2)}} \right)^{\frac{n-2}{2}} |\Lambda|_\Sigma \tag{3.20}$$

in $C^1_{loc}([0, +\infty))$ as $\alpha \to +\infty$. We define $r_\alpha \in [2R_0\mu_\alpha, \rho_\alpha]$ by

$$r_\alpha = \sup \left\{ r \in [2R_0\mu_\alpha, \rho_\alpha] \text{ s.t. } \left(s^{\frac{n-2}{2}} \varphi_\alpha(s) \right)' \le 0 \text{ in } [2R_0\mu_\alpha, r] \right\} \tag{3.21}$$

where $R_0^2 = n(n-2)$. Thanks to (3.20) we have that

$$\frac{r_\alpha}{\mu_\alpha} \to +\infty \tag{3.22}$$

as $\alpha \to +\infty$, while the definition of r_α gives that

$$r^{\frac{n-2}{2}} \varphi_\alpha \text{ is non-increasing in } [2R_0\mu_\alpha, r_\alpha] \tag{3.23}$$

and that

$$\left(r^{\frac{n-2}{2}} \varphi_\alpha(r) \right)' (r_\alpha) = 0 \text{ if } r_\alpha < \rho_\alpha \,. \tag{3.24}$$

Given $R > 0$ we define

$$\eta_{R,\alpha} = \sup_{B_{x_\alpha}(Rr_\alpha) \backslash B_{x_\alpha}(\frac{1}{R} r_\alpha)} |\mathcal{U}_\alpha| \,. \tag{3.25}$$

Now we can prove the following estimate.

Lemma 3.3. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, $p \ge 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1(M, M_p^s(\mathbb{R}))$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of (3.1) such that (3.2) and (3.3) hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that (3.5) and (3.7) hold true, and let $R \ge 6$ be such that $Rr_\alpha \le 6\rho_\alpha$ for all $\alpha \gg 1$. For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, after passing to a subsequence,*

$$|\mathcal{U}_\alpha(x)| \le C_\varepsilon \left(\mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)} d_g(x_\alpha, x)^{(2-n)(1-\varepsilon)} + \eta_{R,\alpha} \left(\frac{r_\alpha}{d_g(x_\alpha, x)} \right)^{(n-2)\varepsilon} \right) \tag{3.26}$$

for all $x \in B_{x_\alpha}(Rr_\alpha) \backslash \{x_\alpha\}$ and all α, where $\eta_{R,\alpha}$ is as in (3.25), μ_α is as in (3.6), and r_α is as in (3.21).

Proof of Lemma 3.3. By Lemma 3.1 there exists $C > 1$ such that

$$\frac{1}{C} \sup_{B_{x_\alpha}(Rs_\alpha) \backslash B_{x_\alpha}(\frac{1}{R} s_\alpha)} |\mathcal{U}_\alpha| \le \varphi_\alpha(s_\alpha) \le C \inf_{B_{x_\alpha}(Rs_\alpha) \backslash B_{x_\alpha}(\frac{1}{R} s_\alpha)} |\mathcal{U}_\alpha| \tag{3.27}$$

for all $0 < s_\alpha \le r_\alpha$ and all α. By (3.23) and (3.27) we then get that for $D \gg 1$ sufficiently large,

$$\begin{aligned}\sup_{x\in B_{x_\alpha}(Rr_\alpha)\backslash B_{x_\alpha}(D\mu_\alpha)} d_g(x_\alpha,x)^{\frac{n-2}{2}}|\mathcal{U}_\alpha(x)| &\le C \sup_{D\mu_\alpha\le r\le r_\alpha} r^{\frac{n-2}{2}}\varphi_\alpha(r)\\ &\le C(D\mu_\alpha)^{\frac{n-2}{2}}\varphi_\alpha(D\mu_\alpha)\end{aligned} \tag{3.28}$$

and it follows from (3.20) and (3.28) that

$$\lim_{D\to+\infty}\lim_{\alpha\to+\infty}\sup_{x\in B_{x_\alpha}(Rr_\alpha)\backslash B_{x_\alpha}(D\mu_\alpha)} d_g(x_\alpha,x)^{\frac{n-2}{2}}|\mathcal{U}_\alpha(x)| = 0\ . \tag{3.29}$$

In particular, by (3.22) and (3.29),

$$r_\alpha^{\frac{n-2}{2}}\eta_{R,\alpha}\to 0 \tag{3.30}$$

as $\alpha\to+\infty$. Let G be the Green's function of Δ_g in M, where we choose G such that $G\ge 1$. Then, see for instance Aubin [4, 5],

$$\left|d_g(x,y)^{n-2}G(x,y)-\frac{1}{(n-2)\omega_{n-1}}\right|\le\tau\left(d_g(x,y)\right) \tag{3.31}$$

and

$$\left|d_g(x,y)^{n-1}|\nabla G(x,y)|-\frac{1}{\omega_{n-1}}\right|\le\tau\left(d_g(x,y)\right) \tag{3.32}$$

for some continuous function $\tau:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $\tau(0)=0$. We fix $0<\varepsilon<\frac{1}{2}$ and set

$$\Phi_\alpha^\varepsilon(x)=\mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)}G(x_\alpha,x)^{1-\varepsilon}+\eta_{R,\alpha}r_\alpha^{(n-2)\varepsilon}G(x_\alpha,x)^\varepsilon\ .$$

By (3.31) it suffices, in order to get Lemma 3.3, to prove that

$$\sup_{B_{x_\alpha}(Rr_\alpha)\backslash\{x_\alpha\}}\frac{|\mathcal{U}_\alpha|_\Sigma}{\Phi_\alpha^\varepsilon}=O(1)\ . \tag{3.33}$$

We have $\Phi_\alpha^\varepsilon(x)\to+\infty$ as $x\to x_\alpha$. Let $y_\alpha\in\overline{B_{x_\alpha}(Rr_\alpha)}\backslash\{x_\alpha\}$ be such that

$$\sup_{B_{x_\alpha}(Rr_\alpha)\backslash\{x_\alpha\}}\frac{|\mathcal{U}_\alpha|_\Sigma}{\Phi_\alpha^\varepsilon}=\frac{|\mathcal{U}_\alpha(y_\alpha)|_\Sigma}{\Phi_\alpha^\varepsilon(y_\alpha)}\ . \tag{3.34}$$

First we assume that $d_g(x_\alpha,y_\alpha)\not\to 0$ as $\alpha\to+\infty$. Then $r_\alpha\not\to 0$ since there holds $d_g(x_\alpha,y_\alpha)\le Rr_\alpha$ and we get that $\Phi_\alpha^\varepsilon(y_\alpha)\ge C\eta_{R,\alpha}$ for some $C>0$ independent of α. By Lemma 3.1 we can also write that $|\mathcal{U}_\alpha(y_\alpha)|\le C\eta_{R,\alpha}$ for some $C>0$ independent of α. This proves (3.33) when $d_g(x_\alpha,y_\alpha)\not\to 0$ as $\alpha\to+\infty$. From now on we assume that

$$d_g(x_\alpha,y_\alpha)\to 0 \tag{3.35}$$

as $\alpha\to+\infty$ and we distinguish three different cases:

Case 1. $\frac{d_g(x_\alpha,y_\alpha)}{\mu_\alpha}\to D$ as $\alpha\to+\infty$,

Case 2. $y_\alpha\in\partial B_{x_\alpha}(Rr_\alpha)$ for all α,

Case 3. $y_\alpha\in B_{x_\alpha}(Rr_\alpha)$ and $\frac{d_g(x_\alpha,y_\alpha)}{\mu_\alpha}\to+\infty$ as $\alpha\to+\infty$.

Assume first that we are in case 1. Then, by Lemma 3.2,

$$\mu_\alpha^{\frac{n-2}{2}}\mathcal{U}_\alpha(y_\alpha)\to\left(\frac{1}{1+\frac{D^2}{n(n-2)}}\right)^{\frac{n-2}{2}}\Lambda \tag{3.36}$$

as $\alpha \to +\infty$, where $\Lambda \in S_+^{p-1}$. By (3.22), (3.30), and (3.31),

$$\begin{aligned}
\mu_\alpha^{\frac{n-2}{2}} \Phi_\alpha^\varepsilon(y_\alpha) &= \left(\frac{1}{(n-2)\omega_{n-1}}\right)^{1-\varepsilon} \left(\frac{\mu_\alpha}{d_g(x_\alpha, y_\alpha)}\right)^{(n-2)(1-\varepsilon)} + o(1) \\
&\quad + O\left(\eta_{R,\alpha} \mu_\alpha^{\frac{n-2}{2}} r_\alpha^{(n-2)\varepsilon} d_g(x_\alpha, y_\alpha)^{(2-n)\varepsilon}\right) \\
&= \left(\frac{1}{(n-2)\omega_{n-1} D^{n-2}}\right)^{1-\varepsilon} + o(1) \\
&\quad + O\left(\eta_{R,\alpha} r_\alpha^{(n-2)\varepsilon} \mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)}\right) \\
&= \left(\frac{1}{(n-2)\omega_{n-1} D^{n-2}}\right)^{1-\varepsilon} + o(1) \\
&\quad + o\left(r_\alpha^{-\frac{n-2}{2}(1-2\varepsilon)} \mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)}\right) \\
&= \left(\frac{1}{(n-2)\omega_{n-1} D^{n-2}}\right)^{1-\varepsilon} + o(1)
\end{aligned}$$

if $D \neq 0$, and if $D = 0$, noting that by (3.31),

$$\mu_\alpha^{\frac{n-2}{2}} \Phi_\alpha^\varepsilon(y_\alpha) \geq C \mu_\alpha^{(n-2)(1-\varepsilon)} d_g(x_\alpha, y_\alpha)^{-(n-2)(1-\varepsilon)} ,$$

we get that

$$\lim_{\alpha \to +\infty} \mu_\alpha^{\frac{n-2}{2}} \Phi_\alpha^\varepsilon(y_\alpha) = +\infty .$$

It follows that in case 1, for $D = 0$ or $D > 0$, using (3.36),

$$\frac{|\mathcal{U}_\alpha(y_\alpha)|}{\Phi_\alpha^\varepsilon(y_\alpha)} \to \left((n-2)\omega_{n-1} D^{n-2}\right)^{1-\varepsilon} \left(\frac{1}{1 + \frac{D^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \tag{3.37}$$

as $\alpha \to +\infty$, and (3.33) follows from (3.37). Now we assume we are in case 2. Then, by the definition of $\eta_{R,\alpha}$, we have that $|\mathcal{U}_\alpha(y_\alpha)| \leq \eta_{R,\alpha}$ and since by (3.31),

$$\begin{aligned}
\Phi_\alpha^\varepsilon(y_\alpha) &\geq \eta_{R,\alpha} r_\alpha^{(n-2)\varepsilon} G(x_\alpha, y_\alpha)^\varepsilon \\
&\geq \eta_{R,\alpha} r_\alpha^{(n-2)\varepsilon} \left(\frac{1}{(n-2)\omega_{n-1}} + o(1)\right)^\varepsilon d_g(x_\alpha, y_\alpha)^{-(n-2)\varepsilon} \\
&= \eta_{R,\alpha} \left(\frac{1}{(n-2)\omega_{n-1} R^{n-2}} + o(1)\right)^\varepsilon
\end{aligned}$$

we get that, here again, (3.33) holds true. At this point it remains to discuss case 3. Since $y_\alpha \in B_{x_\alpha}(Rr_\alpha)$ in case 3, it follows from (3.34) and (3.42) below that

$$\frac{\Delta_g |\mathcal{U}_\alpha|_\Sigma(y_\alpha)}{|\mathcal{U}_\alpha|_\Sigma(y_\alpha)} \geq \frac{\Delta_g \Phi_\alpha^\varepsilon(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} . \tag{3.38}$$

Since

$$\Delta_g |\mathcal{U}_\alpha|_\Sigma \leq C_1 |\mathcal{U}_\alpha|_\Sigma + C_2 |\mathcal{U}_\alpha|_\Sigma^{2^\star - 1} ,$$

where $C_1, C_2 > 0$ are independent of α, we get by (3.35) and (3.29) that

$$\lim_{\alpha \to +\infty} d_g(x_\alpha, y_\alpha)^2 \frac{\Delta_g |\mathcal{U}_\alpha|_\Sigma(y_\alpha)}{|\mathcal{U}_\alpha|_\Sigma(y_\alpha)} = 0 . \tag{3.39}$$

On the other hand, we compute

$$\Delta_g \Phi_\alpha^\varepsilon = \varepsilon(1-\varepsilon)\frac{|\nabla G_{x_\alpha}|^2}{G_{x_\alpha}^2}\Phi_\alpha^\varepsilon \tag{3.40}$$

and by (3.31), (3.32), and (3.40) we get that

$$\lim_{\alpha\to+\infty} d_g(x_\alpha,y_\alpha)^2\frac{\Delta_g\Phi_\alpha^\varepsilon(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} = \varepsilon(1-\varepsilon)(n-2)^2\ . \tag{3.41}$$

Combining (3.38), (3.39), and (3.41) we get a contradiction so that only cases 1 and 2 can occur. This ends the proof of Lemma 3.3. □

In the above process we used that if Ω is an open subset of M, u, v are C^2-positive functions in Ω, and $x_0 \in \Omega$ is a point where $\frac{v}{u}$ achieves its supremum in Ω, then

$$\frac{\Delta_g v(x_0)}{v(x_0)} \geq \frac{\Delta_g u(x_0)}{u(x_0)}\ . \tag{3.42}$$

Indeed, $\nabla\left(\frac{v}{u}\right) = \frac{u\nabla v - v\nabla u}{u^2}$ so that $u(x_0)\nabla v(x_0) = v(x_0)\nabla u(x_0)$. Then,

$$\Delta_g\left(\frac{v}{u}\right)(x_0) = \frac{u(x_0)\Delta_g v(x_0) - v(x_0)\Delta_g u(x_0)}{u^2(x_0)}$$

and we get (3.42) by writing that $\Delta_g\left(\frac{v}{u}\right)(x_0) \geq 0$. At this point, thanks to Lemma 3.3, we can prove the following sharp estimate.

Lemma 3.4. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\geq 3$, $p \geq 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M, M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of* (3.1) *such that* (3.2) *and* (3.3) *hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that* (3.5) *and* (3.7) *hold true, and let $R \geq 6$ be such that $Rr_\alpha \leq 6\rho_\alpha$ for all $\alpha \gg 1$. There exists $C > 0$ such that, after passing to a subsequence,*

$$|\mathcal{U}_\alpha(x)| + d_g(x_\alpha,x)\,\|\nabla\mathcal{U}_\alpha(x)\| \leq C\mu_\alpha^{\frac{n-2}{2}}d_g(x_\alpha,x)^{2-n} \tag{3.43}$$

for all $x \in B_{x_\alpha}(\frac{R}{2}r_\alpha)\backslash\{x_\alpha\}$ and all α, where, for $\mathcal{U} = (u_1,\dots,u_p)$ and $x\in M$, $\|\nabla\mathcal{U}(x)\| = \max_i|\nabla u_i(x)|$, where μ_α is as in (3.6), *and where r_α is as in* (3.21).

Proof of Lemma 3.4. We prove that there exist $C, C' > 0$ such that

$$|\mathcal{U}_\alpha(x)| \leq C\left(\mu_\alpha^{\frac{n-2}{2}}d_g(x_\alpha,x)^{2-n} + \eta_{R,\alpha}\right) \tag{3.44}$$

for all $x \in B_{x_\alpha}(\frac{R}{2}r_\alpha)\backslash\{x_\alpha\}$ and all α, and

$$\eta_{R,\alpha} \leq C'\mu_\alpha^{\frac{n-2}{2}}r_\alpha^{2-n} \tag{3.45}$$

for all α. Lemma 3.4 follows from Lemma 3.1, (3.44), and (3.45). In particular, it suffices to prove (3.44) and (3.45). We start with the proof of (3.45) assuming (3.44). By (3.23), for any $\eta\in(0,1)$,

$$(\eta r_\alpha)^{\frac{n-2}{2}}\varphi_\alpha(\eta r_\alpha) \geq r_\alpha^{\frac{n-2}{2}}\varphi_\alpha(r_\alpha)$$

for all $\alpha \gg 1$. By (3.27) we then get that

$$\frac{1}{C}r_\alpha^{\frac{n-2}{2}}\eta_{R,\alpha} \leq (\eta r_\alpha)^{\frac{n-2}{2}}\sup_{\partial B_{x_\alpha}(\eta r_\alpha)}|\mathcal{U}_\alpha|\ .$$

Assuming (3.44) it follows that

$$\frac{1}{C}\eta_{R,\alpha} \le \eta^{\frac{n-2}{2}}\left(\mu_\alpha^{\frac{n-2}{2}}(\eta r_\alpha)^{2-n} + \eta_{R,\alpha}\right)$$

and if we choose $\eta \in (0,1)$ sufficiently small such that $C\eta^{\frac{n-2}{2}} \le \frac{1}{2}$, we obtain that

$$\eta_{R,\alpha} \le \eta^{2-n}\mu_\alpha^{\frac{n-2}{2}} r_\alpha^{2-n} .$$

This proves (3.45) when we assume (3.44). Now it remains to prove (3.44). For this it suffices to prove that for any sequence $(y_\alpha)_\alpha$ such that

$$y_\alpha \in \overline{B_{x_\alpha}(\frac{R}{2}r_\alpha)}\backslash\{x_\alpha\} \tag{3.46}$$

for all α, there exists $C > 0$ such that, up to a subsequence,

$$|\mathcal{U}_\alpha(y_\alpha)| \le C\left(\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, y_\alpha)^{2-n} + \eta_{R,\alpha}\right) . \tag{3.47}$$

Let $(y_\alpha)_\alpha$ be such that y_α satisfies (3.46) for all α. As a preliminary remark one can note that (3.47) directly follows from Lemma 3.2 if $d_g(x_\alpha, y_\alpha) = O(\mu_\alpha)$. In a similar way, (3.47) follows from Lemma 3.1 if $r_\alpha^{-1} d_g(x_\alpha, y_\alpha) \not\to 0$ as $\alpha \to +\infty$. From now on we assume that

$$\lim_{\alpha\to+\infty}\frac{1}{\mu_\alpha}d_g(x_\alpha, y_\alpha) = +\infty \text{ and } \lim_{\alpha\to+\infty}\frac{1}{r_\alpha}d_g(x_\alpha, y_\alpha) = 0 . \tag{3.48}$$

Let $\lambda > 1$ be such that $\lambda p\|A\|_\infty \notin \mathrm{Sp}(\Delta_g)$, where $\mathrm{Sp}(\Delta_g)$ is the spectrum of Δ_g, and let G be the Green's function of $\Delta_g - \lambda p\|A\|_\infty$. There exist, see for instance Robert [36], positive constants $C_1 > 1$ and $C_2, C_3 > 0$ such that

$$\begin{aligned} &\frac{1}{C_1}d_g(x,y)^{2-n} - C_2 \le G(x,y) \le C_1 d_g(x,y)^{2-n} \text{ , and} \\ &|\nabla G(x,y)| \le C_3 d_g(x,y)^{1-n} \end{aligned} \tag{3.49}$$

for all $x \ne y$. By (3.49) there exists $\delta > 0$ such that $G \ge 0$ in $B_{x_\alpha}(\delta r_\alpha)$ for all α. By (3.48), $y_\alpha \in B_{x_\alpha}(\frac{\delta}{2}r_\alpha)$ for $\alpha \gg 1$. By the Green's representation formula,

$$\begin{aligned} |\mathcal{U}_\alpha|_\Sigma(y_\alpha) = &\int_{B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x)\left(\Delta_g|\mathcal{U}_\alpha|_\Sigma - \lambda p\|A\|_\infty|\mathcal{U}_\alpha|_\Sigma\right)(x)dv_g(x) \\ &+ \int_{\partial B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x)\left(\partial_\nu|\mathcal{U}_\alpha|_\Sigma\right)(x)d\sigma_g(x) \\ &- \int_{\partial B_{x_\alpha}(\delta r_\alpha)} \left(\partial_\nu G(y_\alpha, x)\right)|\mathcal{U}_\alpha|_\Sigma(x)d\sigma_g(x) , \end{aligned} \tag{3.50}$$

where ν is the unit outward normal to $\partial B_{x_\alpha}(\delta r_\alpha)$. Since $\lambda > 1$,

$$\begin{aligned} \Delta_g|\mathcal{U}_\alpha|_\Sigma - \lambda p\|A\|_\infty|\mathcal{U}_\alpha|_\Sigma &\le |\mathcal{U}_\alpha|^{2^\star-2}|\mathcal{U}_\alpha|_\Sigma \\ &\le \sqrt{p}|\mathcal{U}_\alpha|^{2^\star-1} \end{aligned}$$

and since $G \ge 0$ in $B_{x_\alpha}(\delta r_\alpha)$ we get with (3.49) that

$$\begin{aligned} &\int_{B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x)\left(\Delta_g|\mathcal{U}_\alpha|_\Sigma - \lambda p\|A\|_\infty|\mathcal{U}_\alpha|_\Sigma\right)(x)dv_g(x) \\ &\le C\int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) . \end{aligned} \tag{3.51}$$

Independently, by (3.49) and Lemma 3.1,

$$\begin{aligned}&\int_{\partial B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha,x)\left|\partial_\nu|\mathcal{U}_\alpha|_\Sigma(x)\right| d\sigma_g(x) \le C\eta_{R,\alpha}\ , \text{ and}\\ &\int_{\partial B_{x_\alpha}(\delta r_\alpha)} \left|\partial_\nu G(y_\alpha,x)\right| |\mathcal{U}_\alpha|_\Sigma(x) d\sigma_g(x) \le C\eta_{R,\alpha}\end{aligned} \tag{3.52}$$

for some $C>0$. Combining (3.50)–(3.52), we get that

$$\frac{1}{C}|\mathcal{U}_\alpha|_\Sigma(y_\alpha) \le \int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) + \eta_{R,\alpha}\ . \tag{3.53}$$

We fix $\varepsilon = \frac{2}{n+2}$. By Lemmas 3.2 and 3.3, and by (3.48), we can write that

$$\begin{aligned}&\int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x)\\ &= O\left(\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha,y_\alpha)^{2-n}\right)\\ &\quad + O\left(\mu_\alpha^{\frac{n+2}{2}(1-2\varepsilon)}\int_{B_{x_\alpha}(\delta r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} d_g(y_\alpha,x)^{2-n}d_g(x_\alpha,x)^{-(n+2)(1-\varepsilon)}dv_g(x)\right)\\ &\quad + O\left(\eta_{R,\alpha}^{2^\star-1}r_\alpha^{(n+2)\varepsilon}\int_{B_{x_\alpha}(\delta r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} d_g(y_\alpha,x)^{2-n}d_g(x_\alpha,x)^{-(n+2)\varepsilon}dv_g(x)\right)\\ &= O\left(\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha,y_\alpha)^{2-n}\right) + O\left(\eta_{R,\alpha}^{2^\star-1}r_\alpha^2\right)\end{aligned}$$

and we thus get from (3.5), that

$$\begin{aligned}&\int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x)\\ &= O\left(\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha,y_\alpha)^{2-n}\right) + O\left(\eta_{R,\alpha}\right)\ .\end{aligned} \tag{3.54}$$

Indeed $r_\alpha^2\eta_{R,\alpha}^{2^\star-1} = \left(r_\alpha^2\eta_{R,\alpha}^{2^\star-2}\right)\eta_{R,\alpha}$, and by (3.5),

$$\begin{aligned}\left(r_\alpha^2\eta_{R,\alpha}^{2^\star-2}\right)^{\frac{1}{2^\star-2}} &= r_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(Rr_\alpha)\backslash B_{x_\alpha}(\frac{1}{R}r_\alpha)} |\mathcal{U}_\alpha|\\ &\le C \sup_{x\in B_{x_\alpha}(Rr_\alpha)} d_g(x_\alpha,x)^{\frac{n-2}{2}}|\mathcal{U}_\alpha(x)| \le C\ .\end{aligned}$$

Then (3.47) follows from (3.53) and (3.54). This ends the proof of Lemma 3.4. □

At this point we define B_α by

$$B_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha,x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \tag{3.55}$$

for all α, where $x \in M$. As a last estimate in this section we prove Lemma 3.5 below.

Lemma 3.5. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge 3$, $p\ge 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M, M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of* (3.1) *such that* (3.2) *and* (3.3) *hold true.*

Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that (3.5) *and* (3.7) *hold true. There exist $C > 0$ and $(\varepsilon_\alpha)_\alpha$ such that, up to a subsequence,*

$$|\mathcal{U}_\alpha - B_\alpha \Lambda| \le C\mu_\alpha^{\frac{n-2}{2}} \left(r_\alpha^{2-n} + S_\alpha\right) + \varepsilon_\alpha B_\alpha \tag{3.56}$$

in $B_{x_\alpha}(2r_\alpha)\backslash\{x_\alpha\}$ for all α, where $\Lambda \in S_+^{p-1}$ is as in Lemma 3.2, $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, $S_\alpha(x) = d_g(x_\alpha, x)^{3-n}$ for all x, μ_α is as in (3.6)*, and r_α is as in* (3.21).

Proof of Lemma 3.5. Let G be the Green's function of $\Delta_g + 1$ in M. Let $(y_\alpha)_\alpha$ be any sequence of points in $B_{x_\alpha}(2r_\alpha)\backslash\{x_\alpha\}$. By the Green's representation formula, for any $i = 1, \dots, p$,

$$\begin{aligned} (u_\alpha)_i = &\int_{B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)\left(\Delta_g (u_\alpha)_i + (u_\alpha)_i\right)(x) dv_g(x) \\ &+ \int_{\partial B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)\left(\partial_\nu (u_\alpha)_i\right)(x) d\sigma_g(x) \\ &- \int_{\partial B_{x_\alpha}(2r_\alpha)} \left(\partial_\nu G(y_\alpha, x)\right)(u_\alpha)_i(x) d\sigma_g(x) \ , \end{aligned} \tag{3.57}$$

where ν is the unit outward normal to $B_{x_\alpha}(2r_\alpha)$ and $\mathcal{U}_\alpha = ((u_\alpha)_1, \dots, (u_\alpha)_p)$. We have, see, for instance, Druet, Hebey and Robert [20], that $G \ge 0$ and that there exist positive constants $C_1, C_2 > 0$ such that

$$\begin{aligned} &\left| d_g(x,y)^{n-2} G(x,y) - \frac{1}{(n-2)\omega_{n-1}} \right| \le C_1 d_g(x,y) \ , \text{ and} \\ &|\nabla G(x,y)| \le C_2 d_g(x,y)^{1-n} \end{aligned} \tag{3.58}$$

for all $x \ne y$. By (3.58) and Lemma 3.1,

$$\begin{aligned} &\left| \int_{\partial B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)\left(\partial_\nu (u_\alpha)_i\right)(x) d\sigma_g(x) \right| \le C\eta_{6,\alpha} \ , \\ &\left| \int_{\partial B_{x_\alpha}(2r_\alpha)} \left(\partial_\nu G(y_\alpha, x)\right)(u_\alpha)_i(x) d\sigma_g(x) \right| \le C\eta_{6,\alpha} \end{aligned} \tag{3.59}$$

and by (3.45),

$$\eta_{6,\alpha} \le C\mu_\alpha^{\frac{n-2}{2}} r_\alpha^{2-n} \ . \tag{3.60}$$

By Lemma 3.4 and (3.58),

$$\int_{B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)|\mathcal{U}_\alpha| dv_g(x) \le C\mu_\alpha^{\frac{n-2}{2}} \int_{B_{x_\alpha}(2r_\alpha)} d_g(y_\alpha, x)^{2-n} d_g(x_\alpha, x)^{2-n} dv_g(x)$$

and by Giraud's lemma we get that

$$\begin{aligned} &\int_{B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)|\mathcal{U}_\alpha| dv_g(x) \le C\mu_\alpha^{\frac{1}{2}} \ \text{ if } n = 3 \ , \\ &\int_{B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)|\mathcal{U}_\alpha| dv_g(x) \le C\mu_\alpha \left(1 + |\ln d_g(x_\alpha, y_\alpha)|\right) \ \text{ if } n = 4 \ , \text{ and} \\ &\int_{B_{x_\alpha}(2r_\alpha)} G(y_\alpha, x)|\mathcal{U}_\alpha| dv_g(x) \le C\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, y_\alpha)^{4-n} \ \text{ if } n \ge 5 \ . \end{aligned} \tag{3.61}$$

Now we let $\mathcal{R}_\alpha : M \to \mathbb{R}^p$ be given by

$$\mathcal{R}_\alpha(x) = \int_{B_{x_\alpha}(2r_\alpha)} G(x,y)|\mathcal{U}_\alpha(y)|^{2^\star-2}\mathcal{U}_\alpha(y)dv_g(y) \tag{3.62}$$

for $x \in M$, and let $f : M \to \mathbb{R}$ be given by

$$f(x) = (n-2)\omega_{n-1}d_g(x_0,x)^{n-2}G(x_0,x)$$

if $x \neq x_0$ and $f(x_0) = 1$, where, up to a subsequence, $x_\alpha \to x_0$ as $\alpha \to +\infty$. By (3.58), f is continuous at x_0 and

$$|f(x) - 1| \le Cd_g(x_0,x) \; . \tag{3.63}$$

We claim that

$$\lim_{\alpha\to+\infty}\left\|\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} - f(y_\alpha)\Lambda\right\| = 0 \; . \tag{3.64}$$

As is easily checked, Lemma 3.5 follows from (3.64). Indeed, by (3.64), since $(y_\alpha)_\alpha$ is arbitrary in $B_{x_\alpha}(2r_\alpha)\backslash\{x_\alpha\}$, for any $x \in M$

$$\begin{aligned}\|\mathcal{R}_\alpha(x) - B_\alpha(x)\Lambda\|_{\mathbb{R}^p} &\le \|\mathcal{R}_\alpha(x) - f(x)B_\alpha(x)\Lambda\|_{\mathbb{R}^p} + |f(x)-1|\,B_\alpha(x)\\ &\le \left\|\frac{\mathcal{R}_\alpha}{B_\alpha} - f\Lambda\right\|_{L^\infty} B_\alpha(x) + |f(x)-1|\,B_\alpha(x)\\ &\le \varepsilon_\alpha B_\alpha(x) + |f(x)-1|\,B_\alpha(x) \; ,\end{aligned} \tag{3.65}$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, and by (3.63) we can write that

$$\begin{aligned}|f(x)-1| &\le Cd_g(x_0,x)\\ &\le \varepsilon'_\alpha + Cd_g(x_\alpha,x) \; ,\end{aligned}$$

where $\varepsilon'_\alpha = Cd_g(x_0,x_\alpha)$ is such that $\varepsilon'_\alpha \to 0$ as $\alpha \to +\infty$. Moreover,

$$d_g(x_\alpha,x)B_\alpha(x) \le \mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha,x)^{3-n} \tag{3.66}$$

and we thus get (3.56) by combining (3.57), (3.59), (3.60), (3.61), (3.65), and (3.66). Summarizing, at this point, it remains to prove (3.64). Up to passing to a subsequence we may assume that $y_\alpha \to y_0$ as $\alpha \to +\infty$. Suppose first that $y_0 \neq x_0$. By Lemmas 3.2, Lemma 3.4, and the Lebesgue's dominated convergence theorem, writing that

$$\mathcal{R}_\alpha(y_\alpha) = \mu_\alpha^{\frac{n-2}{2}} \int_{B_0(2\frac{r_\alpha}{\mu_\alpha})} G\left(y_\alpha, \exp_{x_\alpha}(\mu_\alpha x)\right)|\tilde{\mathcal{U}}_\alpha(x)|^{2^\star-2}\tilde{\mathcal{U}}_\alpha(x)dv_{\tilde{g}_\alpha}(x) \; ,$$

where

$$\begin{aligned}&\tilde{\mathcal{U}}_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}}\mathcal{U}_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right) \text{ and}\\ &\tilde{g}_\alpha(x) = \left(\exp^\star_{x_\alpha} g\right)(\mu_\alpha x) \; ,\end{aligned} \tag{3.67}$$

we get that

$$\lim_{\alpha\to+\infty}\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} = \left(\frac{d_g(x_0,y_0)^2}{n(n-2)}\right)^{\frac{n-2}{2}}\left(\int_{\mathbb{R}^n} u_0^{2^\star-1}dx\right)G(x_0,y_0)\Lambda \; ,$$

where

$$u_0(x) = \left(\frac{1}{1+\frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \; . \tag{3.68}$$

Since

$$\int_{\mathbb{R}^n} u_0^{2^\star-1}dx = (n-2)\omega_{n-1}\left(n(n-2)\right)^{\frac{n-2}{2}}$$

we get that if $y_0 \neq x_0$, then

$$\begin{aligned}\lim_{\alpha\to+\infty}\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} &= (n-2)\omega_{n-1}d_g(x_0,y_0)^{n-2}G(x_0,y_0)\Lambda \\ &= f(y_0)\Lambda\ .\end{aligned}$$

This proves (3.64) when $y_0 \neq x_0$. Now we assume that $y_0 = x_0$. In addition, as a first case to consider, we assume also that

$$\frac{d_g(x_\alpha,y_\alpha)}{\mu_\alpha} \to D \tag{3.69}$$

as $\alpha \to +\infty$ for some $D \geq 0$. Let z_α be such that $y_\alpha = \exp_{x_\alpha}(\mu_\alpha z_\alpha)$. Then

$$\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} = \left(1+\frac{|z_\alpha|^2}{n(n-2)}\right)^{\frac{n-2}{2}}\mu_\alpha^{n-2}\int_{B_{x_\alpha}(\frac{2r_\alpha}{\mu_\alpha})}\tilde{G}_\alpha|\tilde{\mathcal{U}}_\alpha|^{2^\star-2}\tilde{\mathcal{U}}_\alpha dv_{\tilde{g}_\alpha}\ , \tag{3.70}$$

where $\tilde{\mathcal{U}}_\alpha$ and $\tilde{g}_\alpha$ are as in (3.67), and

$$\tilde{G}_\alpha(x) = G\left(\exp_{x_\alpha}(\mu_\alpha z_\alpha), \exp_{x_\alpha}(\mu_\alpha x)\right)\ .$$

By (3.58),

$$d_g\left(\exp_{x_\alpha}(\mu_\alpha z_\alpha), \exp_{x_\alpha}(\mu_\alpha x)\right)\tilde{G}_\alpha(x) \to \frac{1}{(n-2)\omega_{n-1}} \tag{3.71}$$

as $\alpha \to +\infty$ for all x, and we also have that

$$d_g\left(\exp_{x_\alpha}(\mu_\alpha z_\alpha), \exp_{x_\alpha}(\mu_\alpha x)\right) = \mu_\alpha d_{\tilde{g}_\alpha}(z_\alpha, x)\ . \tag{3.72}$$

Combining (3.70), (3.71), and (3.72), by Lemmas 3.2 and 3.4, and by the Lebesgue's dominated convergence theorem we get that

$$\lim_{\alpha\to+\infty}\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} = \left(1+\frac{|z_0|^2}{n(n-2)}\right)^{\frac{n-2}{2}}\left(\int_{\mathbb{R}^n}\frac{u_0(x)^{2^\star-1}dx}{(n-2)\omega_{n-1}|x-z_0|^{n-2}}\right)\Lambda\ , \tag{3.73}$$

where $z_\alpha \to z_0$ as $\alpha \to +\infty$, and u_0 is as in (3.68). We have that $\Delta u_0 = u_0^{2^\star-1}$, and since

$$G_0(x,y) = \frac{1}{(n-2)\omega_{n-1}|y-x|^{n-2}}$$

is the Green's function of Δ, we get from (3.73) that

$$\lim_{\alpha\to+\infty}\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} = \Lambda\ .$$

This proves (3.64) when $y_0 \neq x_0$ and we assume (3.69). Now it remains to considser the case where $y_0 = x_0$ and

$$\frac{d_g(x_\alpha,y_\alpha)}{\mu_\alpha} \to +\infty \tag{3.74}$$

as $\alpha \to +\infty$. Then

$$\frac{\mathcal{R}_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} = \left(\frac{1}{n(n-2)}+o(1)\right)^{\frac{n-2}{2}}d_g(x_\alpha,y_\alpha)^{n-2}\mu_\alpha^{-\frac{n-2}{2}}I_\alpha\ , \tag{3.75}$$

where

$$I_\alpha = \int_{B_{x_\alpha}(2r_\alpha)}G(y_\alpha,x)|\mathcal{U}_\alpha(x)|^{2^\star-2}\mathcal{U}_\alpha(x)dv_g(x)\ .$$

We write that

$$\begin{aligned} I_\alpha &= \int_{\Omega_\alpha} G(y_\alpha,x)|\mathcal{U}_\alpha(x)|^{2^\star-2}\mathcal{U}_\alpha(x)dv_g(x)\\ &+\int_{\Omega_\alpha^c} G(y_\alpha,x)|\mathcal{U}_\alpha(x)|^{2^\star-2}\mathcal{U}_\alpha(x)dv_g(x)\ , \end{aligned} \tag{3.76}$$

where

$$\Omega_\alpha = \left\{x\in B_{x_\alpha}(2r_\alpha) \text{ s.t. } d_g(y_\alpha,x)\geq \frac{1}{2}d_g(x_\alpha,y_\alpha)\right\}$$

and $\Omega_\alpha^c = B_{x_\alpha}(2r_\alpha)\backslash\Omega_\alpha$. We have that

$$\begin{aligned} &\mu_\alpha^{-\frac{n-2}{2}}\int_{\Omega_\alpha} G(y_\alpha,x)|\mathcal{U}_\alpha(x)|^{2^\star-2}\mathcal{U}_\alpha(x)dv_g(x)\\ &=\int_{\frac{1}{\mu_\alpha}\exp_{x_\alpha}^{-1}(\Omega_\alpha)} G\left(y_\alpha,\exp_{x_\alpha}(\mu_\alpha x)\right)|\tilde{\mathcal{U}}_\alpha(x)|^{2^\star-2}\tilde{\mathcal{U}}_\alpha(x)dv_{\tilde{g}_\alpha}(x)\ , \end{aligned}$$

where $\tilde{\mathcal{U}}_\alpha$ and $\tilde{g}_\alpha$ are as in (3.67). Let $z_\alpha = \exp_{x_\alpha}(\mu_\alpha x)$. For $x\in\frac{1}{\mu_\alpha}\exp_{x_\alpha}^{-1}(\Omega_\alpha)$,

$$\frac{d_g(y_\alpha,z_\alpha)}{\mu_\alpha}\to+\infty$$

as $\alpha\to+\infty$, and since

$$d_g(y_\alpha,z_\alpha)-d_g(x_\alpha,z_\alpha)\leq d_g(x_\alpha,y_\alpha)\leq d_g(y_\alpha,z_\alpha)+d_g(x_\alpha,z_\alpha)$$

and $d_g(x_\alpha,z_\alpha)=\mu_\alpha|x|$, we get that

$$\lim_{\alpha\to+\infty}\frac{d_g(x_\alpha,y_\alpha)}{d_g(y_\alpha,z_\alpha)}=1\ . \tag{3.77}$$

By (3.58) and (3.77),

$$\lim_{\alpha\to+\infty} d_g(x_\alpha,y_\alpha)^{n-2}G_\alpha\left(y_\alpha,\exp_{x_\alpha}(\mu_\alpha x)\right)=\frac{1}{(n-2)\omega_{n-1}}\ .$$

By Lemmas 3.2 and 3.4, and by the Lebesgue's dominated convergence theorem, we then get that

$$\begin{aligned} &\lim_{\alpha\to+\infty} d_g(x_\alpha,y_\alpha)^{n-2}\mu_\alpha^{-\frac{n-2}{2}}\int_{\Omega_\alpha} G(y_\alpha,x)|\mathcal{U}_\alpha(x)|^{2^\star-2}\mathcal{U}_\alpha(x)dv_g(x)\\ &=\frac{1}{(n-2)\omega_{n-1}}\left(\int_{\mathbb{R}^n}u_0^{2^\star-1}dx\right)\Lambda\\ &=\left(n(n-2)\right)^{\frac{n-2}{2}}\Lambda\ . \end{aligned} \tag{3.78}$$

Independently, by (3.58) and by Lemma 3.4,

$$\begin{aligned} &d_g(x_\alpha,y_\alpha)^{n-2}\mu_\alpha^{-\frac{n-2}{2}}\int_{\Omega_\alpha^c} G(y_\alpha,x)|\mathcal{U}_\alpha(x)|^{2^\star-2}\mathcal{U}_\alpha(x)dv_g(x)\\ &\leq Cd_g(x_\alpha,y_\alpha)^{-4}\mu_\alpha^2\int_{\Omega_\alpha^c} d_g(y_\alpha,x)^{2-n}dv_g(x)\\ &\leq C\left(\frac{\mu_\alpha}{d_g(x_\alpha,y_\alpha)}\right)^2=o(1) \end{aligned} \tag{3.79}$$

since

$$d_g(x_\alpha,x)\geq d_g(x_\alpha,y_\alpha)-d_g(y_\alpha,x)\geq\frac{1}{2}d_g(x_\alpha,y_\alpha)$$

for $x \in \Omega_\alpha^c$. Noting that (3.64) follows from (3.75), (3.76), (3.78), and (3.79), we get that (3.64) holds true when $y_0 = x_0$ and we assume (3.74). This ends the proof of Lemma 3.5. $\square$

4. Sharp pointwise blow-up estimates

In this section we prove sharp blow-up estimates for sequences of solutions of perturbed equations like (3.1) when we assume (0.2). The main result of this section is Lemma 4.3. Lemmas 4.1 and 4.2 are preliminary lemmas for the proof of Lemma 4.3. In what follows we let X_α be the 1-form given by

$$X_\alpha(x) = \left(1 - \frac{1}{6(n-1)} Rc_g^\sharp(x)\left(\nabla f_\alpha(x), \nabla f_\alpha(x)\right)\right) \nabla f_\alpha(x) \;, \tag{4.1}$$

where $f_\alpha(x) = \frac{1}{2} d_g(x_\alpha, x)^2$ and, in local coordinates, $(Rc_g^\sharp)^{ij} = g^{i\mu} g^{j\nu} R_{\mu\nu}$, where the R_{ij}'s are the components of the Ricci curvature Rc_g of g.

Lemma 4.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M, M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of (3.1) such that (3.2) and (3.3) hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that (3.5) and (3.7) hold true. Let $\mathcal{R}_{1,\alpha}$ be given by*

$$\mathcal{R}_{1,\alpha} = \sum_{i=1}^p \int_{B_{x_\alpha}(r_\alpha)} \left(\nabla X_\alpha - \frac{1}{n}(div_g X_\alpha) g\right)^\sharp \left(\nabla(u_\alpha)_i, \nabla(u_\alpha)_i\right) dv_g \;, \tag{4.2}$$

where $\mathcal{U}_\alpha = ((u_\alpha)_1, \ldots, (u_\alpha)_p)$, X_α is as in (4.1), and $A^\sharp$ is the musical isomorphism of A. Then

$$\begin{aligned} &\mathcal{R}_{1,\alpha} = \mu_\alpha r_\alpha \quad \text{if } n = 3 \;, \\ &\mathcal{R}_{1,\alpha} = o\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) + o\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) \quad \text{if } n = 4 \;, \\ &\mathcal{R}_{1,\alpha} = o\left(\mu_\alpha^2\right) + o\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) \quad \text{if } n \geq 5 \;, \end{aligned} \tag{4.3}$$

where μ_α is as in (3.6) and r_α is as in (3.21).

Proof of Lemma 4.1. Thanks to the expression of X_α,

$$\left(\nabla X_\alpha\right)_{ij} - \frac{1}{n}\left(\mathrm{div}_g X_\alpha\right) g_{ij} = O\left(d_g(x_\alpha, x)^2\right) \tag{4.4}$$

for all i, j. Assuming $n = 3$ we can write by Lemma 3.4 that

$$\begin{aligned} |\mathcal{R}_{1,\alpha}| &\leq C \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha, x)^2 \|\nabla \mathcal{U}_\alpha(x)\|^2 dv_g(x) \\ &\leq C\mu_\alpha \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha, x)^{-2} dv_g(x) \\ &\leq C\mu_\alpha r_\alpha \;. \end{aligned}$$

This proves (4.3) when $n = 3$. From now on we assume that $n \geq 4$. We have that

$$\left(\nabla X_\alpha - \frac{1}{n}\left(\mathrm{div}_g X_\alpha\right) g\right)^\sharp \left(\nabla B_\alpha, \nabla B_\alpha\right) = O\left(d_g(x_\alpha, \cdot)^3 |\nabla B_\alpha|^2\right) \;, \tag{4.5}$$

where B_α is as in (3.55). Thanks to (4.4) and (4.5) we can write that

$$
\begin{aligned}
\mathcal{R}_{1,\alpha} &= O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3 |\nabla B_\alpha(x)|^2 dv_g(x)\right)\\
&+ O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 |\nabla B_\alpha(x)| \times \|\nabla(\mathcal{U}_\alpha - B_\alpha\Lambda)(x)\| dv_g(x)\right)\\
&+ O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \|\nabla(\mathcal{U}_\alpha - B_\alpha\Lambda)(x)\|^2 dv_g(x)\right) .
\end{aligned}
\tag{4.6}
$$

We have that

$$
\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3 |\nabla B_\alpha(x)|^2 dv_g(x) = o\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) \quad \text{if } n=4 ,\\
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3 |\nabla B_\alpha(x)|^2 dv_g(x) = o\left(\mu_\alpha^2\right) \quad \text{if } n\ge 5 .
\end{aligned}
\tag{4.7}
$$

Moreover, given $i \in \{1,\dots,p\}$, integrating by parts,

$$
\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\nabla\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right|^2 dv_g(x)\\
= \quad & O\left(\int_{\partial B_{x_\alpha}(r_\alpha)} \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| d_g(x_\alpha,x)^2 \left|\nabla\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| d\sigma_g(x)\right)\\
&+O\left(\int_{\partial B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x) \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right|^2 d\sigma_g(x)\right)\\
&+O\left(\int_{B_{x_\alpha}(r_\alpha)} \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right|^2 dv_g(x)\right)\\
&+\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x) \left(\Delta_g\left((u_\alpha)_i - B_\alpha\Lambda_i\right)\right)(x) dv_g(x) ,
\end{aligned}
$$

and we get by Lemma 3.4 that

$$
\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\nabla\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right|^2 dv_g(x)\\
&= \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x) \left(\Delta_g\left((u_\alpha)_i - B_\alpha\Lambda_i\right)\right)(x) dv_g(x)\\
&+ O\left(\mu_\alpha^{n-2} r_\alpha^{4-n}\right) + O\left(\int_{B_{x_\alpha}(r_\alpha)} \left|(u_\alpha)_i - B_\alpha\Lambda_i\right|^2 dv_g\right) .
\end{aligned}
\tag{4.8}
$$

We have that

$$
\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} B_\alpha^2 dv_g = 64\omega_3 \mu_\alpha^2 \ln \frac{r_\alpha}{\mu_\alpha} + o\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) \quad \text{if } n=4 , \text{ and}\\
&\int_{B_{x_\alpha}(r_\alpha)} B_\alpha^2 dv_g = \left(\int_{\mathbb{R}^n} u_0^2 dx\right) \mu_\alpha^2 + o\left(\mu_\alpha^2\right) \quad \text{if } n\ge 5 ,
\end{aligned}
\tag{4.9}
$$

where u_0 is as in (3.68). Moreover,

$$\int_{B_{x_\alpha}(\mu_\alpha)} |(u_\alpha)_i - B_\alpha\Lambda_i|^2 \, dv_g = o\left(\mu_\alpha^2\right) \tag{4.10}$$

by Lemma 3.2, while if S_α is as in Lemma 3.5, we can write that

$$\mu_\alpha^{n-2}\int_{B_{x_\alpha}(r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} S_\alpha^2 dv_g = O\left(\mu_\alpha^{n-2} r_\alpha^{4-n}\right) + o\left(\mu_\alpha^2\right) \ . \tag{4.11}$$

By (3.22),

$$\mu_\alpha^{n-2} r_\alpha^{4-n} = o\left(\mu_\alpha^2\right) \tag{4.12}$$

if $n \geq 5$. By Lemma 3.5 and (4.9)–(4.12) we then get that

$$\begin{aligned} &\int_{B_{x_\alpha}(r_\alpha)} |(u_\alpha)_i - B_\alpha\Lambda_i|^2 \, dv_g = o\left(\mu_\alpha^2 \ln\frac{1}{\mu_\alpha}\right) \ \text{ if } n = 4 \ , \text{ and} \\ &\int_{B_{x_\alpha}(r_\alpha)} |(u_\alpha)_i - B_\alpha\Lambda_i|^2 \, dv_g = o\left(\mu_\alpha^2\right) \ \text{ if } n \geq 5 \ , \end{aligned} \tag{4.13}$$

and coming back to (4.8) we get that

$$\begin{aligned} &\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\nabla\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right|^2 dv_g(x) \\ &= \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x) \left(\Delta_g\left((u_\alpha)_i - B_\alpha\Lambda_i\right)\right)(x) dv_g(x) \\ &+ o\left(\mu_\alpha^2 \ln\frac{1}{\mu_\alpha}\right) \ \text{ if } n = 4 \ , \text{ and} \\ &= \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x) \left(\Delta_g\left((u_\alpha)_i - B_\alpha\Lambda_i\right)\right)(x) dv_g(x) \\ &+ o\left(\mu_\alpha^2\right) \ \text{ if } n \geq 5 \ . \end{aligned} \tag{4.14}$$

Thanks to the equations (3.1) satisfied by the $\mathcal{U}_\alpha$'s, and thanks to the expression of Δ_g in geodesic polar coordinates,

$$\begin{aligned} &\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x) \left(\Delta_g\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right) dv_g(x) \\ &= O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x)\right) \\ &+ O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| B_\alpha(x)^{2^\star-1} dv_g(x)\right) \\ &+ O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times |\mathcal{U}_\alpha(x)| \, dv_g(x)\right) \\ &+ O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times |\nabla B_\alpha(x)| \, dv_g(x)\right) \ . \end{aligned} \tag{4.15}$$

By Lemmas 3.2, 3.4, and 3.5, letting $F_\alpha = |\mathcal{U}_\alpha|^{2^\star-1} + B_\alpha^{2^\star-1}$, we can write that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| F_\alpha(x) dv_g(x)\\
&= \int_{B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| F_\alpha(x) dv_g(x)\\
&+ \int_{B_{x_\alpha}(r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| F_\alpha(x) dv_g(x)\\
&= o\left(\mu_\alpha^2\right) + o\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) .
\end{aligned} \tag{4.16}$$

In a similar way, by Lemmas 3.4 and 3.5,

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times \left|\mathcal{U}_\alpha(x)\right| dv_g(x)\\
&= \int_{B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times \left|\mathcal{U}_\alpha(x)\right| dv_g(x)\\
&+ \int_{B_{x_\alpha}(r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)^2 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times \left|\mathcal{U}_\alpha(x)\right| dv_g(x)\\
&= o\left(\mu_\alpha^2\right) + O\left(\mu_\alpha^{n-2} r_\alpha^{4-n}\right) ,
\end{aligned} \tag{4.17}$$

and since

$$\left|\nabla B_\alpha(x)\right| \le C\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha,x)^{1-n} ,$$

we also have that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3 \left|\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right| \times \left|\nabla B_\alpha(x)\right| dv_g(x)\\
&= o\left(\mu_\alpha^2\right) + O\left(\mu_\alpha^{n-2} r_\alpha^{4-n}\right) .
\end{aligned} \tag{4.18}$$

Plugging (4.15)–(4.18) into (4.14), we get that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 \left|\nabla\left((u_\alpha)_i - B_\alpha\Lambda_i\right)(x)\right|^2 dv_g(x)\\
&= o\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) + o\left(\mu_\alpha^2 \ln\frac{1}{\mu_\alpha}\right) \text{ if } n = 4 \text{ , and}\\
&= o\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) + o\left(\mu_\alpha^2\right) \text{ if } n \ge 5 .
\end{aligned} \tag{4.19}$$

Noting that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 |\nabla B_\alpha(x)|^2 dv_g(x) = O\left(\mu_\alpha^2 \ln\frac{1}{\mu_\alpha}\right) \text{ if } n = 4 ,\\
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2 |\nabla B_\alpha(x)|^2 dv_g(x) = O\left(\mu_\alpha^2\right) \text{ if } n \ge 5 ,
\end{aligned}$$

and since

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2|\nabla B_\alpha(x)|\times\|\nabla(\mathcal{U}_\alpha-B_\alpha\Lambda)(x)\|dv_g(x)\\
&\le\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2|\nabla B_\alpha(x)|^2dv_g(x)\right)^{\frac12}\\
&\times\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2\|\nabla(\mathcal{U}_\alpha-B_\alpha\Lambda)(x)\|^2dv_g(x)\right)^{\frac12},
\end{aligned}$$

we get (4.3) by plugging (4.19) into (4.6). This ends the proof of Lemma 4.1. □

Another lemma we need for the proof of Lemma 4.3 is as follows.

Lemma 4.2. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge 3$, $p\ge 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M,M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of* (3.1) *such that* (3.2) *and* (3.3) *hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that* (3.5) *and* (3.7) *hold true. Let $\mathcal{R}_{2,\alpha}$ be given by*

$$\begin{aligned}
\mathcal{R}_{2,\alpha}=&\int_{B_{x_\alpha}(r_\alpha)}\langle A_\alpha\mathcal{U}_\alpha,X_\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g\\
&+\frac{n-2}{4n}\int_{B_{x_\alpha}(r_\alpha)}\left(\Delta_g\,div_gX_\alpha\right)|\mathcal{U}_\alpha|^2dv_g\\
&+\frac{n-2}{2n}\int_{B_{x_\alpha}(r_\alpha)}\left(div_gX_\alpha\right)\langle A_\alpha\mathcal{U}_\alpha,\mathcal{U}_\alpha\rangle_{\mathbb{R}^p}dv_g\,,
\end{aligned}\tag{4.20}$$

where $\mathcal{U}_\alpha=((u_\alpha)_1,\dots,(u_\alpha)_p)$, $\langle\cdot,\cdot\rangle_{\mathbb{R}^p}$ is the scalar product in $\mathbb{R}^p$, $X_\alpha(\nabla\mathcal{U}_\alpha)_i=(X_\alpha,\nabla(u_\alpha)_i)$, and X_α is as in (4.1)*. Then*

$$\begin{aligned}
&\mathcal{R}_{2,\alpha}=O\left(\mu_\alpha r_\alpha\right)\quad\text{if } n=3\,,\\
&\mathcal{R}_{2,\alpha}=C(4)\mathcal{L}_{A,\Lambda}(x_0)\mu_\alpha^2\ln\frac{r_\alpha}{\mu_\alpha}+o\left(\mu_\alpha^2\ln\frac{1}{\mu_\alpha}\right)\quad\text{if } n=4\,,\\
&\mathcal{R}_{2,\alpha}=C(n)\mathcal{L}_{A,\Lambda}(x_0)\mu_\alpha^2+o\left(\mu_\alpha^2\right)\quad\text{if } n\ge 5\,,
\end{aligned}\tag{4.21}$$

where

$$\mathcal{L}_{A,\Lambda}(x)=\langle A(x)\Lambda,\Lambda\rangle_{\mathbb{R}^p}-\frac{n-2}{4(n-1)}S_g(x)\,,$$

μ_α is as in (3.6)*, r_α is as in* (3.21)*, Λ is as in Lemma 3.2, $C(4)=-64\omega_3$, $C(n)=-\int_{\mathbb{R}^n}u_0^2dx$ when $n\ge5$, u_0 is as in* (3.68)*, and $x_\alpha\to x_0$ as $\alpha\to+\infty$.*

Proof of Lemma 4.2. By the expression of X_α,

$$\begin{aligned}
&|X_\alpha(x)|=O\left(d_g(x_\alpha,x)\right)\,,\\
&\mathrm{div}_gX_\alpha(x)=n+O\left(d_g(x_\alpha,x)^2\right)\,,\text{ and}\\
&\Delta_g\left(\mathrm{div}_gX_\alpha\right)(x)=\frac{n}{n-1}S_g(x_\alpha)+O\left(d_g(x_\alpha,x)\right)\,.
\end{aligned}\tag{4.22}$$

Assume first that $n=3$. By (4.22),

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)}\langle A_\alpha\mathcal{U}_\alpha,X_\alpha(\nabla\mathcal{U}_\alpha)\rangle dv_g\\
&=O\left(\int_{B_{x_\alpha}(r_\alpha)}|\mathcal{U}_\alpha(x)|\times\|\nabla\mathcal{U}_\alpha(x)\|d_g(x_\alpha,x)dv_g(x)\right)
\end{aligned}$$

and by Lemma 3.4 we get that

$$\begin{aligned}&\left|\int_{B_{x_\alpha}(r_\alpha)}\langle A_\alpha\mathcal{U}_\alpha, X_\alpha(\nabla\mathcal{U}_\alpha)\rangle dv_g\right|\\&\le C\mu_\alpha\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^{-2}dv_g(x)\le C\mu_\alpha r_\alpha\ .\end{aligned}\tag{4.23}$$

Similarly, it follows from (4.22) and Lemma 3.4 that

$$\begin{aligned}&\left|\int_{B_{x_\alpha}(r_\alpha)}(\Delta_g\mathrm{div}_g X_\alpha)\,|\mathcal{U}_\alpha|^2dv_g\right|\le C\mu_\alpha r_\alpha \text{ and that}\\&\left|\int_{B_{x_\alpha}(r_\alpha)}(\mathrm{div}_g X_\alpha)\,\langle A_\alpha\mathcal{U}_\alpha,\mathcal{U}_\alpha\rangle_{\mathbb{R}^p}dv_g\right|\le C\mu_\alpha r_\alpha\ .\end{aligned}\tag{4.24}$$

It follows from (4.23) and (4.24) that (4.21) holds true when $n=3$. From now on we assume that $n\ge 4$. We write that

$$A_\alpha(x)=A_\alpha(x_\alpha)+O\left(d_g(x_\alpha,x)\right)\ .$$

Then, by (4.22),

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}\langle A_\alpha\mathcal{U}_\alpha, X_\alpha(\nabla\mathcal{U}_\alpha)\rangle dv_g\\&=\sum_{i,j=1}^p A_{ij}^\alpha(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i X_\alpha(\nabla\mathcal{U}_\alpha)_j dv_g\\&+O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2|\mathcal{U}_\alpha(x)|\times\|\nabla\mathcal{U}_\alpha(x)\|dv_g(x)\right)\ .\end{aligned}\tag{4.25}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2|\mathcal{U}_\alpha(x)|\times\|\nabla\mathcal{U}_\alpha(x)\|dv_g(x)\\&\le\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)|\mathcal{U}_\alpha(x)|^2dv_g(x)\right)^{\frac12}\\&\times\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3\|\nabla\mathcal{U}_\alpha(x)\|^2dv_g(x)\right)^{\frac12}\ .\end{aligned}\tag{4.26}$$

By Lemmas 3.2 and 3.4,

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)|\mathcal{U}_\alpha(x)|^2dv_g(x)\\&=\int_{B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)|\mathcal{U}_\alpha(x)|^2dv_g(x)\\&+\int_{B_{x_\alpha}(r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)|\mathcal{U}_\alpha(x)|^2dv_g(x)\\&=o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.27}$$

Independently, thanks to the equations (3.1) satisfied by the $\mathcal{U}_\alpha$'s, integrating by parts,

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3|\nabla(u_\alpha)_i(x)|^2dv_g(x)\\&=O\left(\int_{\partial B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3|(u_\alpha)_i(x)|\times|\nabla(u_\alpha)_i(x)|d\sigma_g(x)\right)\\&+O\left(\int_{\partial B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2|(u_\alpha)_i(x)|^2d\sigma_g(x)\right)\\&+O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3|\mathcal{U}_\alpha(x)|^{2^\star}dv_g(x)\right)\\&+O\left(\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)|\mathcal{U}_\alpha(x)|^2dv_g(x)\right)\end{aligned}\tag{4.28}$$

for all $i=1,\dots,p$. By Lemma 3.4,

$$\begin{aligned}&\int_{\partial B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3|\mathcal{U}_\alpha(x)|\times\|\nabla\mathcal{U}_\alpha(x)\|d\sigma_g(x)=O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ ,\text{ and}\\&\int_{\partial B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^2|\mathcal{U}_\alpha(x)|^2d\sigma_g(x)=O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.29}$$

By Lemmas 3.2 and 3.4,

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha,x)^3|\mathcal{U}_\alpha(x)|^{2^\star}dv_g(x)\\&=\int_{B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)^3|\mathcal{U}_\alpha(x)|^{2^\star}dv_g(x)\\&+\int_{B_{x_\alpha}(r_\alpha)\backslash B_{x_\alpha}(\mu_\alpha)} d_g(x_\alpha,x)^3|\mathcal{U}_\alpha(x)|^{2^\star}dv_g(x)\\&=o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.30}$$

In particular, we get from (4.25)–(4.30) that

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}\langle A_\alpha\mathcal{U}_\alpha,X_\alpha(\nabla\mathcal{U}_\alpha)\rangle dv_g\\&=\sum_{i,j=1}^pA_{ij}^\alpha(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_iX_\alpha(\nabla\mathcal{U}_\alpha)_jdv_g+o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.31}$$

Integrating by parts, by (4.22) and (4.27),

$$\begin{aligned}&\sum_{i,j=1}^pA_{ij}^\alpha(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_iX_\alpha(\nabla\mathcal{U}_\alpha)_jdv_g\\&=-\frac{n}{2}\sum_{i,j=1}^pA_{ij}^\alpha(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i(u_\alpha)_jdv_g+o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.32}$$

By (4.22) and (4.27) we also have that

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}\left(\Delta_g(\mathrm{div}_g X_\alpha)\right)|\mathcal{U}_\alpha|^2dv_g\\&=\frac{nS_g(x_\alpha)}{n-1}\int_{B_{x_\alpha}(r_\alpha)}|\mathcal{U}_\alpha|^2dv_g+o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ ,\end{aligned}\tag{4.33}$$

and that

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}(\mathrm{div}_g X_\alpha)\left\langle A_\alpha\mathcal{U}_\alpha,\mathcal{U}_\alpha\right\rangle_{\mathbb{R}^p}dv_g\\&=n\sum_{i,j=1}^p A_{ij}^\alpha(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i(u_\alpha)_jdv_g+o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.34}$$

By (4.31)–(4.34),

$$\begin{aligned}\mathcal{R}_{2,\alpha}=&-\sum_{i,j=1}^p A_{ij}^\alpha(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i(u_\alpha)_jdv_g\\&+\frac{n-2}{4(n-1)}S_g(x_\alpha)\int_{B_{x_\alpha}(r_\alpha)}|\mathcal{U}_\alpha|^2dv_g+o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.35}$$

Let S_α be as in Lemma 3.5. We can write that

$$\begin{aligned}&\mu_\alpha^{\frac{n-2}{2}}r_\alpha^{2-n}\int_{B_{x_\alpha}(r_\alpha)}B_\alpha dv_g=O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ ,\text{ and}\\&\mu_\alpha^{\frac{n-2}{2}}\int_{B_{x_\alpha}(r_\alpha)}B_\alpha S_\alpha dv_g=o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ .\end{aligned}\tag{4.36}$$

By (4.9) and (4.13), and by Lemma 3.5, we get (4.21) from (4.35) and (4.36). This ends the proof of Lemma 4.2. □

Now, at this point, we can state the main result of this section. This is the subject of the following lemma. We assume (0.2) in the lemma.

Lemma 4.3. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge 3$, $p\ge 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M,M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence of nonnegative solutions of* (3.1) *such that* (0.2), (3.2), *and* (3.3) *hold true. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that* (3.5) *and* (3.7) *hold true. Assume $r_\alpha\to 0$ as $\alpha\to+\infty$, where r_α is as in* (3.21)*. Then $\rho_\alpha=O(r_\alpha)$ and*

$$r_\alpha^{n-2}\mu_\alpha^{1-\frac{n}{2}}\mathcal{U}_\alpha\left(\exp_{x_\alpha}(r_\alpha x)\right)\to\frac{(n(n-2))^{\frac{n-2}{2}}\Lambda}{|x|^{n-2}}+\mathcal{H}(x)\tag{4.37}$$

in $C^2_{loc}\left(B_0(2)\backslash\{0\}\right)$ as $\alpha\to+\infty$, where μ_α is as in (3.6)*, Λ is as in Lemma 3.2, and $\mathcal{H}$ is a harmonic function in $B_0(2)$ which satisfies that $\langle\Lambda,\mathcal{H}(0)\rangle_{\mathbb{R}^p}\le 0$ with equality if and only if $\mathcal{H}(0)=0$. Moreover, assuming $n\ge 4$, it is necessarily the case that $r_\alpha\to 0$ as $\alpha\to+\infty$.*

Proof of Lemma 4.3. Let $R\ge 6$ be such that $Rr_\alpha\le 6\rho_\alpha$ for $\alpha\gg 1$. We assume first that $r_\alpha\to 0$ as $\alpha\to+\infty$. Then we set, for $x\in B_0(3)$,

$$\begin{aligned}\mathcal{W}_\alpha(x)&=r_\alpha^{n-2}\mu_\alpha^{1-\frac{n}{2}}\mathcal{U}_\alpha\left(\exp_{x_\alpha}(r_\alpha x)\right)\ ,\\g_\alpha(x)&=\left(\exp_{x_\alpha}^\star g\right)(r_\alpha x)\ ,\text{ and}\\\tilde{A}_\alpha(x)&=A_\alpha\left(\exp_{x_\alpha}(r_\alpha x)\right)\ .\end{aligned}$$

Since $r_\alpha \to 0$ as $\alpha \to +\infty$, we have that $\tilde{g}_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$, where ξ is the Euclidean metric. Thanks to Lemma 3.4 we also have that

$$|\mathcal{W}_\alpha(x)| \leq C\,|x|^{2-n} \tag{4.38}$$

in $B_0(\frac{R}{2})\backslash\{0\}$. By (3.1),

$$\Delta_{g_\alpha}(w_\alpha)_i + r_\alpha^2 \sum_{j=1}^{p} \tilde{A}^\alpha_{ij}(w_\alpha)_j = \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 |\mathcal{W}_\alpha|^{2^\star-2}(w_\alpha)_i \tag{4.39}$$

in $B_0(\frac{R}{2})$, for all i, where $\mathcal{W}_\alpha = ((w_\alpha)_1, \ldots, (w_\alpha)_p)$ and $\tilde{A}_\alpha = (\tilde{A}^\alpha_{ij})_{i,j}$. Thanks to (3.22) and by standard elliptic theory, we then deduce that, after passing to a subsequence,

$$\mathcal{W}_\alpha \to \mathcal{W} \tag{4.40}$$

in $C^2_{loc}\left(B_0(\frac{R}{2})\backslash\{0\}\right)$ as $\alpha \to +\infty$, where $\mathcal{W}$ satisfies

$$\Delta\mathcal{W} = 0 \tag{4.41}$$

in $B_0(\frac{R}{2})\backslash\{0\}$. Moreover, thanks to (4.38), we know that

$$|\mathcal{W}(x)| \leq C\,|x|^{2-n} \tag{4.42}$$

in $B_0(\frac{R}{2})\backslash\{0\}$. Thus we can write that

$$\mathcal{W}(x) = \frac{\tilde{\Lambda}}{|x|^{n-2}} + \mathcal{H}(x) \tag{4.43}$$

where $\tilde{\Lambda} \in \mathbb{R}^p$ has nonnegative components and $\mathcal{H}$ satisfies $\Delta\mathcal{H} = 0$ in $B_0(\frac{R}{2})$. In order to see that $\tilde{\Lambda} = (n(n-2))^{(n-2)/2}\,\Lambda$, it is sufficient to integrate (4.39) in $B_0(1)$ to get that

$$\begin{aligned} &-\int_{\partial B_0(1)} \partial_\nu \mathcal{W}_\alpha d\sigma_{g_\alpha} \\ &= \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} |\mathcal{W}_\alpha|^{2^\star-2}\mathcal{W}_\alpha dv_{g_\alpha} - r_\alpha^2 \int_{B_0(1)} \tilde{A}_\alpha \mathcal{W}_\alpha dv_{g_\alpha}\ . \end{aligned} \tag{4.44}$$

By (4.38),

$$\int_{B_0(1)} |\mathcal{W}_\alpha| dv_{\tilde{g}_\alpha} \leq C \tag{4.45}$$

and by changing x into $\frac{\mu_\alpha}{r_\alpha}x$, we can write that

$$\int_{B_0(1)} |\mathcal{W}_\alpha|^{2^\star-2}\mathcal{W}_\alpha dv_{g_\alpha} = r_\alpha^2\mu_\alpha^{-2} \int_{B_0(\frac{r_\alpha}{\mu_\alpha})} |\tilde{\mathcal{U}}_\alpha|^{2^\star-2}\tilde{\mathcal{U}}_\alpha dv_{\tilde{g}_\alpha}\ ,$$

where $\tilde{\mathcal{U}}_\alpha$ and $\tilde{g}_\alpha$ are as in (3.67). By Lemmas 3.2 and 3.4, we then get that

$$\begin{aligned} &\lim_{\alpha\to+\infty} \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} |\mathcal{W}_\alpha|^{2^\star-2}\mathcal{W}_\alpha dv_{g_\alpha} \\ &= \left(\int_{\mathbb{R}^n} u_0^{2^\star-1} dx\right)\Lambda \\ &= (n-2)\omega_{n-1}\left(n(n-2)\right)^{\frac{n-2}{2}}\Lambda\ , \end{aligned} \tag{4.46}$$

where u_0 is as in (3.68). Noting that by (4.40) and (4.43),

$$\lim_{\alpha\to+\infty}\int_{\partial B_0(1)}\partial_\nu\mathcal{W}_\alpha d\sigma_{g_\alpha}=-(n-2)\omega_{n-1}\tilde{\Lambda}\ , \tag{4.47}$$

we get that

$$\tilde{\Lambda}=(n(n-2))^{\frac{n-2}{2}}\Lambda \tag{4.48}$$

thanks to (4.45)–(4.47) by passing into the limit in (4.44) as $\alpha\to+\infty$. Now we prove that $\langle\Lambda,\mathcal{H}(0)\rangle\le 0$ and that $r_\alpha\to 0$ if $n\ge 4$. For that purpose, we let X_α be the vector field given by (4.1) and we apply the Pohozaev identity in Druet and Hebey [19] to $\mathcal{U}_\alpha$ in $B_{x_\alpha}(r_\alpha)$. We get that

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}\langle A_\alpha\mathcal{U}_\alpha,X_\alpha(\nabla\mathcal{U}_\alpha)\rangle dv_g+\frac{n-2}{4n}\int_{B_{x_\alpha}(r_\alpha)}(\Delta_g\mathrm{div}_gX_\alpha)\,|\mathcal{U}_\alpha|^2dv_g\\&+\frac{n-2}{2n}\int_{B_{x_\alpha}(r_\alpha)}(\mathrm{div}_gX_\alpha)\,\langle A_\alpha\mathcal{U}_\alpha,\mathcal{U}_\alpha\rangle dv_g\\&=Q_{1,\alpha}+Q_{2,\alpha}+Q_{3,\alpha}\ ,\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}Q_{1,\alpha}&=\frac{n-2}{2n}\int_{\partial B_{x_\alpha}(r_\alpha)}(\mathrm{div}_gX_\alpha)\,\langle\partial_\nu\mathcal{U}_\alpha,\mathcal{U}_\alpha\rangle d\sigma_g\\&\quad-\int_{\partial B_{x_\alpha}(r_\alpha)}\left(\frac{1}{2}X_\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2-\langle X_\alpha(\nabla\mathcal{U}_\alpha),\partial_\nu\mathcal{U}_\alpha\rangle\right)d\sigma_g\ ,\end{aligned}$$

$$Q_{2,\alpha}=-\sum_{i=1}^p\int_{B_{x_\alpha}(r_\alpha)}\left(\nabla X_\alpha-\frac{1}{n}(\mathrm{div}_gX_\alpha)\,g\right)^\sharp\left((\nabla\mathcal{U}_\alpha)_i\,,(\nabla\mathcal{U}_\alpha)_i\right)dv_g\ ,$$

$$\begin{aligned}Q_{3,\alpha}&=\frac{n-2}{2n}\int_{\partial B_{x_\alpha}(r_\alpha)}X_\alpha(\nu)\,|\mathcal{U}_\alpha|^{2^\star}d\sigma_g\\&\quad-\frac{n-2}{4n}\int_{\partial B_{x_\alpha}(r_\alpha)}(\partial_\nu\,(div_gX_\alpha))\,|\mathcal{U}_\alpha|^2\,d\sigma_g\ ,\end{aligned}$$

and ν is the unit outward normal derivative to $B_{x_\alpha}(r_\alpha)$. We have that

$$|X_\alpha(x)|=O\left(d_g(x_\alpha,x)\right)\quad\text{and}\quad|\nabla\left(\mathrm{div}_gX_\alpha\right)(x)|=O\left(d_g(x_\alpha,x)\right)\ .$$

It follows that

$$Q_{3,\alpha}=O\left(\mu_\alpha^nr_\alpha^{-n}\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\ . \tag{4.50}$$

By Lemmas 4.1 and 4.2, by (4.49) and (4.50), we can write that

$$\begin{aligned}&Q_{1,\alpha}=O\left(\mu_\alpha^3r_\alpha^{-3}\right)+O\left(\mu_\alpha r_\alpha\right)\quad\text{if }n=3\ ,\\&Q_{1,\alpha}=C(4)\left(\langle A(x_0)\Lambda,\Lambda\rangle_{\mathbb{R}^p}-\frac{1}{6}S_g(x_0)\right)\mu_\alpha^2\ln\frac{r_\alpha}{\mu_\alpha}\\&\qquad+o\left(\mu_\alpha^2\ln\frac{1}{\mu_\alpha}\right)+o\left(\mu_\alpha^2r_\alpha^{-2}\right)\quad\text{if }n=4\ ,\\&Q_{1,\alpha}=C(n)\left(\langle A(x_0)\Lambda,\Lambda\rangle_{\mathbb{R}^p}-\frac{n-2}{4(n-1)}S_g(x_0)\right)\mu_\alpha^2\\&\qquad+o\left(\mu_\alpha^2\right)+o\left(\mu_\alpha^{n-2}r_\alpha2-n\right)\quad\text{if }n\ge5\ ,\end{aligned} \tag{4.51}$$

where the constants $C(4)$ and $C(n)$ are as in Lemma 4.2. We wrote here that

$$\mu_\alpha^n r_\alpha^{-n} = \mu_\alpha^{n-2} r_\alpha^{2-n} \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 = o\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) .$$

By Lemma 3.4, (4.22), and the expression of $Q_{1,\alpha}$, we have that

$$Q_{1,\alpha} = O\left(\mu_\alpha^{n-2} r_\alpha^{2-n}\right) .$$

Thanks to (0.2), this clearly implies that $r_\alpha \to 0$ as $\alpha \to +\infty$ if $n \geq 4$. Now, assuming that $r_\alpha \to 0$ as $\alpha \to +\infty$, it is easily checked thanks to (4.40), (4.42), and (4.43), that

$$Q_{1,\alpha} = -\left(\frac{n^{n-2}(n-2)^n}{2}\omega_{n-1}\langle\tilde{\Lambda}, \mathcal{H}(0)\rangle_{\mathbb{R}^p} + o(1)\right)\mu_\alpha^{n-2} r_\alpha^{2-n} . \tag{4.52}$$

Coming back to (4.51), it follows from (4.52) that

$$\begin{aligned}
&\frac{n^{n-2}(n-2)^n}{2}\omega_{n-1}\langle\tilde{\Lambda}, \mathcal{H}(0)\rangle_{\mathbb{R}^p} = 0 \ \text{ if } n = 3 , \\
&\frac{n^{n-2}(n-2)^n}{2}\omega_{n-1}\langle\tilde{\Lambda}, \mathcal{H}(0)\rangle_{\mathbb{R}^p} = -C_4(x_0) \lim_{\alpha\to+\infty} r_\alpha^2 \ln\frac{r_\alpha}{\mu_\alpha} \ \text{ if } n = 4 , \\
&\frac{n^{n-2}(n-2)^n}{2}\omega_{n-1}\langle\tilde{\Lambda}, \mathcal{H}(0)\rangle_{\mathbb{R}^p} = -C_n(x_0) \lim_{\alpha\to+\infty} \left(\mu_\alpha^{4-n} r_\alpha^{n-2}\right) \ \text{ if } n \geq 5 ,
\end{aligned} \tag{4.53}$$

where

$$C_n(x_0) = C(n)\left(\langle A(x_0)\Lambda, \Lambda\rangle_{\mathbb{R}^p} - \frac{n-2}{4(n-1)}S_g(x_0)\right) ,$$

and $C(n)$ is as in Lemma 4.2. Since $C_n(x_0) > 0$ by (0.2), we get with (4.48) and (4.53) that

$$\langle\Lambda, \mathcal{H}(0)\rangle_{\mathbb{R}^p} \leq 0 \tag{4.54}$$

in all dimensions. In what follows we still assume that $r_\alpha \to 0$ as $\alpha \to +\infty$. We multiply line i of the system (3.1) by $(u_\alpha)_j$ and integrate over $B_{x_\alpha}(r_\alpha)$. We obtain that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} (u_\alpha)_j \left(\Delta_g (u_\alpha)_i\right) dv_g + \sum_{k=1}^p \int_{B_{x_\alpha}(r_\alpha)} A_{ik}^\alpha (u_\alpha)_j (u_\alpha)_k dv_g \\
&= \int_{B_{x_\alpha}(r_\alpha)} |\mathcal{U}_\alpha|^{2^\star-2} (u_\alpha)_j (u_\alpha)_i dv_g .
\end{aligned}$$

Inverting i and j and substracting one to the other, we get that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r_\alpha)} \big((u_\alpha)_j \left(\Delta_g (u_\alpha)_i\right) - (u_\alpha)_i \left(\Delta_g (u_\alpha)_j\right)\big) dv_g \\
&= \sum_{k=1}^p \int_{B_{x_\alpha}(r_\alpha)} \big((u_\alpha)_i A_{jk}^\alpha - (u_\alpha)_j A_{ik}^\alpha\big)(u_\alpha)_k dv_g .
\end{aligned}$$

Integrating by parts, this leads to

$$\begin{aligned}
&\int_{\partial B_{x_\alpha}(r_\alpha)} \left((u_\alpha)_i \partial_\nu (u_\alpha)_j - (u_\alpha)_j \partial_\nu (u_\alpha)_i\right) d\sigma_g \\
&= \sum_{k=1}^p \int_{B_{x_\alpha}(r_\alpha)} \big((u_\alpha)_i A_{jk}^\alpha - (u_\alpha)_j A_{ik}^\alpha\big) (u_\alpha)_k dv_g .
\end{aligned}$$

Thanks to (4.40) and (4.43), since $\mathcal{H}$ is harmonic in $B_0(r)$ with $r>1$, it is easily checked that

$$\begin{aligned}&\int_{\partial B_{x_\alpha}(r_\alpha)}\left((u_\alpha)_i\partial_\nu(u_\alpha)_j-(u_\alpha)_j\partial_\nu(u_\alpha)_i\right)d\sigma_g\\&=\mu_\alpha^{n-2}r_\alpha^{2-n}\left(\int_{\partial B_0(1)}\left(\mathcal{W}_i\partial_\nu\mathcal{W}_j-\mathcal{W}_j\partial_\nu\mathcal{W}_i\right)d\sigma+o(1)\right)\\&=\left((n-2)\omega_{n-1}\left(\tilde\Lambda_i\mathcal{H}_j(0)-\tilde\Lambda_j\mathcal{H}_i(0)\right)+o(1)\right)\mu_\alpha^{n-2}r_\alpha^{2-n}\ .\end{aligned}\tag{4.55}$$

Suppose first that $n=3$. Then, by Lemma 3.4,

$$\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i(u_\alpha)_jdv_g=O\left(\mu_\alpha r_\alpha\right)\ .$$

Suppose now that $n\geq 4$. By Lemma 3.5, by (4.9), and by (4.13), with similar computations to those developed in the proof of Lemma 4.1, we have that

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i(u_\alpha)_jdv_g\\&=\left(\int_{B_{x_\alpha}(r_\alpha)}B_\alpha^2dv_g\right)\left(\Lambda_i\Lambda_j+o(1)\right)+o\left(\mu_\alpha^2\ln\frac{1}{\mu_\alpha}\right)\\&=\left(64\omega_3\Lambda_i\Lambda_j+o(1)\right)\mu_\alpha^2\ln\frac{r_\alpha}{\mu_\alpha}+o\left(\mu_\alpha^2\ln\frac{1}{\mu_\alpha}\right)\end{aligned}$$

if $n=4$, while

$$\begin{aligned}&\int_{B_{x_\alpha}(r_\alpha)}(u_\alpha)_i(u_\alpha)_jdv_g\\&=\left(\int_{B_{x_\alpha}(r_\alpha)}B_\alpha^2dv_g\right)\left(\Lambda_i\Lambda_j+o(1)\right)+o\left(\mu_\alpha^2\right)+O\left(\mu_\alpha^{n-2}r_\alpha^{4-n}\right)\\&=\left(\left(\int_{\mathbb{R}^n}u_0^2dx\right)\Lambda_i\Lambda_j+o(1)\right)\mu_\alpha^2\end{aligned}$$

if $n\geq 5$, where u_0 is as in (3.68). In particular, we get that

$$\begin{aligned}&\sum_{k=1}^p\int_{B_{x_\alpha}(r_\alpha)}\left((u_\alpha)_iA_{jk}^\alpha-(u_\alpha)_jA_{ik}^\alpha\right)(u_\alpha)_kdv_g\\&=O\left(\mu_\alpha r_\alpha\right)\ \text{ if } n=3\ ,\\&=64\omega_3\big(A_{jk}(x_0)\Lambda_i\Lambda_k-A_{ik}(x_0)\Lambda_j\Lambda_k+o(1)\big)\mu_\alpha^2\ln\frac{r_\alpha}{\mu_\alpha}\ \text{ if } n=4\ ,\\&=\big(A_{jk}(x_0)\Lambda_i\Lambda_k-A_{ik}(x_0)\Lambda_j\Lambda_k+o(1)\big)\mu_\alpha^2\int_{\mathbb{R}^n}u_0^2dx\ \text{ if } n\geq 5\ .\end{aligned}\tag{4.56}$$

Assuming that $\langle\Lambda,\mathcal{H}(0)\rangle_{\mathbb{R}^p}=0$, we get from (4.53) that

$$\lim_{\alpha\to+\infty}r_\alpha^2\ln\frac{r_\alpha}{\mu_\alpha}=0\ \text{ if } n=4\ ,\text{ and}$$
$$\lim_{\alpha\to+\infty}\mu_\alpha^{4-n}r_\alpha^{n-2}=0\ \text{ if } n=4\ .$$

Coming back to (4.55) and (4.56) we get that

$$\Lambda_i\mathcal{H}_j(0)=\Lambda_j\mathcal{H}_i(0)$$

for all $i, j \in \{1, \dots, p\}$. Multiplying by Λ_i and summing over i, we then deduce that

$$\mathcal{H}_j(0) = \langle \Lambda, \mathcal{H}(0) \rangle \Lambda_j \ ,$$

which proves that $\mathcal{H}(0) = 0$ if $\langle \Lambda, \mathcal{H}(0) \rangle_{\mathbb{R}^p} = 0$. At this point it remains to prove that $\rho_\alpha = O(r_\alpha)$. We still assume that $r_\alpha \to 0$ as $\alpha \to +\infty$ and we proceed by contradiction so that we also assume that

$$\frac{r_\alpha}{\rho_\alpha} \to 0 \tag{4.57}$$

as $\alpha \to \infty$. Then (4.43) holds true in $B_0(R)$ for all R. Since $\mathcal{H}$ is harmonic we then get from (4.43) that

$$\begin{aligned} &\frac{(n(n-2))^{\frac{n-2}{2}}}{R^{n-2}} \Lambda + \frac{1}{|\partial B_0(R)|} \int_{\partial B_0(R)} \mathcal{H} d\sigma \\ &= \frac{(n(n-2))^{\frac{n-2}{2}}}{R^{n-2}} \Lambda + \mathcal{H}(0) \\ &= \frac{1}{|\partial B_0(R)|} \int_{\partial B_0(R)} \mathcal{W} d\sigma \end{aligned}$$

and hence, since $\mathcal{W} \geq 0$, and since $|\Lambda| = 1$, we get that

$$\frac{(n(n-2))^{\frac{n-2}{2}}}{R^{n-2}} + \langle \Lambda, \mathcal{H}(0) \rangle_{\mathbb{R}^p} \geq 0 \ . \tag{4.58}$$

Passing into the limit in (4.58) as $R \to +\infty$ we get that $\langle \Lambda, \mathcal{H}(0) \rangle_{\mathbb{R}^p} \geq 0$. By (4.54) we also have that $\langle \Lambda, \mathcal{H}(0) \rangle_{\mathbb{R}^p} \leq 0$. It follows that $\langle \Lambda, \mathcal{H}(0) \rangle_{\mathbb{R}^p} = 0$. Hence, $\mathcal{H}(0) = 0$. However, since $r_\alpha < \rho_\alpha$ by (4.57), we get with (3.24) that there holds $\left((r^{(n-2)/2} \varphi(r)\right)'(1) = 0$, where

$$\begin{aligned} \varphi(r) &= \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_0(r)} |\mathcal{W}|_\Sigma d\sigma \\ &= \frac{(n(n-2))^{\frac{n-2}{2}}}{r^{n-2}} |\Lambda|_\Sigma + |\mathcal{H}(0)|_\Sigma \ . \end{aligned}$$

Hence,

$$|\mathcal{H}(0)|_\Sigma = (n(n-2))^{\frac{n-2}{2}} \, |\Lambda|_\Sigma$$

and since $\mathcal{H}(0) = 0$, we get a contradiction with the fact that $|\Lambda| = 1$. In particular, (4.57) is false, and thus, $\rho_\alpha = O(r_\alpha)$. This ends the proof of the lemma. ☐

5. Construction of a parametrix for $\Delta_g + A$ when $n = 3$

Let (M, g) be a smooth compact Riemannian manifold of dimension $n = 3$, $p \geq 1$ be an integer, and A be a map in $C^1\left(M, M_p^s(\mathbb{R})\right)$. We prove the existence of a parametrix for multi-valued Schrödinger operators like $\Delta_g + A$ and get a positive mass theorem for such parametrix from the positive mass theorem of Schoen and Yau [37] (see also Witten [43]). We assume here that

$$\Delta_g + A \text{ is coercive and } -A \text{ is cooperative} \tag{5.1}$$

and we also assume that

$$A < \frac{S_g}{8} \mathrm{Id}_p \tag{5.2}$$

in the sense of bilinear forms. Let $\eta \in C^\infty(M \times M)$, $0 \le \eta \le 1$, be such that $\eta(x,y) = 1$ if $d_g(x,y) \le \delta$ and $\eta(x,y) = 0$ if $d_g(x,y) \ge 2\delta$, where $\delta > 0$ is small. For $x \ne y$ we define

$$H(x,y) = \frac{\eta(x,y)}{\omega_2 d_g(x,y)} , \tag{5.3}$$

where ω_2 is the volume of the unit 2-sphere. The result we prove in this section is as follows. We refer to the end of the section for a remark on how to get the Green's matrix from Proposition 5.1.

Proposition 5.1. *Let (M,g) be a smooth compact Riemannian 3-manifold, $p \ge 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map satisfying (5.1). Let $\Lambda \ge 0$ be a nonnegative vector in $\mathbb{R}^p$. There exists $G : M \times M \backslash D \to \mathbb{R}^p$, $G \ge 0$, such that for any $x \in M$, and any $i = 1,\dots,p$,*

$$\Delta_g (G_x)_i + \sum_{j=1}^p A_{ij} (G_x)_j = \Lambda_i \delta_x , \tag{5.4}$$

where D is the diagonal in $M \times M$, $G_x(y) = G(x,y)$, $G = (G_1,\dots,G_p)$, δ_x is the Dirac mass at x, and G can be written as

$$G(x,y) = H(x,y)\Lambda + \mathcal{R}(x,y) \tag{5.5}$$

for all $x, y \in M \times M\backslash D$, where $\mathcal{R} : M \times M \to \mathbb{R}^p$ is continuous in $M \times M$. Moreover, there exists $C > 0$ such that $\mathcal{R}(x,x) \ge C\Lambda$ for all $x \in M$ if we also assume that A satisfies (5.2). In particular, $\mathcal{R}(x,x)_i > 0$ for at least one i if $\Lambda \not\equiv 0$ and (5.1)–(5.2) hold true.

Proof of Proposition 5.1. (i) First we construct G such that (5.4) holds true. We have that, see, for instance, Aubin [4, 5],

$$|\Delta_{g,y} H(x,y)| + |H(x,y)| \le \frac{C}{d_g(x,y)} \tag{5.6}$$

and

$$\Delta_{g,y,dist.} H(x,y) = \delta_x + \Delta_{g,y} H(x,y) \tag{5.7}$$

in the sense of distributions, where δ_x is the Dirac mass at x. We define the maps $\Gamma_1, \Gamma_2 : M \times M \to \mathbb{R}^p$ by

$$\begin{aligned} \Gamma_1(x,y)_i &= - \left(\Delta_{g,y} H(x,y)\right) \Lambda_i - H(x,y) \sum_{j=1}^p A_{ij}(y) \Lambda_j , \\ \Gamma_2(x,y)_i &= - \int_M \Gamma_1(x,z)_i \Delta_{g,y} H(z,y) dv_g(z) \\ &\quad - \sum_{j=1}^p A_{ij}(y) \int_M \Gamma_1(x,z)_j H(z,y) dv_g(z) , \end{aligned} \tag{5.8}$$

for all $(x,y) \in M \times M\backslash D$ and all $i = 1,\dots,p$. By Giraud's lemma and (5.6), Γ_2 is continuous in $M \times M$. Given $x \in M$, we let $\mathcal{S}_x : M \to \mathbb{R}^p$ be the solution of the linear system

$$\Delta_g (\mathcal{S}_x)_i + \sum_{j=1}^p A_{ij} (\mathcal{S}_x)_j = (\Gamma_{2,x})_i \tag{5.9}$$

for all $i = 1, \ldots, p$, where $\Gamma_{2,x}(\cdot) = \Gamma_2(x, \cdot)$. The existence of $\mathcal{S}_x$ easily follows from the variational theory and the coercivity of $\Delta_g + A$. In particular, $\mathcal{S}_x \in H^{2,q}$ for all q. We define $G : M \times M \backslash D \to \mathbb{R}^p$ by

$$G(x, y) = H(x, y)\Lambda + \int_M H(z, y)\Gamma_1(x, z) dv_g(z) + \mathcal{S}(x, y) \ , \tag{5.10}$$

where $\mathcal{S}(x, y) = \mathcal{S}_x(y)$. By Giraud's lemma and (5.6),

$$(x, y) \to \int_M H(z, y)\Gamma_1(x, z) dv_g(z) \tag{5.11}$$

is continuous in $M \times M$. Let $\varphi \in C^2(M)$, $x \in M$, and $i \in \{1, \ldots, p\}$. Thanks to (5.7)–(5.9) we get that

$$\int_M G(x, y)_i \Delta_g \varphi(y) dv_g(y) + \sum_{j=1}^{p} \int_M A_{ij}(y) G(x, y)_j \varphi(y) dv_g(y) = \varphi(x)\Lambda_i \ ,$$

where G is as in (5.10). This proves (5.4).

(ii) We prove (5.5) and that $G \geq 0$. Let $x, x' \in M$. By (5.9),

$$\Delta_g \left((\mathcal{S}_x)_i - (\mathcal{S}_{x'})_i \right) + \sum_{j=1}^{p} A_{ij} \left((\mathcal{S}_x)_j - (\mathcal{S}_{x'})_j \right) = (\Gamma_{2,x})_i - (\Gamma_{2,x'})_i \ . \tag{5.12}$$

Mutiplying (5.12) by $(\mathcal{S}_x)_i - (\mathcal{S}_{x'})_i$, integrating over M, and summing over i, it follows that

$$\begin{aligned} &\int_M \left| \nabla \left(\mathcal{S}_x - \mathcal{S}_{x'} \right) \right|^2 dv_g + \int_M A \left(\mathcal{S}_x - \mathcal{S}_{x'}, \mathcal{S}_x - \mathcal{S}_{x'} \right) dv_g \\ &\leq \left\| \Gamma_{2,x} - \Gamma_{2,x'} \right\|_{L^2} \left\| \mathcal{S}_x - \mathcal{S}_{x'} \right\|_{C^0} \end{aligned}$$

and by the coercivity of $\Delta_g + A$ we get that $\left\| \mathcal{S}_x - \mathcal{S}_{x'} \right\|_{L^2} \leq C \left\| \Gamma_{2,x} - \Gamma_{2,x'} \right\|_{L^2}$. Then, by standard elliptic theory, we obtain that

$$\left\| \mathcal{S}_x - \mathcal{S}_{x'} \right\|_{C^0} \leq C \left\| \Gamma_{2,x} - \Gamma_{2,x'} \right\|_{L^2} \ . \tag{5.13}$$

In a similar way, we get by (5.9) that $\|\mathcal{S}_x\|_{L^2} \leq C \|\Gamma_{2,x}\|_{C^0} \leq C'$ and then, by standard elliptic theory, we can write that $\|\mathcal{S}_x\|_{C^1} \leq C$. Writing that

$$\begin{aligned} |\mathcal{S}(x', y') - \mathcal{S}(x, y)| &\leq |\mathcal{S}(x', y') - \mathcal{S}(x', y)| + |\mathcal{S}(x', y) - \mathcal{S}(x, y)| \\ &\leq \left\| \mathcal{S}_x - \mathcal{S}_{x'} \right\|_{C^0} + \left\| \nabla \mathcal{S}_{x'} \right\|_{C^0} d_g(y, y') \end{aligned}$$

we get from (5.13), the above estimate on $\mathcal{S}_x$, and the continuity of Γ_2, that $\mathcal{S}$ is continuous in $M \times M$. Together with (5.10), and the above remark that the map in (5.11) is continuous, this proves (5.5). Now we prove that $G \geq 0$. Given $u : M \to \mathbb{R}$ a continuous function, we let $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$ so that $u = u^+ + u^-$. By (5.5), there exists $\delta > 0$ such that for any i, if $\Lambda_i > 0$ then $(G_x)_i^-$ has its support in $M \backslash B_x(\delta)$. On the other hand, if $\Lambda_i = 0$ then, by (5.4), (5.5), standard elliptic theory, and the Sobolev embedding theorem, we can write that $\Delta_g (G_x)_i \in L^q$ for all $q < 3$, then that $(G_x)_i \in H^{2,q}$ for all such q, and at last that $(G_x)_i \in H^{1,s}$ for all s. In both cases we can multiply (5.4) by $(G_x)_i^-$, integrate over M, sum over i, and get that

$$\int_M \left| \nabla G_x^- \right|^2 dv_g + \int_M A \left(G_x^-, G_x^- \right) dv_g + \int_M A \left(G_x^+, G_x^- \right) dv_g = 0 \ . \tag{5.14}$$

Noting that for any $u, v \in C^0$, $u^- v^+ + u^+ v^- \le 0$ in M, it follows from the fact that $-A$ is cooperative that $\int_M A\left(G_x^+, G_x^-\right) dv_g \ge 0$. Coming back to (5.14), we get that $G_x^- \equiv 0$ for all x, and thus that $G \ge 0$.

(iii) We prove the last part of the proposition that there exists $C > 0$ such that $\mathcal{R}(x,x) \ge C\Lambda$ for all $x \in M$ if we also assume that A satisfies (5.2). By (5.2), and since $\Delta_g + A$ is coercive, the Schrödinger operators $\Delta_g + A_{ii}$ and $\Delta_g + \frac{S_g}{8}$ are also coercive. We let $\tilde{G}_i$ be the Green's function of $\Delta_g + A_{ii}$ and G_g be the Green's function of $\Delta_g + \frac{S_g}{8}$. By (5.4), for any $x \in M$ and any $i \in \{1, \ldots, p\}$,

$$\Delta_g \left((G_x)_i - (\tilde{G}_i)_x \Lambda_i \right) + A_{ii} \left((G_x)_i - (\tilde{G}_i)_x \Lambda_i \right) = -\sum_{j \ne i} A_{ij} (G_x)_j \ge 0 \tag{5.15}$$

since $G \ge 0$ and $-A$ is cooperative. By (5.5), $(G_x)_i - (\tilde{G}_i)_x \Lambda_i$ is continuous in M. Then, by the maximum principle, we get with (5.15) that

$$G_i \ge \tilde{G}_i \Lambda_i \tag{5.16}$$

for all i. By (5.2), given $i \in \{1, \ldots, p\}$, and $x \in M$, there exists $h_i > 0$ smooth and such that

$$h_i \le \left(\frac{S_g}{8} - A_{ii} \right) (G_g)_x \tag{5.17}$$

in M. By the coercivity of $\Delta_g + A_{ii}$ there exists $\theta_i \in C^2$, $\theta_i > 0$, such that

$$\Delta_g \theta_i + A_{ii} \theta_i = h_i \ . \tag{5.18}$$

Noting that by (5.17) and (5.18),

$$\Delta_g \left((\tilde{G}_i)_x - (G_g)_x - \theta_i \right) + A_{ii} \left((\tilde{G}_i)_x - (G_g)_x - \theta_i \right) \ge 0 \ ,$$

and that $(\tilde{G}_i)_x - (G_g)_x - \theta_i$ is continuous in M, we get that

$$(\tilde{G}_i)_x \ge (G_g)_x + \theta_i \tag{5.19}$$

for all i and all x. Combining (5.16) and (5.19) it follows from the positive mass theorem of Schoen and Yau [37, 38, 39] that there exists $C > 0$ such that $\mathcal{R}(x,x) \ge C\Lambda$ for all $x \in M$. This ends the proof of Proposition 5.1. □

Fix $x \in M$. As a remark there holds that there exists $C > 0$ such that

$$d_g(x,y) \left| \nabla \mathcal{R}_x(y) \right| \le C \tag{5.20}$$

for all $y \in M \backslash \{x\}$, where $\mathcal{R}_x(y) = \mathcal{R}(x,y)$. By (5.4) and (5.6) we get that there exists $C > 0$ such that $d_g(x,y) \left| \Delta_g \mathcal{R}_x(y) \right| \le C$ for all $y \in M \backslash \{x\}$. In order to get (5.20) it suffices to prove that for any sequence $(y_\alpha)_\alpha$ in $M \backslash \{x\}$ such that $y_\alpha \to x$ as $\alpha \to +\infty$,

$$d_g(x, y_\alpha) \left| \nabla \mathcal{R}_x(y_\alpha) \right| = O(1) \ . \tag{5.21}$$

Let $s_\alpha = d_g(x, y_\alpha)$ and set $\mathcal{R}_\alpha(y) = \mathcal{R}_x\left(\exp_x(s_\alpha y)\right)$. Let also g_α be given by $g_\alpha(y) = \left(\exp_x^\star g\right)(s_\alpha y)$, and $\tilde{y}_\alpha$ be such that $y_\alpha = \exp_x(s_\alpha \tilde{y}_\alpha)$. We can write that $\left| \Delta_{g_\alpha} \mathcal{R}_\alpha(y) \right| \le C s_\alpha |y|^{-1}$ while $\mathcal{R}_\alpha$ is bounded and $g_\alpha \to \xi$ as $\alpha \to +\infty$ in $C^1_{loc}(\mathbb{R}^3)$ since $s_\alpha \to 0$ as $\alpha \to +\infty$. Moreover $|\tilde{y}_\alpha| = 1$ for all α. Let y_0 be such that $\tilde{y}_\alpha \to y_0$ as $\alpha \to +\infty$. Since $|y_0| = 1$, we can write by the above estimates and standard elliptic theory that $\mathcal{R}_\alpha$ is bounded in the C^1-topology in the Euclidean ball of center y_0 and radius 1/4. This proves (5.21) and thus (5.20). It also follows from the proof that

$$s_\alpha \max_{y \in \partial B_x(s_\alpha)} |\nabla \mathcal{R}_x(y)| = o(1) \tag{5.22}$$

for all sequences $(s_\alpha)_\alpha$ of positive real numbers such that $s_\alpha \to 0$ as $\alpha \to +\infty$. Indeed, there holds that $\Delta_{g_\alpha} \mathcal{R}_\alpha \to 0$ uniformly in compact subsets of $\mathbb{R}^3 \backslash \{0\}$ as $\alpha \to +\infty$. Hence $\mathcal{R}_\alpha \to \mathcal{R}$ in $C^1_{loc}(\mathbb{R}^3 \backslash \{0\})$, where $\mathcal{R}$ is a bounded harmonic map in $\mathbb{R}^3 \backslash \{0\}$. By Liouville's theorem we get that $\mathcal{R}$ is constant and (5.22) follows.

Given $j \in \{1, \dots, p\}$, let $\Lambda_j \in \mathbb{R}^p$ be defined by $(\Lambda_j)_i = \delta_{ij}$ for all $i = 1, \dots, p$, where the δ_{ij}'s are the Kronecker symbols. Also let G_j be the parametrix given by Proposition 5.1 when $\Lambda = \Lambda_j$ and $\mathcal{G} = (G_{ij})_{i,j}$ be the matrix given by $G_{ij} = (G_j)_i$ for all $i, j = 1, \dots, p$. Then

$$\Delta_{g,y} \sum_{\alpha=1}^{p} \int_M G_{i\alpha}(x,y) f_\alpha(x) dv_g(x) + \sum_{\alpha,j=1}^{p} A_{ij}(y) \int_M G_{j\alpha}(x,y) f_\alpha(x) dv_g(x) = f_i(y)$$

for all $f \in C^\infty(M, \mathbb{R}^p)$, all $i \in \{1, \dots, p\}$, and all $y \in M$. In other words, $\mathcal{G}$ is the Green's matrix of $\Delta_g + A$.

6. Proof of the theorem

We prove our theorem in what follows. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence in $C^1\left(M, M_p^s(\mathbb{R})\right)$, and $(\mathcal{U}_\alpha)_\alpha$ be a sequence nonnegative solutions of (3.1) such that (0.2), (3.2), and (3.3) hold true. As a preliminary remark we claim that there exists $C > 0$ such that for any α the following holds true. Namely that there exist $N_\alpha \in \mathbb{N}^\star$ and N_α critical points of $|\mathcal{U}_\alpha|$, denoted by $(x_{1,\alpha}, x_{2,\alpha}, \dots, x_{N_\alpha,\alpha})$, such that

$$d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)^{\frac{n-2}{2}} |\mathcal{U}_\alpha(x_{i,\alpha})| \geq 1 \tag{6.1}$$

for all $i, j \in \{1, \dots, N_\alpha\}$, $i \neq j$, and

$$\left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x\right)\right)^{\frac{n-2}{2}} |\mathcal{U}_\alpha(x)| \leq C \tag{6.2}$$

for all $x \in M$ and all α. We prove (6.1) and (6.2). Clearly $|\mathcal{U}_\alpha|_\Sigma$ satisfies the maximum principle since, summing the equations in (3.1),

$$\Delta_g |\mathcal{U}_\alpha|_\Sigma + p\|A_\alpha\|_\infty |\mathcal{U}_\alpha|_\Sigma \geq 0 \ ,$$

where $|\mathcal{U}_\alpha|_\Sigma$ is given by (3.4). Hence, $|\mathcal{U}_\alpha|_\Sigma > 0$ and we also get that $|\mathcal{U}_\alpha| > 0$ in M. In particular, we can use Lemma 1.1 of Druet and Hebey [18] and we get the existence of $N_\alpha \in \mathbb{N}^\star$ and of $(x_{1,\alpha}, x_{2,\alpha}, \dots, x_{N_\alpha,\alpha})$ a family of critical points of $|\mathcal{U}_\alpha|$ such that (6.1) holds true for all $i, j \in \{1, \dots, N_\alpha\}$, $i \neq j$, and

$$\left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x\right)\right)^{\frac{n-2}{2}} |\mathcal{U}_\alpha(x)| \leq 1 \tag{6.3}$$

for all critical points of $|\mathcal{U}_\alpha|$. We claim now that there exists $C > 0$ such that (6.2) holds true for all $x \in M$ and all α. We proceed by contradiction and assume that

$$\left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x_\alpha\right)\right)^{\frac{n-2}{2}} |\mathcal{U}_\alpha(x_\alpha)| \to +\infty \tag{6.4}$$

as $\alpha \to +\infty$, where

$$\begin{aligned}&\left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x_\alpha\right)\right)^{\frac{n-2}{2}} \left|\mathcal{U}_\alpha(x_\alpha)\right| \\ &= \sup_M \left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x\right)\right)^{\frac{n-2}{2}} \left|\mathcal{U}_\alpha(x)\right| .\end{aligned} \tag{6.5}$$

We set $|\mathcal{U}_\alpha(x_\alpha)| = \mu_\alpha^{1-\frac{n}{2}}$. Thanks to (6.4) and (6.5), since M is compact so that the distance between two points in M is always bounded, $\mu_\alpha \to 0$ as $\alpha \to +\infty$. We let $\mathcal{S}_\alpha$ be the above set of critical points $x_{i,\alpha}$ of $|\mathcal{U}_\alpha|$. By (6.4),

$$\frac{d_g\left(x_\alpha, \mathcal{S}_\alpha\right)}{\mu_\alpha} \to +\infty \tag{6.6}$$

as $\alpha \to +\infty$. We set, for $x \in \Omega_\alpha = B_0\left(\frac{\delta}{\mu_\alpha}\right)$, where $0 < \delta < \frac{1}{2} i_g$ is fixed,

$$\begin{aligned}\mathcal{V}_\alpha(x) &= \mu_\alpha^{\frac{n-2}{2}} \mathcal{U}_\alpha\big(\exp_{x_\alpha}(\mu_\alpha x)\big) , \\ g_\alpha(x) &= \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x) \text{ , and} \\ \tilde{A}_\alpha(x) &= A_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right) .\end{aligned}$$

We have that $g_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$, where ξ is the Euclidean metric, since $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Thanks to (3.1),

$$\Delta_{g_\alpha}(v_\alpha)_i + \mu_\alpha^2 \sum_{j=1}^p \tilde{A}_{ij}^\alpha (v_\alpha)_j = |\mathcal{V}_\alpha|^{2^\star-2}(v_\alpha)_i \tag{6.7}$$

in Ω_α, for all i, where $\mathcal{V}_\alpha = ((v_\alpha)_1, \dots, (v_\alpha)_p)$, and $\tilde{A}_\alpha = (\tilde{A}^\alpha_{ij})_{i,j}$. We have that $|\mathcal{V}_\alpha(0)| = 1$ and also that, thanks to (6.5) and (6.6), for any $R > 0$,

$$\limsup_{\alpha\to+\infty} \sup_{B_0(R)} |\mathcal{V}_\alpha| = 1 . \tag{6.8}$$

Indeed, for any $x \in B_{x_\alpha}(R\mu_\alpha)$, for any $i = 1, \dots, N_\alpha$,

$$\begin{aligned}d_g(x_{i,\alpha}, x) &\geq d_g(x_{i,\alpha}, x_\alpha) - R\mu_\alpha \\ &\geq d_g(x_\alpha, \mathcal{S}_\alpha) - R\mu_\alpha \\ &\geq d_g(x_\alpha, \mathcal{S}_\alpha)\left(1 - \frac{R\mu_\alpha}{d_g(x_\alpha, \mathcal{S}_\alpha)}\right) .\end{aligned}$$

By standard elliptic theory we then get by (6.7) that, after passing to a subsequence,

$$\mathcal{V}_\alpha \to \mathcal{U} \tag{6.9}$$

in $C^1_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$, where $\mathcal{U}$ has nonnegative components and satisfies

$$\Delta\mathcal{U} = |\mathcal{U}|^{2^\star-2}\mathcal{U}$$

in $\mathbb{R}^n$ with $|\mathcal{U}| \leq |\mathcal{U}(0)| = 1$. It follows from Proposition 1.1 that

$$\mathcal{U} = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}} \Lambda$$

for some $\Lambda \in \mathbb{R}^p$ with nonnegative components such that $|\Lambda| = 1$. In particular, $|\mathcal{U}|$ has a strict local maximum at 0 which proves that $|\mathcal{U}_\alpha|$ has a local maximum, and hence a critical point, y_α with $d_g(x_\alpha, y_\alpha) = o(\mu_\alpha)$ and $\mu_\alpha^{(n-2)/2}|\mathcal{U}_\alpha(y_\alpha)| \to 1$ as $\alpha \to +\infty$. This clearly violates (6.3) thanks to (6.6) since for any $i = 1, \dots, N_\alpha$,

$$\begin{aligned} d_g(x_{i,\alpha}, y_\alpha) &\geq d_g(x_{i,\alpha}, x_\alpha) - d_g(x_\alpha, y_\alpha) \\ &\geq d_g(x_\alpha, \mathcal{S}_\alpha) + o(\mu_\alpha) \\ &\geq R_\alpha \mu_\alpha + o(\mu_\alpha) \\ &= R_\alpha \mu_\alpha (1 + o(1)) \end{aligned}$$

where $R_\alpha \to +\infty$ as $\alpha \to +\infty$ by (6.6). Thus we have contradicted (6.4). This concludes the proof of (6.1) and (6.2).

Now we consider the family $(x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})$ given by (6.1) and (6.2) and we define d_α by

$$d_\alpha = \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}) \ . \tag{6.10}$$

If $N_\alpha = 1$, we set $d_\alpha = \frac{1}{4} i_g$, where i_g is the injectivity radius of (M, g). We claim that

$$d_\alpha \not\to 0 \tag{6.11}$$

as $\alpha \to +\infty$. In order to prove this claim, we proceed by contradiction. Assuming on the contrary that $d_\alpha \to 0$ as $\alpha \to +\infty$, we see that $N_\alpha \geq 2$ for α large, and we can thus assume that the concentration points are ordered in such a way that

$$d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha}) \leq d_g(x_{1,\alpha}, x_{3,\alpha}) \leq \dots \leq d_g(x_{1,\alpha}, x_{N_\alpha,\alpha}) \ . \tag{6.12}$$

We set, for $x \in B_0(\delta d_\alpha^{-1})$, $0 < \delta < \frac{1}{2} i_g$ fixed,

$$\begin{aligned} \hat{\mathcal{U}}_\alpha(x) &= d_\alpha^{\frac{n-2}{2}} \mathcal{U}_\alpha\left(\exp_{x_{1,\alpha}}(d_\alpha x)\right) , \\ \hat{A}_\alpha(x) &= A_\alpha\left(\exp_{x_{1,\alpha}}(d_\alpha x)\right) , \text{ and} \\ \hat{g}_\alpha(x) &= \left(\exp_{x_{1,\alpha}}^\star g\right)(d_\alpha x) . \end{aligned}$$

It is clear that $\hat{g}_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$ since $d_\alpha \to 0$ as $\alpha \to +\infty$. Thanks to (3.1) we have that

$$\Delta_{\hat{g}_\alpha}(\hat{u}_\alpha)_i + d_\alpha^2 \sum_{j=1}^p \hat{A}_{ij}^\alpha (\hat{u}_\alpha)_j = |\hat{\mathcal{U}}_\alpha|^{2^\star - 2}(\hat{u}_\alpha)_i \tag{6.13}$$

in $B_0(\delta d_\alpha^{-1})$, for all i, where $\hat{\mathcal{U}}_\alpha = ((\hat{u}_\alpha)_1, \dots, (\hat{u}_\alpha)_p)$, and $\hat{A}_\alpha = (\hat{A}_{ij}^\alpha)_{i,j}$. For any $R > 0$, we also let $1 \leq N_{R,\alpha} \leq N_\alpha$ be such that

$$\begin{aligned} &d_g(x_{1,\alpha}, x_{i,\alpha}) \leq R d_\alpha \text{ for } 1 \leq i \leq N_{R,\alpha} \ , \text{ and} \\ &d_g(x_{1,\alpha}, x_{i,\alpha}) > R d_\alpha \text{ for } N_{R,\alpha} + 1 \leq i \leq N_\alpha \ . \end{aligned}$$

Such a $N_{R,\alpha}$ does exist thanks to (6.12). We also have that $N_{R,\alpha} \geq 2$ for all $R > 1$ and that $(N_{R,\alpha})_\alpha$ is uniformly bounded for all $R > 0$ thanks to (6.10). Indeed, suppose there are k_α points $x_{i,\alpha}$, $i = 1, \dots, k_\alpha$, such that $d_g(x_{1,\alpha}, x_{i,\alpha}) \leq R d_\alpha$ for all $i = 1, \dots, k_\alpha$. By (6.10),

$$B_{x_{i,\alpha}}\left(\frac{d_\alpha}{2}\right) \cap B_{x_{j,\alpha}}\left(\frac{d_\alpha}{2}\right) = \emptyset$$

for all $i \neq j$. Then,

$$\mathrm{Vol}_g\left(B_{x_{1,\alpha}}\left(\frac{3R}{2}d_\alpha\right)\right) \geq \sum_{i=1}^{k_\alpha} \mathrm{Vol}_g\left(B_{x_{1,\alpha}}\left(\frac{d_\alpha}{2}\right)\right)$$

and we get an upper bound for k_α depending only on R. In the sequel, we set

$$\hat{x}_{i,\alpha} = d_\alpha^{-1} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha})$$

for all $1 \leq i \leq N_\alpha$ such that $d_g(x_{1,\alpha}, x_{i,\alpha}) \leq \frac{1}{2} i_g$. Thanks to (6.2), for any $R > 1$, there exists $C_R > 0$ such that

$$\sup_{B_0(R)\setminus \bigcup_{i=1}^{N_{2R,\alpha}} B_{\hat{x}_{i,\alpha}}\left(\frac{1}{R}\right)} |\hat{\mathcal{U}}_\alpha| \leq C_R \ . \tag{6.14}$$

Mimicking the proof of Lemma 3.1, one easily gets that, for any $R > 1$, there exists $D_R > 1$ such that

$$\left\|\nabla \hat{\mathcal{U}}_\alpha\right\|_{L^\infty(\Omega_{R,\alpha})} \leq D_R \sup_{\Omega_{R,\alpha}} |\hat{\mathcal{U}}_\alpha| \leq D_R^2 \inf_{\Omega_{R,\alpha}} |\hat{\mathcal{U}}_\alpha| \tag{6.15}$$

where

$$\Omega_{R,\alpha} = B_0(R) \setminus \bigcup_{i=1}^{N_{2R,\alpha}} B_{\hat{x}_{i,\alpha}}\left(\frac{1}{R}\right) .$$

Assume first that, for some $R > 0$, there exists $1 \leq i \leq N_{R,\alpha}$ such that

$$|\hat{\mathcal{U}}_\alpha(\hat{x}_{i,\alpha})| = O(1) \ . \tag{6.16}$$

Since (3.5) is satisfied by the sequences $x_\alpha = x_{i,\alpha}$ and $\rho_\alpha = \frac{1}{8} d_\alpha$, it follows from Lemma 3.2 that (3.7) cannot hold and thus that $(|\hat{\mathcal{U}}_\alpha|)_\alpha$ is uniformly bounded in $B_{\hat{x}_{i,\alpha}}(\frac{3}{4})$. In particular, by standard elliptic theory, and thanks to (6.13), $(\hat{\mathcal{U}}_\alpha)_\alpha$ is uniformly bounded in $C^1\left(B_{\hat{x}_{i,\alpha}}(\frac{1}{2})\right)$. Since, by (6.1), we have that

$$|\hat{x}_{i,\alpha}|^{\frac{n-2}{2}} |\hat{\mathcal{U}}_\alpha(\hat{x}_{i,\alpha})| \geq 1 \ ,$$

we get the existence of some $\delta_i > 0$ such that

$$|\hat{\mathcal{U}}_\alpha| \geq \frac{1}{2} |\hat{x}_{i,\alpha}|^{1-\frac{n}{2}} \geq \frac{1}{2} R^{1-\frac{n}{2}}$$

in $B_{\hat{x}_{i,\alpha}}(\delta_i)$. Assume now that, for some $R > 0$, there exists $1 \leq i \leq N_{R,\alpha}$ such that

$$|\hat{\mathcal{U}}_\alpha(\hat{x}_{i,\alpha})| \to +\infty \tag{6.17}$$

as $\alpha \to +\infty$. Since (3.5) and (3.7) are satisfied by the sequences $x_\alpha = x_{i,\alpha}$ and $\rho_\alpha = \frac{1}{8} d_\alpha$, it follows from Lemma 4.3 that the sequence $(|\hat{\mathcal{U}}_\alpha(\hat{x}_{i,\alpha})| \times |\hat{\mathcal{U}}_\alpha|)_\alpha$ is uniformly bounded in

$$\hat{\Omega}_\alpha = B_{\hat{x}_{i,\alpha}}(\tilde{\delta}_i) \setminus B_{\hat{x}_{i,\alpha}}(\frac{\tilde{\delta}_i}{2})$$

for some $\tilde{\delta}_i > 0$. Thus, using (6.15), we can deduce that these two situations are mutually exclusive in the sense that either (6.16) holds true for all i or (6.17) holds true for all i. Now we split the conclusion of the proof into two cases.

In the first case we assume that there exist $R > 0$ and $1 \leq i \leq N_{R,\alpha}$ such that $|\hat{\mathcal{U}}_\alpha(\hat{x}_{i,\alpha})| = O(1)$. Then, thanks to the above discussion, we get that

$$|\hat{\mathcal{U}}_\alpha(\hat{x}_{j,\alpha})| = O(1)$$

for all $1 \le j \le N_{R,\alpha}$ and all $R > 0$. Now, as above, we get that $(|\hat{\mathcal{U}}_\alpha|)_\alpha$ is uniformly bounded in $C^1_{loc}(\mathbb{R}^n)$. Thus, by standard elliptic theory, there exists a subsequence of $(\hat{\mathcal{U}}_\alpha)_\alpha$ which converges in $C^1_{loc}(\mathbb{R}^n)$ to some $\hat{\mathcal{U}}$ solution of

$$\Delta \hat{\mathcal{U}} = |\hat{\mathcal{U}}|^{2^\star-2}\hat{\mathcal{U}}$$

in $\mathbb{R}^n$. Still thanks to the above discussion, we know that $\mathcal{U} \not\equiv 0$ and has nonnegative components. Moreover, $|\mathcal{U}|$ possesses at least two critical points, namely 0 and $\hat{x}_2$, the limit of $\hat{x}_{2,\alpha}$. This is absurd thanks to the classification of Proposition 1.1.

In the second case we assume that there exist $R > 0$ and $1 \le i \le N_{R,\alpha}$ such that $|\hat{\mathcal{U}}_\alpha(\hat{x}_{i,\alpha})| \to +\infty$ as $\alpha \to +\infty$. Then, thanks to the above discussion,

$$|\hat{\mathcal{U}}_\alpha(\hat{x}_{j,\alpha})| \to +\infty$$

as $\alpha \to +\infty$, for all $1 \le j \le N_{R,\alpha}$ and all $R > 0$. By (6.13) we have that

$$\Delta_{\hat{g}_\alpha}(\hat{v}_\alpha)_i + d_\alpha^2 \sum_{j=1}^p \hat{A}^\alpha_{ij}(\hat{v}_\alpha)_j = \frac{1}{|\hat{\mathcal{U}}_\alpha(0)|^{2^\star-2}}|\hat{\mathcal{V}}_\alpha|^{2^\star-2}(\hat{v}_\alpha)_i \ ,$$

where $\hat{\mathcal{V}}_\alpha = |\hat{\mathcal{U}}_\alpha(0)|\hat{\mathcal{U}}_\alpha$ and $\hat{\mathcal{V}}_\alpha = ((\hat{v}_\alpha)_1, \ldots, (\hat{v}_\alpha)_p)$. Applying Lemma 4.3 and standard elliptic theory, and thanks to (6.15) and to the above discussion, one easily checks that, after passing to a subsequence,

$$|\hat{\mathcal{U}}_\alpha(0)|\hat{\mathcal{U}}_\alpha \to \hat{G}$$

in $C^1_{loc}(\mathbb{R}^n \backslash \{\hat{x}_i\}_{i\in I})$ as $\alpha \to +\infty$, where

$$I = \left\{1, \ldots, \lim_{R\to+\infty} \lim_{\alpha\to+\infty} N_{R,\alpha}\right\}$$

and, for any $R > 0$,

$$\hat{G}(x) = \sum_{i=1}^{\tilde{N}_R} \frac{\tilde{\Lambda}_i}{|x-\hat{x}_i|^{n-2}} + \hat{H}_R(x)$$

in $B_0(R)$, where $2 \le \tilde{N}_R \le N_{2R}$ is such that $|\hat{x}_{\tilde{N}_R}| \le R$ and $|\hat{x}_{\tilde{N}_R+1}| > R$, and where $N_{2R,\alpha} \to N_{2R}$ as $\alpha \to +\infty$. In this expression, the $\tilde{\Lambda}_i$'s are nonzero vectors with nonnegative components and $\hat{H}_R$ is a harmonic function in $B_0(R)$. We have that

$$\hat{H}_{R_1}(x) - \hat{H}_{R_2}(x) = \sum_{i=\tilde{N}_{R_1}+1}^{\tilde{N}_{R_2}} \frac{\tilde{\Lambda}_i}{|x-\hat{x}_i|^{n-2}}$$

for all $0 < R_1 < R_2$. We can write that

$$\hat{G}(x) = \frac{\tilde{\Lambda}_1}{|x|^{n-2}} + X(x)$$

in $B_0\left(\frac{1}{2}\right)$ where, for any $R > 1$,

$$X(x) = \sum_{i=2}^{\tilde{N}_R} \frac{\tilde{\Lambda}_i}{|x-\hat{x}_i|^{n-2}} + \hat{H}_R(x) \ .$$

Let $\hat{G} = (\hat{G}_1, \ldots, \hat{G}_p)$, $X = (X_1, \ldots, X_p)$, and $\tilde{\Lambda}_1 = ((\tilde{\Lambda}_1)_1, \ldots, (\tilde{\Lambda}_1)_p)$. We have that $\hat{G}_i \ge 0$ for all $1 \le i \le p$. Hence, by the maximum principle, we get that $X_i(0) \ge -(\tilde{\Lambda}_1)_i R^{2-n}$ for all $R > 1$, so that $X_i(0) \ge 0$ for all $1 \le i \le p$. By Lemma

4.3 we now have that $\langle \tilde{\Lambda}_1, X(0)\rangle_{\mathbb{R}^p} \le 0$ with equality if and only if $X(0) = 0$. Since all the components of $X(0)$ and of $\tilde{\Lambda}_1$ are nonnegative, we are actually in the case of equality so that $X(0) = 0$. Let i be such that $(\tilde{\Lambda}_2)_i > 0$. By the maximum principle,

$$X_i(0) \ge (\tilde{\Lambda}_2)_i - \frac{(\tilde{\Lambda}_1)_i}{R^{n-2}} - \frac{(\tilde{\Lambda}_2)_i}{(R-1)^{n-2}} .$$

Choosing $R \gg 1$ sufficiently large we get that $X_i(0) > 0$ and this is in contradiction with $X(0) = 0$.

By the above discussion we get that (6.11) holds true. Clearly, this implies that $(N_\alpha)_\alpha$ is uniformly bounded. Now we let $(x_\alpha)_\alpha$ be a sequence of maximal points of $|\mathcal{U}_\alpha|$. Thanks to (3.3) and to (6.11), we clearly have that (3.5) and (3.7) hold for the sequences $(x_\alpha)_\alpha$ and $\rho_\alpha = \delta$ for some $\delta > 0$ fixed. This clearly contradicts Lemma 4.3 in dimensions $n \ge 4$ and thus concludes the proof of the theorem in dimensions $n \ge 4$.

Suppose now that $n = 3$. In addition to (0.2), (3.2), and (3.3) we assume that $\Delta_g + A$ is coercive and that $-A$ is cooperative. Up to a subsequence, since $(N_\alpha)_\alpha$ is bounded, there holds that $N_\alpha = N$ for all α. Let

$$x_i = \lim_{\alpha\to+\infty} x_{i,\alpha} \tag{6.18}$$

for all $i = 1, \dots, N$. Let also $\mu_{i,\alpha}$ be given by (3.6) with $x_{i,\alpha}$ instead of x_α. By the above discussion, $\mu_{i,\alpha} \to 0$ for all $i = 1, \dots, N$. Up to a subsequence we can assume that $\mu_{1,\alpha} = \max_i \mu_{i,\alpha}$ for all α. Still up to a subsequence we define $\mu_i \ge 0$ by

$$\mu_i = \lim_{\alpha\to+\infty} \frac{\mu_{i,\alpha}}{\mu_{1,\alpha}} . \tag{6.19}$$

By Lemma 3.4, there exist $C, \delta > 0$ such that

$$|\mathcal{U}_\alpha(x)| \le C\mu_{i,\alpha}^{1/2} d_g(x_{i,\alpha}, x)^{-1} \tag{6.20}$$

in $B_{x_{i,\alpha}}(2\delta)$ for all i. By (6.20) and Harnack's inequality we thus get that

$$|\mathcal{U}_\alpha| \le C\mu_{1,\alpha}^{1/2} \tag{6.21}$$

in $M\backslash \bigcup_{i=1}^N B_{x_{i,\alpha}}(\delta)$. Let $\tilde{\mathcal{U}}_\alpha$ be given by $\tilde{\mathcal{U}}_\alpha = \mu_{1,\alpha}^{-1/2}\mathcal{U}_\alpha$. Then

$$\Delta_g(\tilde{u}_\alpha)_i + \sum_{j=1}^p A_{ij}^\alpha(x)(\tilde{u}_\alpha)_j = \mu_{1,\alpha}^2 |\tilde{\mathcal{U}}_\alpha|^{2^\star-2}(\tilde{u}_\alpha)_i \tag{6.22}$$

for all i, where the $(\tilde{u}_\alpha)_i$'s are the components of $\tilde{\mathcal{U}}_\alpha$. By (6.21), (6.22), and standard elliptic theory, we then get that, up to a subsequence,

$$\mu_{1,\alpha}^{-1/2}\mathcal{U}_\alpha \to \mathcal{Z} \tag{6.23}$$

in $C^1_{loc}(M\backslash\mathcal{S})$ as $\alpha \to +\infty$, where $\mathcal{S}$ is the finite set consisting of the x_i's defined in (6.18). Let $\Phi \in C^\infty(M, \mathbb{R}^p)$ be given. By (3.1),

$$\int_M \langle \mathcal{U}_\alpha, (\Delta_g\Phi + A\Phi)\rangle\, dv_g = \int_M |\mathcal{U}_\alpha|^4 \langle \mathcal{U}_\alpha, \Phi\rangle\, dv_g + o\left(\int_M |\mathcal{U}_\alpha| dv_g\right) . \tag{6.24}$$

For any $R > 0$,

$$\begin{aligned}\int_M |\mathcal{U}_\alpha|^4 \langle \mathcal{U}_\alpha, \Phi\rangle\, dv_g &= \sum_{i=1}^N \left\langle \Phi(x_i), \int_{B_{x_{i,\alpha}}(R\mu_{i,\alpha})} |\mathcal{U}_\alpha|^4 \mathcal{U}_\alpha dv_g \right\rangle \\ &+ o\left(\sum_{i=1}^N \int_{B_{x_{i,\alpha}}(R\mu_{i,\alpha})} |\mathcal{U}_\alpha|^5 dv_g\right) + \int_{M\setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})} |\mathcal{U}_\alpha|^4 \langle \mathcal{U}_\alpha, \Phi\rangle\, dv_g\ .\end{aligned} \tag{6.25}$$

By Lemma 3.2,

$$\begin{aligned}&\lim_{R\to+\infty}\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1/2} \sum_{i=1}^N \left\langle \Phi(x_i), \int_{B_{x_{i,\alpha}}(R\mu_{i,\alpha})} |\mathcal{U}_\alpha|^4 \mathcal{U}_\alpha dv_g \right\rangle \\ &= \sqrt{3}\omega_2 \sum_{i=1}^N \mu_i^{1/2} \langle \Lambda_i, \Phi(x_i)\rangle\ ,\end{aligned} \tag{6.26}$$

where the Λ_i's are vectors in S_+^{p-1} given by (3.15), and

$$\mu_{1,\alpha}^{-1/2} \sum_{i=1}^N \int_{B_{x_{i,\alpha}}(R\mu_{i,\alpha})} |\mathcal{U}_\alpha|^5 dv_g \le C \tag{6.27}$$

for some $C > 0$ independent of α and R. By (6.20) and (6.21) we can also write that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1/2} \int_{M\setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})} |\mathcal{U}_\alpha|^4 \langle \mathcal{U}_\alpha, \Phi\rangle\, dv_g = 0 \tag{6.28}$$

and that

$$\int_M |\mathcal{U}_\alpha| dv_g = O\left(\mu_{1,\alpha}^{1/2}\right)\ . \tag{6.29}$$

Plugging (6.25)–(6.29) into (6.24) it follows that

$$\mu_{1,\alpha}^{-1/2} \int_M \langle \mathcal{U}_\alpha, (\Delta_g \Phi + A\Phi)\rangle\, dv_g = \sqrt{3}\omega_2 \sum_{i=1}^N \mu_i^{1/2} \langle \Lambda_i, \Phi(x_i)\rangle + o(1)\ . \tag{6.30}$$

Since $\Phi \in C^\infty(M, \mathbb{R}^p)$ is arbitrary, it follows from (6.23) and (6.30) that

$$\Delta_g \mathcal{Z} + A\mathcal{Z} = \sqrt{3}\omega_2 \sum_{i=1}^N \mu_i^{1/2} \Lambda_i \delta_{x_i}\ . \tag{6.31}$$

Since $\Delta_g + A$ is coercive, by Proposition 5.1 and (6.31), there holds that

$$\mathcal{Z}(x) = \sqrt{3}\omega_2 \sum_{i=1}^N \mu_i^{1/2} \left(H(x_i, x)\Lambda_i + \mathcal{R}_i(x_i, x)\right)\ , \tag{6.32}$$

where H is as in (5.3), and $\mathcal{R}_i$ is a continuous function in $M \times M$ such that $\mathcal{R}_i(x_i, x_i) \ge C\Lambda_i$ for some $C > 0$. Let $i = 1, \ldots, N$ be arbitrary and X_α be the vector field given by $X_\alpha = \nabla f_\alpha$, where $f_\alpha(x) = \frac{1}{2} d_g(x_{i,\alpha} x)^2$. We apply the Pohozaev identity in Druet and Hebey [19] to $\mathcal{U}_\alpha$ in $B_{x_{i,\alpha}}(r)$ for $r > 0$ small. We get that

$$\begin{aligned}&\int_{B_{x_{i,\alpha}}(r)} \langle A_\alpha \mathcal{U}_\alpha, X_\alpha(\nabla \mathcal{U}_\alpha)\rangle dv_g + \frac{1}{12}\int_{B_{x_{i,\alpha}}(r)} (\Delta_g \mathrm{div}_g X_\alpha)\, |\mathcal{U}_\alpha|^2 dv_g \\ &+ \frac{1}{6}\int_{B_{x_{i,\alpha}}(r)} (\mathrm{div}_g X_\alpha)\, \langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle dv_g = Q_{1,\alpha} + Q_{2,\alpha} + Q_{3,\alpha}\ ,\end{aligned} \tag{6.33}$$

where $X_\alpha(\nabla \mathcal{U}_\alpha)$ is as in Lemma 4.2,

$$\begin{aligned}Q_{1,\alpha} \;&=\; \frac{1}{6}\int_{\partial B_{x_{i,\alpha}}(r)} (\mathrm{div}_g X_\alpha)\, \langle \partial_\nu \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle d\sigma_g \\ &\quad - \int_{\partial B_{x_{i,\alpha}}(r)} \left(\frac{1}{2} X_\alpha(\nu) |\nabla \mathcal{U}_\alpha|^2 - \langle X_\alpha(\nabla \mathcal{U}_\alpha), \partial_\nu \mathcal{U}_\alpha\rangle\right) d\sigma_g\ ,\end{aligned}$$

$$Q_{2,\alpha} = -\sum_{j=1}^{p} \int_{B_{x_{i,\alpha}}(r)} \left(\nabla X_\alpha - \frac{1}{3}(\mathrm{div}_g X_\alpha)\, g\right)^\sharp \left((\nabla \mathcal{U}_\alpha)_j\, , (\nabla \mathcal{U}_\alpha)_j\right) dv_g\ ,$$

$$Q_{3,\alpha} \;=\; \frac{1}{6}\int_{\partial B_{x_{i,\alpha}}(r)} X_\alpha(\nu)\, |\mathcal{U}_\alpha|^{2^\star} d\sigma_g - \frac{1}{12}\int_{\partial B_{x_{i,\alpha}}(r)} (\partial_\nu (div_g X_\alpha))\, |\mathcal{U}_\alpha|^2 d\sigma_g\ ,$$

and ν is the unit outward normal derivative to $B_{x_{i,\alpha}}(r)$. By (6.23),

$$\begin{aligned}&\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1} (Q_{1,\alpha} + Q_{3,\alpha}) \\ &= \frac{1}{6}\int_{\partial B_{x_i}(r)} (\mathrm{div}_g X)\, \langle \partial_\nu \mathcal{Z}, \mathcal{Z}\rangle d\sigma_g \\ &- \int_{\partial B_{x_i}(r)} \left(\frac{1}{2} X(\nu) |\nabla \mathcal{Z}|^2 - \langle X(\nabla \mathcal{Z}), \partial_\nu \mathcal{Z}\rangle\right) d\sigma_g \\ &- \frac{1}{12}\int_{\partial B_{x_i}(r)} (\partial_\nu (div_g X))\, |\mathcal{Z}|^2 d\sigma_g\ ,\end{aligned} \tag{6.34}$$

where $X = \nabla f$ and $f(x) = \frac{1}{2} d_g(x_i, x)^2$. We have that

$$\mathrm{div}_g X = 3 + O\left(d_g(x_i,x)^2\right) \quad \text{and} \quad |\nabla \mathrm{div}_g X| = O\left(d_g(x_i,x)\right)$$

while, by (6.20), there also holds that $|\mathcal{Z}| \le C d_g(x_i,x)^{-1}$ in a neighbourhood of x_i. From (5.20) we have in addition that $d_g(x_i,x)|\nabla \mathcal{R}_i(x)| \le C$ for all $x \ne x_i$. It follows that

$$\lim_{r\to 0} \int_{\partial B_{x_i}(r)} (\partial_\nu (div_g X))\, |\mathcal{Z}|^2 d\sigma_g = 0 \tag{6.35}$$

and that

$$\frac{1}{6}\int_{\partial B_{x_i}(r)} (\mathrm{div}_g X)\, \langle \partial_\nu \mathcal{Z}, \mathcal{Z}\rangle d\sigma_g = \frac{1}{2}\int_{\partial B_{x_i}(r)} \langle \partial_\nu \mathcal{Z}, \mathcal{Z}\rangle d\sigma_g + o(1) \tag{6.36}$$

as $r \to 0$. We choose $\delta > 0$ in the definition of η in (5.3) such that $d_g(x_j, x_k) \ge 4\delta$ for all $j,k = 1,\dots,N$ such that $x_j \ne x_k$. Since the parametrix in Proposition 5.1 are nonnegative, it follows from our choice of δ that $\mathcal{R}_j(x_j, x_i) \ge 0$ for all $j \ne i$. In a neighbourhood of x_i we get from (6.32) that

$$\mathcal{Z}(x) = \sqrt{3} d_g(x_i,x)^{-1} \mu_i^{1/2} \Lambda_i + \sqrt{3}\omega_2 \sum_{j=1}^{N} \mu_j^{1/2} \mathcal{R}_j(x_j, x)\ . \tag{6.37}$$

By (5.22) and (6.37) we compute

$$\begin{aligned} &\frac{1}{2}\int_{\partial B_{x_i}(r)} \langle \partial_\nu \mathcal{Z}, \mathcal{Z}\rangle d\sigma_g - \int_{\partial B_{x_i}(r)} \left(\frac{1}{2}X(\nu)|\nabla \mathcal{Z}|^2 - \langle X(\nabla \mathcal{Z}), \partial_\nu \mathcal{Z}\rangle\right) d\sigma_g \\ &= -\frac{3\omega_2}{2}\left\langle \mu_i^{1/2}\Lambda_i, \sum_{j=1}^N \mu_j^{1/2}\mathcal{R}_j(x_j,x_i)\right\rangle + o(1)\ . \end{aligned} \tag{6.38}$$

Combining (6.36) and (6.38) it follows that

$$\begin{aligned} &\frac{1}{6}\int_{\partial B_{x_i}(r)} (\mathrm{div}_g X)\, \langle \partial_\nu \mathcal{Z}, \mathcal{Z}\rangle d\sigma_g \\ &- \int_{\partial B_{x_i}(r)} \left(\frac{1}{2}X(\nu)|\nabla \mathcal{Z}|^2 - \langle X(\nabla \mathcal{Z}), \partial_\nu \mathcal{Z}\rangle\right) d\sigma_g \\ &= -\frac{3\omega_2}{2}\left\langle \mu_i^{1/2}\Lambda_i, \sum_{j=1}^N \mathcal{R}_j(x_j,x_i)\right\rangle + o(1)\ . \end{aligned} \tag{6.39}$$

Noting that

$$(\nabla X_\alpha)_{\mu\nu} - \frac{1}{3}(\mathrm{div}_g X_\alpha)\, g_{\mu\nu} = O\left(d_g(x_{i,\alpha},x)^2\right)$$

for all i and all μ,ν, we can write with Lemma 3.4 that $|Q_{2,\alpha}| \le C\mu_{1,\alpha} r$. It follows that

$$\lim_{r\to 0}\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1} Q_{2,\alpha} = 0\ . \tag{6.40}$$

Still by Lemma 3.4, we also have that

$$\begin{aligned} &\lim_{r\to 0}\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1}\int_{B_{x_{i,\alpha}}(r)} \langle A_\alpha \mathcal{U}_\alpha, X_\alpha(\nabla \mathcal{U}_\alpha)\rangle dv_g = 0\ , \\ &\lim_{r\to 0}\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1}\int_{B_{x_{i,\alpha}}(r)} (\Delta_g \mathrm{div}_g X_\alpha)\, |\mathcal{U}_\alpha|^2 dv_g = 0\ ,\ \text{and} \\ &\lim_{r\to 0}\lim_{\alpha\to+\infty} \mu_{1,\alpha}^{-1}\int_{B_{x_{i,\alpha}}(r)} (\mathrm{div}_g X_\alpha)\, \langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle dv_g = 0\ . \end{aligned} \tag{6.41}$$

Multiplying (6.33) by $\mu_{1,\alpha}^{-1}$, passing to the limit as $\alpha \to +\infty$, and then as $r \to 0$, we get with (6.34), (6.35), (6.39), (6.40), and (6.41), that

$$\left\langle \mu_i^{1/2}\Lambda_i, \sum_{j=1}^N \mu_j^{1/2}\mathcal{R}_j(x_j,x_i)\right\rangle = 0 \tag{6.42}$$

for all i. We fix $i = 1$. Then $\mu_1 = 1$. As already mentioned, according to our choice of δ in the definition of η in (5.3), we get that $\mathcal{R}_j(x_j,x_1) \ge 0$ for all $j \ne 1$. By Proposition 5.1 we also have that $\mathcal{R}_1(x_1,x_1) \ge C\Lambda_1$ for some $C > 0$. Since the Λ_j's are nonnegative vectors, it follows that

$$\left\langle \Lambda_1, \omega_2 \sum_{j=1}^N \mu_j^{1/2}\mathcal{R}_j(x_j,x_1)\right\rangle \ge \langle \Lambda_1, \mathcal{R}_1(x_1,x_1)\rangle \ge C|\Lambda_1|^2 \tag{6.43}$$

and we get a contradiction by combining (6.42) and (6.43) since the Λ_i's are nonzero vectors. This concludes the proof of the theorem when $n = 3$.

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