BLOW-UP SOLUTIONS FOR ASYMPTOTICALLY CRITICAL ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. Given (M,g) a smooth, compact Riemannian n-manifold, we consider equations like $\Delta_g u + hu = u^{2^*-1-\varepsilon}$, where h is a C^1-function on M, the exponent $2^* = 2n/(n-2)$ is critical from the Sobolev viewpoint, and ε is a small real parameter such that $\varepsilon \to 0$. We prove the existence of blowing-up families of positive solutions in the subcritical and supercritical case when the graph of h is distinct at some point from the graph of $\frac{n-2}{4(n-1)} \operatorname{Scal}_g$.

1. Introduction

We let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$. We are interested in the asymptotically critical equation

$$\Delta_g u + hu = u^{2^*-1-\varepsilon} \quad \text{in } M\,, \quad u > 0 \quad \text{in } M\,, \tag{1.1}$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator, h is a C^1-function on M, ε is a small real parameter such that $\varepsilon \to 0$, and $2^* = \frac{2n}{n-2}$ is the critical exponent for the embeddings of the Riemannian Sobolev space $H_g^1(M)$ into Lebesgue spaces. The equation with $\varepsilon > 0$ is subcritical, and the equation with $\varepsilon < 0$ is supercritical. In case $\varepsilon = 0$ and

$$h \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g\,, \tag{1.2}$$

where Scal_g is the scalar curvature of the manifold, (1.1) is the intensively studied Yamabe equation (see Aubin [1], Schoen [23], Trudinger [26], and Yamabe [27] for early references on the subject).

We say that a family of solutions $(u_\varepsilon)_\varepsilon$ of equations (1.1) *blows up* at a point ξ_0 if there exists a family of points $(\xi_\varepsilon)_\varepsilon$ in M such that $\xi_\varepsilon \to \xi_0$ and $u_\varepsilon(\xi_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. The question of whether solutions of equations like (1.1) with $\varepsilon > 0$ blow up or not as $\varepsilon \to 0$ have been intensively studied in recent years. In case of the Yamabe-type equation

$$\Delta_g u + \frac{n-2}{4(n-1)} \operatorname{Scal}_g u = u^{2^*-1-\varepsilon} \quad \text{in } M\,, \quad u > 0 \quad \text{in } M\,, \tag{1.3}$$

with $\varepsilon \geq 0$, Schoen [24, 25] proved that blow-up cannot occur when the manifold is locally conformally flat and (this is a necessary condition) not conformally diffeomorphic to the unit sphere. More precisely, Schoen proved that for any sequence of nonnegative real numbers $(\varepsilon_\alpha)_\alpha$, $\varepsilon_\alpha \ll 1$, any sequence $(u_\alpha)_\alpha$ of solutions of (1.3) with $\varepsilon = \varepsilon_\alpha$ is automatically bounded in $C^{2,\theta}(M)$ for all real numbers θ in $(0,1)$. In particular, up to a subsequence, $(u_\alpha)_\alpha$ converges in $C^2(M)$. We say that equation (1.3) is *compact*. Schoen then conjectured that this result should remain true for non-locally conformally flat manifolds. His conjecture was very recently proved to be true in case $n \leq 24$ by Khuri–Marques–Schoen [13]. Previous contributions on

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the subject, where the conjecture was proved for lower dimensions, are by Druet [7], Li–Zhang [14–16], Li–Zhu [17], and Marques [18]. On the other hand, surprisingly, Schoen's conjecture turns out to be false in general when $n \geq 25$. The existence of blowing-up sequences of solutions for the Yamabe equation (1.3) with $\varepsilon = 0$ in high dimensions have been proved by Brendle [4] in case $n \geq 52$, and by Brendle–Marques [5] in case $25 \leq n \leq 51$. When (1.2) is not anymore an equality, the question of compactness of equations like (1.1) have been investigated, among other possible references, by Druet [6, 7], Druet–Hebey [8], and Li–Zhu [17]. We refer to the survey Druet–Hebey [9] and the references therein for more material on this subject. We point out here the following result from Druet [7]. Namely that for any smooth, compact Riemannian manifolds (M, g) of dimension $n \geq 3$ and any smooth function h on M such that the operator $\Delta_g + h$ is coercive, if there holds

$$h(\xi) < \frac{n-2}{4(n-1)} \operatorname{Scal}_g(\xi) \tag{1.4}$$

for all points ξ in M, then equations (1.1) with $\varepsilon \geq 0$ is compact.

Our first result is that in case the reverse inequality (1.4) holds true at some point together with a nonnedegeneracy assumption at this point, then we can construct a family of solutions of equations (1.1) with $0 \leq \varepsilon \ll 1$ blowing up at the point as $\varepsilon \to 0$. We prove this result when $n \geq 6$ for arbitrary compact manifolds. Given a C^1-function φ on M, we say that a critical point ξ_0 of φ is *C^1-stable* if there exists a small, open neighborhood Ω of ξ_0 such that for any point ξ in $\overline{\Omega}$, there holds $\nabla\varphi(\xi) = 0 \Leftrightarrow \xi = \xi_0$ and such that $\deg\left(\nabla\left(\varphi \circ \exp_{\xi_0}\right), \exp_{\xi_0}^{-1}(\Omega), 0\right) \neq 0$, where deg is the Brouwer degree. If φ is a C^2-function on M, then any nondegenerate critical point of φ is C^1-stable. Our first result states as follows.

Theorem 1.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 6$, let h be a C^1-function on M such that the operator $\Delta_g + h$ is coercive, and let ξ_0 be a C^1-stable critical point of the function $h - \frac{n-2}{4(n-1)} \operatorname{Scal}_g$. If there holds*

$$h(\xi_0) > \frac{n-2}{4(n-1)} \operatorname{Scal}_g(\xi_0) \tag{1.5}$$

and if $\varepsilon > 0$ is small enough, then equation (1.1) admits a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ is bounded in $H_g^1(M)$ and the u_ε's blow up at ξ_0 as $\varepsilon \to 0$.

Thanks to the result of Druet [7] and thanks to the compactness of the Yamabe equation, the assumption (1.5) in Theorem 1.1 is sharp. In particular,when $\varepsilon > 0$, blowing-up solutions can only be constructed for large potentials with respect to the potential of the Yamabe equation. Now that we get Theorem 1.1, it is natural to investigate the supercritical case where $\varepsilon < 0$. In that case, we get a perfect companion to Theorem 1.1 by reversing inequality (1.5). The existence of blowing-up solutions for asymptotically supercritical equations comes with small potentials. Our second result states as follows.

Theorem 1.2. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 6$, let h be a C^1-function on M such that the operator $\Delta_g + h$ is coercive, and let ξ_0 be a C^1-stable critical point of the function $h - \frac{n-2}{4(n-1)} \operatorname{Scal}_g$. If there holds*

$$h(\xi_0) < \frac{n-2}{4(n-1)} \operatorname{Scal}_g(\xi_0)$$

and if $\varepsilon < 0$ is small enough, then equation (1.1) admits a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ is bounded in $H_g^1(M)$ and the u_ε's blow up at ξ_0 as $\varepsilon \to 0$.

Problems like (1.1) with either $\varepsilon > 0$ or $\varepsilon < 0$ have been widely investigated when M is a flat domain of $\mathbb{R}^n$. In the bounded case, with Neumann boundary condition, the problem

$$-\Delta u + \mu u = u^{2^*-1-\varepsilon} \quad \text{in } \Omega\,, \quad u > 0 \quad \text{in } \Omega\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,, \tag{1.6}$$

appears in several branches of applied sciences, like in biological studies or in chemotaxis research. Recent references on (1.6) are by del Pino–Musso–Pistoia [20] and Rey–Wei [21, 22] where blowing-up families of solutions are proved to exist with blow-up points located on the boundary and determined by the mean curvature of $\partial\Omega$. In the unbounded case where $\Omega = \mathbb{R}^n$, a recent reference is by Micheletti–Pistoia [19]. We refer to [19–22] and the references therein for more material on the subject.

The proofs of our results rely on a well known Lĭapunov–Schmidt reduction introduced by Bahri–Coron [2] and Floer–Weinstein [10]. The paper is organized as follows. We describe the proof of Theorems 1.1 and 1.2 in Section 2. We perform the finite dimensional reduction in Section 3. We study the reduced problem in Section 4.

2. The existence result

We first set some notations. Since the operator $\Delta_g + h$ is coercive, we can provide the Sobolev space $H_g^1(M)$ with the scalar product

$$\langle u, v\rangle_h = \int_M \langle \nabla u, \nabla v\rangle_g \, dv_g + \int_M huv dv_g\,, \tag{2.1}$$

where dv_g is the volume element of the manifold. We let $\|\cdot\|_h$ be the norm induced by $\langle\cdot,\cdot\rangle_h$. Moreover, for any u in $L^q(M)$, we denote the L^q-norm of u by $\|u\|_q = \left(\int_M |u|^q dv_g\right)^{1/q}$. We let $i^* : L^{\frac{2n}{n+2}}(M) \to H_g^1(M)$ be the adjoint operator to the embedding $i : H_g^1(M) \hookrightarrow L^{2^*}(M)$, i.e. for any w in $L^{\frac{2n}{n+2}}(M)$, the function $u = i^*(w)$ in $H_g^1(M)$ is the unique solution of the equation $\Delta_g u + hu = w$ in M. By the continuity of the embedding of $H_g^1(M)$ into $L^{2^*}(M)$, we get

$$\|i^*(w)\|_h \le C \|w\|_{\frac{2n}{n+2}} \tag{2.2}$$

for some positive constant C independent of w. In order to study the supercritical case, it is also useful to recall that by standard elliptic estimates (see, for instance, Gilbarg–Trudinger [12]), given a real number $s > \frac{2n}{n-2}$, i.e. $\frac{ns}{n+2s} > \frac{2n}{n+2}$, for any w in $L^{\frac{ns}{n+2s}}(M)$, the function $i^*(w)$ belongs to $L^s(M)$ and satisfies

$$\|i^*(w)\|_s \le C \|w\|_{\frac{ns}{n+2s}} \tag{2.3}$$

for some positive constant C independent of w. For ε small, we then set

$$s_\varepsilon = \begin{cases} 2^* - \dfrac{n}{2}\varepsilon & \text{if } \varepsilon < 0\,, \\ 2^* & \text{if } \varepsilon > 0\,, \end{cases}$$

and we let $\mathcal{H}_\varepsilon = H_g^1(M) \cap L^{s_\varepsilon}(M)$ be the Banach space provided with the norm

$$\|u\|_{h,s_\varepsilon} = \|u\|_h + \|u\|_{s_\varepsilon}\,.$$

We point out that in the subcritical case $\varepsilon > 0$, the space $\mathcal{H}_\varepsilon$ coincides with the Sobolev space $H_g^1(M)$, and the norm $\|\cdot\|_{h,s_\varepsilon}$ is equivalent to the norm $\|\cdot\|_h$. Taking into account that there holds

$$\frac{ns_\varepsilon}{n+2s_\varepsilon}=\begin{cases}\dfrac{s_\varepsilon}{2^*-1-\varepsilon} & \text{if } \varepsilon<0\,,\\ \dfrac{2n}{n+2} & \text{if } \varepsilon>0\,,\end{cases} \tag{2.4}$$

and by (2.2) (and (2.3) in the supercritical case), we can rewrite problem (1.1) as

$$u=i^*\left(f_\varepsilon\left(u\right)\right),\quad u\in\mathcal{H}_\varepsilon\,, \tag{2.5}$$

where $f_\varepsilon\left(u\right)=u_+^{2^*-1-\varepsilon}$ and $u_+=\max\left(u,0\right)$.

We let r_0 be a positive real number less than the injectivity radius of M, and χ be a smooth cutoff function such that $0\le\chi\le 1$ in $\mathbb{R}_+$, $\chi\equiv 1$ in $[0,r_0/2]$, and $\chi\equiv 0$ out of $[r_0,+\infty)$. For any point ξ in M and for any positive real number δ, we define the function $W_{\delta,\xi}$ on M by

$$W_{\delta,\xi}\left(x\right)=\chi\left(d_g\left(x,\xi\right)\right)\delta^{\frac{2-n}{2}}U\left(\delta^{-1}\exp_\xi^{-1}\left(x\right)\right), \tag{2.6}$$

where d_g is the geodesic distance on M with respect to the metric g, and where

$$U\left(x\right)=\left(\frac{\sqrt{n\left(n-2\right)}}{1+\left|x\right|^2}\right)^{\frac{n-2}{2}}. \tag{2.7}$$

In particular, the functions $\delta^{\frac{2-n}{2}}U\left(\delta^{-1}x\right)$ satisfy the equation $\Delta_{\mathrm{Eucl}}U=U^{2^*-1}$, where Δ_{Eucl} is the Laplace–Beltrami operator associated with the Euclidean metric. Moreover, by Bianchi–Egnell [3], any solution of the linear equation

$$\Delta_{\mathrm{Eucl}}v=\left(2^*-1\right)U^{2^*-2}v \tag{2.8}$$

is a linear combination of the functions

$$V_0\left(x\right)=\left.\frac{d\left(\delta^{\frac{2-n}{2}}U\left(\delta^{-1}x\right)\right)}{d\delta}\right|_{\delta=1}=\frac{1}{2}n^{\frac{n-2}{4}}\left(n-2\right)^{\frac{n+2}{4}}\frac{\left|x\right|^2-1}{\left(1+\left|x\right|^2\right)^{\frac{n}{2}}} \tag{2.9}$$

and

$$V_i\left(x\right)=-\frac{\partial U}{\partial x_i}\left(x\right)=n^{\frac{n-2}{4}}\left(n-2\right)^{\frac{n+2}{4}}\frac{x_i}{\left(1+\left|x\right|^2\right)^{\frac{n}{2}}} \tag{2.10}$$

for $i=1,\dots,n$. In reference with this result, for any point ξ in M and for any positive real number δ, we introduce the functions

$$Z_{\delta,\xi}^i\left(x\right)=\chi\left(d_g\left(x,\xi\right)\right)\delta^{\frac{2-n}{2}}V_i\left(\delta^{-1}\exp_\xi^{-1}\left(x\right)\right) \tag{2.11}$$

for $i=0,\dots,n$. We then define the projections $\mathit{\Pi}_{\delta,\xi}$ and $\mathit{\Pi}_{\delta,\xi}^\perp$ of the Sobolev space $H_g^1\left(M\right)$ onto the respective subspaces

$$K_{\delta,\xi}=\operatorname{Span}\left\{Z_{\delta,\xi}^0,\dots,Z_{\delta,\xi}^n\right\}$$

and

$$K_{\delta,\xi}^\perp=\left\{\phi\in H_g^1\left(M\right);\ \left\langle\phi,Z_{\delta,\xi}^i\right\rangle_h=0\quad\forall i\in\left\{0,\dots,n\right\}\right\},$$

where $\langle\cdot,\cdot\rangle_h$ is as in (2.1). We look for solutions of equation (1.1), or equivalently of (2.5), of the form

$$u_\varepsilon=W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}\quad\text{with } \delta_\varepsilon(t_\varepsilon)=\sqrt{\left|\varepsilon\right|t_\varepsilon}\,,\ t_\varepsilon>0\,,\text{ and } \xi_\varepsilon\in M\,, \tag{2.12}$$

where $W_{\delta_\varepsilon(t),\xi_\varepsilon}$ is as in (2.6), and where $\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a function in $\mathcal{H}_\varepsilon\cap K_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}^\perp$. Therefore, we have to solve the couple of equations

$$\mathit{\Pi}_{\delta_\varepsilon(t),\xi}\left(\left\{W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}-i^*\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)\right)\right\}\right)=0\,, \tag{2.13}$$

and

$$\Pi^{\perp}_{\delta_\varepsilon(t),\xi}\left(\left\{W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}-i^*\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)\right)\right\}\right)=0\,. \tag{2.14}$$

We begin with solving equation (2.14) in Proposition 2.1 below which proof is postponed to Section 3.

Proposition 2.1. *If $n\ge 6$ and $\delta_\varepsilon(t)$ is as in* (2.12)*, then for any real numbers a and b satisfying $0<a<b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any point ξ in M, and for any real number t in $[a,b]$, equation* (2.14) *admits a unique solution $\phi_{\delta_\varepsilon(t),\xi}$ in $\mathcal{H}_\varepsilon\cap K^{\perp}_{\delta_\varepsilon(t),\xi}$, which is continuously differentiable with respect to ξ and t, such that*

$$\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon}\le C_{a,b}\left|\varepsilon\right|\left|\ln\left|\varepsilon\right|\right|. \tag{2.15}$$

We then introduce the functional J_ε defined on $H^1_g(M)$ by

$$J_\varepsilon(u)=\frac{1}{2}\int_M\left|\nabla u\right|^2_g dv_g+\frac{1}{2}\int_M hu^2dv_g-\frac{1}{2^*-\varepsilon}\int_M u_+^{2^*-\varepsilon}dv_g\,,$$

where $u_+=\max(u,0)$. Its critical points are the solutions of equation (2.5). We also define the function $\widetilde{J}_\varepsilon$ on $\mathbb{R}^+_*\times M$ by

$$\widetilde{J}_\varepsilon(t,\xi)=J_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right) \tag{2.16}$$

where $W_{\delta_\varepsilon(t),\xi_\varepsilon}$ is as in (2.6) and where $\phi_{\delta_\varepsilon(t),\xi}$ is given by Proposition 2.1. We solve equation (2.13) in Proposition 2.2 below which proof is postponed to Section 4. As a general remark, given some C^1-functions f_ε, we say that the estimate $f_\varepsilon=\mathrm{o}(\varepsilon)$ is C^1-uniform if there hold both $f_\varepsilon=\mathrm{o}(\varepsilon)$ and $\nabla f_\varepsilon=\mathrm{o}(\varepsilon)$ as $\varepsilon\to 0$.

Proposition 2.2. *If $n\ge 6$ and $\delta_\varepsilon(t)$ is as in* (2.12)*, then for any real numbers a and b satisfying $0<a<b$, there holds*

$$\widetilde{J}_\varepsilon(t,\xi)=c_1-c_2\varepsilon-c_3\varepsilon\ln\left(\left|\varepsilon\right|t\right)+c_4\left|\varepsilon\right|t\left(h(\xi)-\frac{n-2}{4(n-1)}\mathrm{Scal}_g(\xi)\right)+\mathrm{o}(\varepsilon) \tag{2.17}$$

as $\varepsilon\to 0$, C^1-uniformly with respect to ξ in M and to t in $[a,b]$, where the c_i's are positive constants. Moreover, for ε small, if $(t_\varepsilon,\xi_\varepsilon)\in[a,b]\times M$ is a critical point of the function $\widetilde{J}_\varepsilon$, then $W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of (2.5)*, or equivalently of equation* (1.1)*.*

We can now prove Theorems 1.1 and 1.2 by using Propositions 2.1 and 2.2 together with the assumption that the function $h-\frac{n-2}{4(n-1)}\mathrm{Scal}_g$ admits a C^1-stable critical point ξ_0 with positive value in the subcritical case and negative value in the supercritical case. This is the only place in our proof where this assumption comes into play.

Proof of Theorem 1.1. We introduce the function $\widetilde{J}$ defined on $\mathbb{R}^*_+\times M$ by

$$\widetilde{J}(t,\xi)=-c_3\ln t+c_4t\varphi(\xi)\,,$$

where φ is the function defined on M by

$$\varphi(\xi)=h(\xi)-\frac{n-2}{4(n-1)}\mathrm{Scal}_g(\xi)\,. \tag{2.18}$$

We let ξ_0 be a C^1-stable critical point of φ satisfying (1.5) and set $t_0=\frac{c_3}{c_4\varphi(\xi_0)}$. Since $\varphi(\xi_0)>0$, we get $t_0>0$. By the continuity of the Brouwer degree via homotopy (see, for instance, Fonseca–Gangbo [11]), considering $H : [0,1] \times \mathbb{R}_+^* \times \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by

$$H(s,t,\xi) = s\left(\frac{d\widetilde{J}}{dt}, \left.\frac{\partial\big(\widetilde{J}\left(t, \exp_\xi(y)\right)\big)}{\partial y_1}\right|_{y=0}, \dots, \left.\frac{\partial\big(\widetilde{J}\left(t, \exp_\xi(y)\right)\big)}{\partial y_n}\right|_{y=0}\right) + (1-s)\left(t - t_0, \left.\frac{\partial\left(\varphi \circ \exp_\xi(y)\right)}{\partial y_1}\right|_{y=0}, \dots, \left.\frac{\partial\left(\varphi \circ \exp_\xi(y)\right)}{\partial y_n}\right|_{y=0}\right),$$

one can easily see that (t_0, ξ_0) is a C^1-stable critical point of $\widetilde{J}$. By Proposition 2.2, we get

$$\left|\frac{d}{dt}\left(\frac{1}{\varepsilon}\widetilde{J}_\varepsilon - \widetilde{J}\right)\right| + \left|\nabla_\xi\left(\frac{1}{\varepsilon}\widetilde{J}_\varepsilon - \widetilde{J}\right)\right| \longrightarrow 0$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. By standard properties of the Brouwer degree, it follows that there exists a family of critical points $(t_\varepsilon, \xi_\varepsilon)$ of $\widetilde{J}_\varepsilon$ converging to (t_0, ξ_0) as $\varepsilon \to 0$. Proposition 2.2 yields that the function $u_\varepsilon = W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of equation (2.5) for ε small. As is easily seen, the u_ε's blow up at ξ_0 as $\varepsilon \to 0$. By coercivity of the operator $\Delta_g + h$ and since $f_\varepsilon(u_\varepsilon) \geq 0$, we get that the u_ε's are positive. By (2.15), (4.2), and (4.4), the u_ε's are bounded in $H_g^1(M)$. This ends the proof of Theorem 1.1. ☐

Proof of Theorem 1.2. We introduce the function $\widetilde{J}$ defined on $\mathbb{R}_+^* \times M$ by

$$\widetilde{J}(t,\xi) = -c_3 \ln t - c_4 t \varphi(\xi),$$

where φ is as in (2.18), and we then proceed in a similar way as in the proof of Theorem 1.1. ☐

3. The finite dimensional reduction

This section is devoted to the proof of Proposition 2.1. For ε small, for any $\delta > 0$, and any point ξ in M, we introduce the map $L_{\varepsilon,\delta,\xi} : \mathcal{H}_\varepsilon \cap K_{\delta,\xi}^\perp \to \mathcal{H}_\varepsilon \cap K_{\delta,\xi}^\perp$ defined by

$$L_{\varepsilon,\delta,\xi}(\phi) = \Pi_{\delta,\xi}^\perp\left(\phi - i^*\left(f_\varepsilon'\left(W_{\delta,\xi}\right)\phi\right)\right).$$

One can easily check that this map is well defined by using (2.2) and (2.3). As a first step, we prove the invertibility of $L_{\varepsilon,\delta,\xi}$.

Lemma 3.1. *If $\delta_\varepsilon(t)$ is as in (2.12), then for any real numbers a and b satisfying $0 < a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any point ξ in M, any real number t in $[a,b]$, and any function ϕ in $\mathcal{H}_\varepsilon \cap K_{\delta_\varepsilon(t),\xi}^\perp$, there holds*

$$\left\|L_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{h,s_\varepsilon} \geq C_{a,b}\|\phi\|_{h,s_\varepsilon}.$$

Proof. We proceed by contradiction and assume that there exist a sequence of real numbers $(\varepsilon_\alpha)_\alpha$ converging to 0, a sequence of points $(\xi_\alpha)_\alpha$ in M, a sequence of real numbers $(t_\alpha)_\alpha$ in $[a,b]$, and a sequence of functions $(\phi_\alpha)_\alpha$ satisfying

$$\phi_\alpha \in \mathcal{H}_{\varepsilon_\alpha} \cap K_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}^\perp, \quad \|\phi_\alpha\|_{h,s_{\varepsilon_\alpha}} = 1, \quad \left\|L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha)\right\|_{h,s_{\varepsilon_\alpha}} \longrightarrow 0 \tag{3.1}$$

as $\alpha \to +\infty$. For any α, we set $\delta_\alpha = \delta_{\varepsilon_\alpha}(t_\alpha)$ and

$$\widetilde{\phi}_\alpha(x) = \chi(\delta_\alpha|x|)\,\delta_\alpha^{\frac{n-2}{2}}\phi_\alpha \circ \exp_{\xi_\alpha}(\delta_\alpha x),$$

where χ is a cutoff function as in (2.6). By (3.1) and by an easy change of variable, we get that the sequence $(\widetilde{\phi}_\alpha)_\alpha$ is bounded in $D^{1,2}(\mathbb{R}^n)$. Passing if necessary to a subsequence, we may assume that $\left(\widetilde{\phi}_\alpha\right)_\alpha$ converges weakly to a function $\widetilde{\phi}$ in $D^{1,2}(\mathbb{R}^n)$, and thus in $L^{2^*}(\mathbb{R}^n)$ by the continuity of the embedding of $D^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$. Since, for any α, the function ϕ_α belongs to $K^\perp_{\delta_\alpha,\xi_\alpha}$, by an easy change of variable, for $j=0,\dots,n$, we get

$$0=\left\langle Z^j_{\delta_\alpha,\xi_\alpha},\phi_\alpha\right\rangle_h=\int_{\mathbb{R}^n}\left\langle\nabla\left(\chi_\alpha V_j\right),\nabla\widetilde{\phi}_\alpha\right\rangle_{g_\alpha}dv_{g_\alpha}+\delta_\alpha^2\int_{\mathbb{R}^n}h_\alpha\chi_\alpha V_j\widetilde{\phi}_\alpha dv_{g_\alpha}\,, \tag{3.2}$$

where $g_\alpha(x)=\exp^*_{\xi_\alpha}g(\delta_\alpha x)$, $\chi_\alpha(x)=\chi(\delta_\alpha|x|)$, $h_\alpha(x)=h\left(\exp_{\xi_\alpha}(\delta_\alpha x)\right)$, and where V_j is as in (2.9)–(2.10). For $j=0,\dots,n$, since the function V_j is a solution in $D^{1,2}(\mathbb{R}^n)$ of equation (2.8) and since the sequence $\left(\widetilde{\phi}_\alpha\right)_\alpha$ converges weakly to $\widetilde{\phi}$ in $D^{1,2}(\mathbb{R}^n)$, passing to the limit into (3.2) as $\alpha\to+\infty$ yields

$$\int_{\mathbb{R}^n}\left\langle\nabla V_j,\nabla\widetilde{\phi}\right\rangle_{\text{Eucl}}dx=(2^*-1)\int_{\mathbb{R}^n}U^{2^*-2}V_j\widetilde{\phi}dx=0\,, \tag{3.3}$$

where the function U is as in (2.7). Taking into account that $\phi_\alpha-i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right)-L_{\varepsilon_\alpha,\delta_\alpha,\xi_\alpha}(\phi_\alpha)$ belongs to $K_{\delta_\alpha,\xi_\alpha}$, for any α, we get that there exist some real numbers $\lambda^0_\alpha,\dots,\lambda^n_\alpha$ such that

$$\phi_\alpha-i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right)-L_{\varepsilon_\alpha,\delta_\alpha,\xi_\alpha}(\phi_\alpha)=\sum_{k=0}^n\lambda^k_\alpha Z^k_{\delta_\alpha,\xi_\alpha}\,. \tag{3.4}$$

We claim that there holds

$$\left\|\phi_\alpha-i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right)\right\|_{h,\varepsilon_\alpha}\longrightarrow 0 \tag{3.5}$$

as $\alpha\to+\infty$. By (3.1), one can easily see that, in order to prove this claim, it suffices to show that, for $j=0,\dots,n$ there holds $\lambda^j_\alpha\to0$ as $\alpha\to+\infty$. For any α, since the functions ϕ_α and $L_{\varepsilon_\alpha,\delta_\alpha,\xi_\alpha}(\phi_\alpha)$ belong to $K^\perp_{\delta_\alpha,\xi_\alpha}$, multiplying (3.4) by $Z^j_{\delta_\alpha,\xi_\alpha}$ gives

$$\begin{aligned}\sum_{k=0}^n\lambda^k_\alpha\left\langle Z^k_{\delta_\alpha,\xi_\alpha},Z^j_{\delta_\alpha,\xi_\alpha}\right\rangle_h&=-\left\langle i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right),Z^j_{\delta_\alpha,\xi_\alpha}\right\rangle_h\\&=-\int_M f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})Z^j_{\delta_\alpha,\xi_\alpha}\phi_\alpha dv_g\,.\end{aligned} \tag{3.6}$$

For any α and for $j,k=0,\dots,n$, an easy change of variable yields

$$\left\langle Z^k_{\delta_\alpha,\xi_\alpha},Z^j_{\delta_\alpha,\xi_\alpha}\right\rangle_h=\int_{\mathbb{R}^n}\left\langle\nabla(\chi_\alpha V_k),\nabla(\chi_\alpha V_j)\right\rangle_{g_\alpha}dv_{g_\alpha}+\delta_\alpha^2\int_{\mathbb{R}^n}h_\alpha\chi_\alpha^2V_kV_jdv_{g_\alpha}\,, \tag{3.7}$$

and

$$\int_M f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})Z^j_{\delta_\alpha,\xi_\alpha}\phi_\alpha dv_g=\delta_\alpha^{\frac{n-2}{2}\varepsilon_\alpha}\int_{\mathbb{R}^n}f'_{\varepsilon_\alpha}(\chi_\alpha U)\chi_\alpha V_j\widetilde{\phi}_\alpha dv_{g_\alpha}\,. \tag{3.8}$$

Passing to the limit into (3.7) gives

$$\left\langle Z^k_{\delta_\alpha,\xi_\alpha},Z^j_{\delta_\alpha,\xi_\alpha}\right\rangle_h\longrightarrow\begin{cases}\|V_k\|^2_{D^{1,2}(\mathbb{R}^n)}&\text{if }k=j,\\0&\text{if }k\neq j,\end{cases} \tag{3.9}$$

as $\alpha\to+\infty$. Since there holds

$$\delta_\alpha^{\frac{n-2}{2}\varepsilon_\alpha}=\left(|\varepsilon_\alpha|\,t_\alpha\right)^{\frac{n-2}{4}\varepsilon_\alpha}\longrightarrow1 \tag{3.10}$$

as $\alpha\to+\infty$, and since the sequence $\left(\widetilde{\phi}_\alpha\right)_\alpha$ converges weakly to $\widetilde{\phi}$ in $D^{1,2}(\mathbb{R}^n)$ by (3.3), passing to the limit into (3.8) gives

$$\int_M f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})Z^j_{\delta_\alpha,\xi_\alpha}\phi_\alpha dv_g\longrightarrow(2^*-1)\int_{\mathbb{R}^n}U^{2^*-2}V_j\widetilde{\phi}dx=0 \tag{3.11}$$

as $\alpha \to +\infty$. It follows from (3.6), (3.9), and (3.11) that for $j = 0, \dots, n$, there holds $\lambda_\alpha^j \to 0$ as $\alpha \to +\infty$, and our claim (3.5) is proved. For any bounded sequence $(\varphi_\alpha)_\alpha$ in $H_g^1(M)$, by Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\langle \phi_\alpha, \varphi_\alpha \rangle_h &= \int_M f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha}) \phi_\alpha \varphi_\alpha dv_g + \left\langle \phi_\alpha - i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right), \varphi_\alpha \right\rangle_h \\ &= \int_M f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha}) \phi_\alpha \varphi_\alpha dv_g + \mathrm{o}(1) \end{aligned} \tag{3.12}$$

as $\alpha \to +\infty$. For any smooth function φ with compact support in $\mathbb{R}^n$ and for any α, we take

$$\varphi_\alpha(x) = \delta_\alpha^{\frac{2-n}{2}} \varphi\left(\delta_\alpha^{-1} \exp_{\xi_\alpha}^{-1}(x)\right).$$

By (3.12) and by an easy change of the variable, we get

$$\int_{\mathbb{R}^n} \left\langle \nabla \widetilde{\phi}_\alpha, \nabla \varphi \right\rangle_{g_\alpha} dv_{g_\alpha} + \delta_\alpha^2 \int_{\mathbb{R}^n} h_\alpha \widetilde{\phi}_\alpha \varphi dv_{g_\alpha} = \delta_\alpha^{\frac{n-2}{2}\varepsilon_\alpha} \int_{\mathbb{R}^n} f'_{\varepsilon_\alpha}(\chi_\alpha U) \widetilde{\phi}_\alpha \varphi dv_{g_\alpha} + \mathrm{o}(1)$$

as $\alpha \to +\infty$. By (3.10) and since the sequence $\left(\widetilde{\phi}_\alpha\right)_\alpha$ converges weakly to $\widetilde{\phi}$ in $D^{1,2}(\mathbb{R}^n)$, it follows that $\widetilde{\phi}$ is a weak solution of equation (2.8). By (3.3), we then get that the function $\widetilde{\phi}$ is identically zero. Plugging $\varphi_\alpha = \phi_\alpha$ into (3.12) and changing the variable yield

$$\begin{aligned}\|\phi_\alpha\|_h &= \int_M f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha}) \phi_\alpha^2 dv_g + \mathrm{o}(1) \\ &= \delta_\alpha^{\frac{n-2}{2}\varepsilon_\alpha} \int_{\mathbb{R}^n} f'_{\varepsilon_\alpha}(\chi_\alpha U) \widetilde{\phi}_\alpha^2 dv_{g_\alpha} + \mathrm{o}(1)\end{aligned} \tag{3.13}$$

as $\alpha \to +\infty$. As is easily seen, the functions $f'_{\varepsilon_\alpha}(\chi_\alpha U)$ converge strongly to $(2^*-1)U^{2^*-2}$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$. Moreover, since the functions $\widetilde{\phi}_\alpha^2$ are uniformly bounded in $L^{\frac{n}{n-2}}(\mathbb{R}^n)$ and converge up to a subsequence almost everywhere to $\widetilde{\phi}^2 \equiv 0$ in $\mathbb{R}^n$ as $\alpha \to +\infty$, we get that they converge weakly to 0 in $L^{\frac{n}{n-2}}(\mathbb{R}^n)$. We then get

$$\int_{\mathbb{R}^n} f'_{\varepsilon_\alpha}(\chi_\alpha U) \widetilde{\phi}_\alpha^2 dv_{g_\alpha} \longrightarrow 0 \tag{3.14}$$

as $\alpha \to +\infty$. It follows from (3.10), (3.13), and (3.14) that the sequence $(\phi_\alpha)_\alpha$ converges strongly to 0 in $H_g^1(M)$. Moreover, in the supercritical case $\varepsilon_\alpha < 0$ for all α, by (2.3), (3.5), and by Hölder's inequality, we get

$$\begin{aligned}\|\phi_\alpha\|_{s_{\varepsilon_\alpha}} &\le \left\| i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right)\right\|_{s_{\varepsilon_\alpha}} + \left\|\phi_\alpha - i^*\left(f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha\right)\right\|_{s_{\varepsilon_\alpha}} \\ &\le \left\| f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha})\phi_\alpha \right\|_{\frac{ns_{\varepsilon_\alpha}}{n+2s_{\varepsilon_\alpha}}} + \mathrm{o}(1) \\ &\le \left\| f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha}) \right\|_{\theta_\alpha} \|\phi_\alpha\|_{2^*} + \mathrm{o}(1)\end{aligned} \tag{3.15}$$

as $\alpha \to +\infty$, where

$$\theta_\alpha = \frac{2ns_{\varepsilon_\alpha}}{2n - (n-6)s_{\varepsilon_\alpha}}.$$

One can easily compute

$$\left\| f'_{\varepsilon_\alpha}(W_{\delta_\alpha,\xi_\alpha}) \right\|_{\theta_\alpha} = \mathrm{O}\left((|\varepsilon_\alpha| t_\alpha)^{\frac{n}{2\theta_\alpha} - 1 - \frac{n-2}{4}\varepsilon_\alpha} \|U\|_{L^{(2^*-2)\theta_\alpha}(\mathbb{R}^n)}^{2^*-2} \right) = \mathrm{O}(1) \tag{3.16}$$

as $\alpha \to +\infty$. Since the sequence $(\phi_\alpha)_\alpha$ converges strongly to 0 as $\alpha \to +\infty$ in $H_g^1(M)$, and thus in $L^{2^*}(M)$ by the continuity of the embedding of $H_g^1(M)$ into $L^{2^*}(M)$, it follows from (3.15) and (3.16) that in the supercritical case, there also holds $\|\phi_\alpha\|_{h,s_{\varepsilon_\alpha}} \to 0$ as $\alpha \to +\infty$, which contradicts (3.1). This ends the proof of Lemma 3.1. □

For ε small, for any point ξ in M, and any positive real number δ, equation (2.14) is equivalent to

$$L_{\varepsilon,\delta,\xi}(\phi) = N_{\varepsilon,\delta,\xi}(\phi) + R_{\varepsilon,\delta,\xi}\,, \tag{3.17}$$

where

$$N_{\varepsilon,\delta,\xi}(\phi) = \Pi^{\perp}_{\delta,\xi}\left(i^*\left(f_\varepsilon\left(W_{\delta,\xi}+\phi\right) - f_\varepsilon\left(W_{\delta,\xi}\right) - f'_\varepsilon\left(W_{\delta,\xi}\right)\phi\right)\right) \tag{3.18}$$

and

$$R_{\varepsilon,\delta,\xi} = \Pi^{\perp}_{\delta,\xi}\left(i^*\left(f_\varepsilon\left(W_{\delta,\xi}\right)\right) - W_{\delta,\xi}\right). \tag{3.19}$$

In Lemma 3.2 below, we estimate the remainder term $R_{\varepsilon,\delta,\xi}$.

Lemma 3.2. *If $n \geq 6$ and $\delta_\varepsilon(t)$ is as in (2.12), then for any real numbers a and b satisfying $0 < a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any point ξ in M, and any real number t in $[a,b]$, there holds*

$$\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon} \leq C_{a,b}\left|\varepsilon\right|\left|\ln\left|\varepsilon\right|\right|. \tag{3.20}$$

Proof. By (2.2) (and by (2.3) in the supercritical case), we get that there exists $C > 0$ such that for ε small, for any point ξ in M, and any $\delta > 0$, there holds

$$\left\|i^*\left(f_\varepsilon\left(W_{\delta,\xi}\right)\right) - W_{\delta,\xi}\right\|_{h,s_\varepsilon} \leq C\left\|f_\varepsilon\left(W_{\delta,\xi}\right) - \Delta_g W_{\delta,\xi} - hW_{\delta,\xi}\right\|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}.$$

We take $\delta = \delta_\varepsilon(t)$ for some real number t in $[a,b]$. Increasing C if necessary, an easy change of variable yields

$$\begin{aligned}\left\|i^*\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right) - W_{\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon} \leq C\Big\|&\delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}\chi_{\varepsilon,t}^{2^*-1-\varepsilon}U^{2^*-1-\varepsilon} - \chi_{\varepsilon,t}\Delta_{g_{\varepsilon,t,\xi}}U\\ &- U\Delta_{g_{\varepsilon,t,\xi}}\chi_{\varepsilon,t} - 2\left\langle\nabla U,\nabla\chi_{\varepsilon,t}\right\rangle_{g_{\varepsilon,t,\xi}} - \delta_\varepsilon(t)^2 h_{\varepsilon,t}\chi_{\varepsilon,t}U\Big\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)},\end{aligned}$$

where $g_{\varepsilon,t,\xi}(x) = \exp_\xi^* g\left(\delta_\varepsilon(t)x\right)$, $\chi_{\varepsilon,t}(x) = \chi\left(\delta_\varepsilon(t)|x|\right)$, and $h_{\varepsilon,t}(x) = h\left(\exp_\xi\left(\delta_\varepsilon(t)x\right)\right)$, and where the function U is as in (2.7). Taking into account that the function U satisfies the equation $\Delta_{\mathrm{Eucl}}U = U^{2^*-1}$, it follows that

$$\begin{aligned}&\left\|i^*\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right) - W_{\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon} \leq C\bigg(\delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}\left\|\chi_{\varepsilon,t}^{2^*-1-\varepsilon}\left(U^{2^*-1-\varepsilon} - U^{2^*-1}\right)\right\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)}\\ &+ \left\|\left(\delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}\chi_{\varepsilon,t}^{2^*-1-\varepsilon} - \chi_{\varepsilon,t}\right)U^{2^*-1}\right\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)} + \left\|\chi_{\varepsilon,t}\left(\Delta_{g_{\varepsilon,t,\xi}}U - \Delta_{\mathrm{Eucl}}U\right)\right\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)}\\ &+ \left\|U\Delta_{g_{\varepsilon,t,\xi}}\chi_{\varepsilon,t}\right\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)} + 2\left\|\left\langle\nabla U,\nabla\chi_{\varepsilon,t}\right\rangle_{g_{\varepsilon,t,\xi}}\right\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)} + \delta_\varepsilon(t)^2\left\|h_{\varepsilon,t}\chi_{\varepsilon,t}U\right\|_{L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathbb{R}^n)}\bigg).\end{aligned} \tag{3.21}$$

We are led to estimate each term in the right hand side of (3.21). First, we compute

$$\begin{aligned}\int_{\mathbb{R}^n}&\left|\chi_{\varepsilon,t}^{2^*-1-\varepsilon}\left(U^{2^*-1-\varepsilon} - U^{2^*-1}\right)\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx \leq \int_{B_0\left(\frac{r_0}{\delta_\varepsilon(t)}\right)}\left|U^{2^*-1-\varepsilon} - U^{2^*-1}\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx\\ &= \mathrm{O}\left(\int_0^{\frac{r_0}{\delta_\varepsilon(t)}}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n(n+2)s_\varepsilon}{2(n+2s_\varepsilon)}}}\left|\left(1+r^2\right)^{\frac{n-2}{2}\varepsilon} - 1\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dr\right)\\ &= \mathrm{O}\left(\int_0^{\frac{r_0}{\delta_\varepsilon(t)}}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n(n+2)s_\varepsilon}{2(n+2s_\varepsilon)}}}\left(\frac{n-2}{2}|\varepsilon|\ln\left(1+r^2\right)\right)^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dr\right) = \mathrm{O}\left(|\varepsilon|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\right),\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\left(\delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}\chi_{\varepsilon,t}^{2^*-1-\varepsilon}-\chi_{\varepsilon,t}\right)U^{2^*-1}\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx\\&\quad=\mathrm{O}\left(\left|\varepsilon\ln\delta_\varepsilon(t)\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\int_0^{\frac{r_0}{2\delta_\varepsilon(t)}}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n(n+2)s_\varepsilon}{2(n+2s_\varepsilon)}}}dx+\int_{\frac{r_0}{2\delta_\varepsilon(t)}}^{+\infty}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n(n+2)s_\varepsilon}{2(n+2s_\varepsilon)}}}dx\right)\\&\quad=\mathrm{O}\left(\left|\varepsilon\ln\delta_\varepsilon(t)\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}+\delta_\varepsilon(t)^{\frac{n^2(s_\varepsilon-1)}{n+2s_\varepsilon}}\right)=\mathrm{O}\left(\left|\varepsilon\ln|\varepsilon|\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\right)\end{aligned}\tag{3.23}$$

as $\varepsilon\to 0$, uniformly with respect to ξ in M and to t in $[a,b]$. By standard properties of the exponential map, we get that there exists a positive constant C such that for any point ξ in M, any real number t in $[a,b]$, any point x in $B_0\left(r_0/\delta_\varepsilon(t)\right)$, and any indices i, j, and k, there hold $|g_{\varepsilon,t,\xi}^{ij}(x)-\mathrm{Eucl}^{ij}|\le C\delta_\varepsilon(t)^2|x|^2$ and $|g_{\varepsilon,t,\xi}^{ij}(x)\left(\mathit{\Gamma}_{\varepsilon,t,\xi}\right)_{ij}^k(x)|\le C\delta_\varepsilon(t)^2|x|$. Taking into account that there holds

$$\Delta_{g_{\varepsilon,t,\xi}}=-g_{\varepsilon,t,\xi}^{ij}\left(\frac{\partial^2}{\partial x_i\partial x_j}-\left(\mathit{\Gamma}_{\varepsilon,t,\xi}\right)_{ij}^k\frac{\partial}{\partial x_k}\right),$$

where $\left(\mathit{\Gamma}_{\varepsilon,t,\xi}\right)_{ij}^k$ stand for the Christoffel symbols of the metric $g_{\varepsilon,t,\xi}$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\chi_{\varepsilon,t}\left(\Delta_{g_{\varepsilon,t,\xi}}U-\Delta_{\mathrm{Eucl}}U\right)\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{2ns_\varepsilon}{n+2s_\varepsilon}}\int_0^{\frac{r_0}{\delta_\varepsilon(t)}}\frac{r^{\frac{n^2-n+2(2n-1)s_\varepsilon}{n+2s_\varepsilon}}}{\left(1+r^2\right)^{\frac{n^2s_\varepsilon}{2(n+2s_\varepsilon)}}}dr\right)\\&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{2ns_\varepsilon}{n+2s_\varepsilon}}\right)=\mathrm{O}\left(|\varepsilon|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\right)\end{aligned}\tag{3.24}$$

as $\varepsilon\to 0$, uniformly with respect to ξ in M and to t in $[a,b]$. Since there hold $\left|\chi_{\varepsilon,t}'\right|\le C\delta_\varepsilon(t)$ and $\left|\chi_{\varepsilon,t}''\right|\le C\delta_\varepsilon^2(t)$ for some positive constant C independent of ε and t, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\left|U\Delta_{g_{\varepsilon,t,\xi}}\chi_{\varepsilon,t}\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{2ns_\varepsilon}{n+2s_\varepsilon}}\int_{B_0\left(\frac{r_0}{\delta_\varepsilon(t)}\right)\backslash B_0\left(\frac{r_0}{2\delta_\varepsilon(t)}\right)}U^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx\right)\\&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{2ns_\varepsilon}{n+2s_\varepsilon}}\int_{\frac{r_0}{2\delta_\varepsilon(t)}}^{\frac{r_0}{\delta_\varepsilon(t)}}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n(n-2)s_\varepsilon}{2(n+2s_\varepsilon)}}}dr\right)\\&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{n((n-2)s_\varepsilon-n)}{n+2s_\varepsilon}}\right)=\mathrm{O}\left(|\varepsilon|^{\frac{n((n-2)s_\varepsilon-n)}{2(n+2s_\varepsilon)}}\right)\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\left\langle\nabla U,\nabla\chi_{\varepsilon,t}\right\rangle_{g_{\varepsilon,t,\xi}}\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\int_{\mathbb{R}^n\backslash B_0\left(\frac{r_0}{2\delta_\varepsilon(t)}\right)}\left|\nabla U(x)\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}dx\right)\\&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\int_{\frac{r_0}{2\delta_\varepsilon(t)}}^{+\infty}\frac{r^{n-1+\frac{ns_\varepsilon}{n+2s_\varepsilon}}}{\left(1+r^2\right)^{\frac{n^2s_\varepsilon}{2(n+2s_\varepsilon)}}}dr\right)\\&=\mathrm{O}\left(\delta_\varepsilon(t)^{\frac{n^2s_\varepsilon}{n+2s_\varepsilon}}\right)=\mathrm{O}\left(\varepsilon^{\frac{n^2s_\varepsilon}{2(n+2s_\varepsilon)}}\right)\end{aligned}\tag{3.26}$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in $[a,b]$. We also compute

$$\begin{aligned}\int_{\mathbb{R}^n} \left|h_{\varepsilon,t}\chi_{\varepsilon,t}U\right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}} dx &= \mathrm{O}\left(\int_{B_0\left(\frac{r_0}{\delta_\varepsilon(t)}\right)} U^{\frac{ns_\varepsilon}{n+2s_\varepsilon}} dx\right)\\ &= \mathrm{O}\left(\int_0^{\frac{r_0}{\delta_\varepsilon(t)}} \frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n(n-2)s_\varepsilon}{2(n+2s_\varepsilon)}}} dr\right)\\ &= \begin{cases} \mathrm{O}\left(\left|\ln \delta_\varepsilon\left(t\right)\right|\right) & \text{if } n=6 \text{ and } \varepsilon>0,\\ \mathrm{O}\left(1\right) & \text{otherwise.}\end{cases}\\ &= \begin{cases} \mathrm{O}\left(\left|\ln\left|\varepsilon\right|\right|\right) & \text{if } n=6 \text{ and } \varepsilon>0,\\ \mathrm{O}\left(1\right) & \text{otherwise.}\end{cases}\end{aligned} \tag{3.27}$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in $[a,b]$. Finally, (3.20) follows from (3.21)–(3.27). ☐

We can now prove Proposition 2.1 by using Lemmas 3.1 and 3.2.

Proof of Proposition 2.1. For ε small, for any point ξ in M, and any positive real number δ, we let $T_{\varepsilon,\delta,\xi} : \mathcal{H}_\varepsilon \cap K_{\delta,\xi}^\perp \to \mathcal{H}_\varepsilon \cap K_{\delta,\xi}^\perp$ be defined by

$$T_{\varepsilon,\delta,\xi}\left(\phi\right) = L_{\varepsilon,\delta,\xi}^{-1}\left(N_{\varepsilon,\delta,\xi}\left(\phi\right) + R_{\varepsilon,\delta,\xi}\right),$$

where $N_{\varepsilon,\delta,\xi}\left(\phi\right)$ and $R_{\varepsilon,\delta,\xi}$ are as in (3.18) and (3.19). We also set

$$\mathcal{B}_{\varepsilon,\delta,\xi}\left(\Lambda\right) = \left\{\phi \in \mathcal{H}_\varepsilon \cap K_{\delta,\xi}^\perp;\ \left\|\phi\right\|_{h,s_\varepsilon} \le \Lambda \left\|R_{\varepsilon,\delta,\xi}\right\|_{h,s_\varepsilon}\right\},$$

where Λ is a positive constant to be chosen large later on. We take $\delta = \delta_\varepsilon\left(t\right)$ for some real number t in $[a,b]$. In order to solve equation (3.17), or equivalently (2.14), it suffices to show that the map $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ admits a fixed point $\phi_{\delta_\varepsilon(t),\xi}$. Therefore, we prove that for ε small, for any point ξ in M, and any real number t in $[a,b]$, there holds $T_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\Lambda\right)\right) \subset \mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\Lambda\right)$ and $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ is a contraction map on the ball $\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\Lambda\right)$. By Lemma 3.1, by (2.2) (and (2.3) in the supercritical case), for ε small, for any point ξ in M, any real number t in $[a,b]$, and any functions ϕ, ϕ_1, and ϕ_2 in $\mathcal{H}_\varepsilon$, we get

$$\begin{aligned}\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\phi\right)\right\|_{h,s_\varepsilon} &\le C\left(\left\|N_{\varepsilon,\delta,\xi}\left(\phi\right)\right\|_{h,s_\varepsilon} + \left\|R_{\varepsilon,\delta,\xi}\right\|_{h,s_\varepsilon}\right)\\ &\le C'\left(\left\|f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right) - f_\varepsilon'\left(W_{\delta_\varepsilon(t),\xi}\right)\phi\right\|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} + \left\|R_{\varepsilon,\delta,\xi}\right\|_{h,s_\varepsilon}\right)\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\phi_1\right) - T_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\phi_2\right)\right\|_{h,s_\varepsilon} &\le C\left\|N_{\varepsilon,\delta,\xi}\left(\phi_1\right) - N_{\varepsilon,\delta,\xi}\left(\phi_2\right)\right\|_{h,s_\varepsilon}\\ &\le C'\left(\left\|f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_1\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_2\right) - f_\varepsilon'\left(W_{\delta_\varepsilon(t),\xi}\right)\left(\phi_1-\phi_2\right)\right\|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\right)\end{aligned} \tag{3.29}$$

for some positive constants C and C' independent of ε, ξ, t, ϕ, ϕ_1, and ϕ_2. By the mean value theorem, one can easily check that there exists a positive constant C such that for ε small, there holds

$$\begin{aligned}&\left|f_\varepsilon\left(x+y\right) - f_\varepsilon\left(x+z\right) - f_\varepsilon'\left(x\right)\left(y-z\right)\right|\\ &\quad\le C\left|y-z\right| \begin{cases} \left(\left|y\right|+\left|z\right|\right)\left(x+\left|y\right|+\left|z\right|\right)^{-\varepsilon} & \text{if } n=6 \text{ and } \varepsilon<0,\\ \min\left(\left(\left|y\right|+\left|z\right|\right)^{2^*-2-\varepsilon}, x^{2^*-3-\varepsilon}\left(\left|y\right|+\left|z\right|\right)\right) & \text{otherwise,}\end{cases}\end{aligned} \tag{3.30}$$

for all positive real numbers x and all real numbers y and z. By Hölder's inequality, for any ϕ_1 and ϕ_2 in $\mathcal{H}_\varepsilon$, it follows that if $n=6$ and $\varepsilon<0$, then there holds

$$\begin{aligned}&\left\|f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_1\right)-f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_2\right)-f_\varepsilon'\left(W_{\delta_\varepsilon(t),\xi}\right)\left(\phi_1-\phi_2\right)\right\|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\\&\le C\left\|\phi_1-\phi_2\right\|_{\frac{3s_\varepsilon(2-\varepsilon)}{3+s_\varepsilon}}\left(\left\|\phi_1\right\|_{\frac{3s_\varepsilon(2-\varepsilon)}{3+s_\varepsilon}}+\left\|\phi_2\right\|_{\frac{3s_\varepsilon(2-\varepsilon)}{3+s_\varepsilon}}\right)\\&\quad\times\left(\left\|W_{\delta_\varepsilon(t),\xi}\right\|_{\frac{3s_\varepsilon(2-\varepsilon)}{3+s_\varepsilon}}+\left\|\phi_1\right\|_{\frac{3s_\varepsilon(2-\varepsilon)}{3+s_\varepsilon}}+\left\|\phi_2\right\|_{\frac{3s_\varepsilon(2-\varepsilon)}{3+s_\varepsilon}}\right)^{-\varepsilon}. \qquad (3.31)\end{aligned}$$

and otherwise, if $n\ge 7$ or $\varepsilon>0$, then there holds

$$\begin{aligned}&\left\|f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_1\right)-f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_2\right)-f_\varepsilon'\left(W_{\delta_\varepsilon(t),\xi}\right)\left(\phi_1-\phi_2\right)\right\|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\\&\le C\left\|\phi_1-\phi_2\right\|_{\frac{ns_\varepsilon(2^*-1-\varepsilon)}{n+2s_\varepsilon}}\left(\left\|\phi_1\right\|_{\frac{ns_\varepsilon(2^*-1-\varepsilon)}{n+2s_\varepsilon}}+\left\|\phi_2\right\|_{\frac{ns_\varepsilon(2^*-1-\varepsilon)}{n+2s_\varepsilon}}\right)^{2^*-2-\varepsilon} \qquad (3.32)\end{aligned}$$

By (2.4), (3.28), (3.29), (3.31), and (3.32), for any functions ϕ, ϕ_1, and ϕ_2 in $\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Lambda)$ and for ε small, we get

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{h,s_\varepsilon}\le C\left(\Lambda^{1+\theta_\varepsilon}\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon}^{1+\theta_\varepsilon}+\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon}\right)$$

and

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_1)-T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_2)\right\|_{h,s_\varepsilon}\le C\Lambda^{\theta_\varepsilon}\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon}^{\theta_\varepsilon}\left\|\phi_1-\phi_2\right\|_{h,s_\varepsilon},$$

where $\theta_\varepsilon=1$ in case $n=6$ and $\varepsilon<0$, and $\theta_\varepsilon=2^*-2-\varepsilon$ otherwise, and where C is a positive constant independent of Λ, ε, ξ, t, ϕ, ϕ_1, and ϕ_2. By Lemma 3.2, it follows that if the constant Λ is fixed large enough, then for ε small, for any point ξ in M, and any real number t in $[a,b]$, $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ is a contraction map on the ball $\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Lambda)$ and satisfies $T_{\varepsilon,\delta_\varepsilon(t),\xi}\left(\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Lambda)\right)\subset\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Lambda)$. As a consequence, the map $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ admits a fixed point $\phi_{\delta_\varepsilon(t),\xi}$ in the ball $\mathcal{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Lambda)$. (2.15) then follows from Lemma 3.2, and the regularity of $\phi_{\delta_\varepsilon(t),\xi}$ with respect to ξ and t can be proved by standard arguments involving the implicit function theorem. This ends the proof of Proposition 2.1. □

4. The reduced problem

This section is devoted to the proof of Proposition 2.2. As a first step, in Lemma 4.1 below, we give the asymptotic expansion of $J_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)$ as $\varepsilon\to 0$, where $\delta_\varepsilon(t)$ is as in (2.12). K_n denotes the sharp constant for the embedding of $D^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$, namely

$$K_n=\sqrt{\frac{4}{n(n-2)\,\omega_n^{2/n}}}\,,$$

where ω_n is the volume of the unit n-sphere.

Lemma 4.1. *If $n\ge 5$ and $\delta_\varepsilon(t)$ is as in (2.12), then there holds*

$$\begin{aligned}J_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)=\frac{K_n^{-n}}{n}\Bigg(&1-\frac{(n-2)^2}{8}\varepsilon\ln(|\varepsilon|\,t)-\alpha_n\varepsilon\\&+\frac{2(n-1)|\varepsilon|\,t}{(n-2)(n-4)}\left(h(\xi)-\frac{n-2}{4(n-1)}\mathrm{Scal}_g(\xi)\right)+\mathrm{o}(\varepsilon)\Bigg) \qquad (4.1)\end{aligned}$$

as $\varepsilon \to 0$, C^1-uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$, where

$$\alpha_n = 2^{n-3} (n-2)^2 \frac{\omega_{n-1}}{\omega_n} \int_0^{+\infty} \frac{r^{\frac{n-2}{2}} \ln(1+r)}{(1+r)^n} dr + \frac{(n-2)^2}{4n} \left(1 - n \ln \sqrt{n(n-2)}\right).$$

Proof. We proceed as in Aubin [1]. For any point ξ in M, there holds

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_\xi(r)} d\sigma_g = 1 - \frac{1}{6n} \operatorname{Scal}_g(\xi) r^2 + \mathrm{O}\left(r^4\right)$$

as $r \to 0$, where $|g|$ is the determinant of the components of the metric g in geodesic normal coordinates. Furthermore, by standard properties of the exponential map, the remainder $\mathrm{O}(r^4)$ can be made C^1-uniform with respect to ξ. For any positive real numbers p and q satisfying $p - q > 1$, we set

$$I_p^q = \int_0^{+\infty} \frac{r^q}{(1+r)^p} dr$$

and

$$\widetilde{I}_p^q = \int_0^{+\infty} \frac{r^q \ln(1+r)}{(1+r)^p} dr .$$

If $n \geq 5$, then we can compute

$$\int_M \left|\nabla W_{\delta_\varepsilon(t),\xi}\right|_g^2 dv_g \tag{4.2}$$
$$= \frac{n^{\frac{n-2}{2}} (n-2)^{\frac{n+2}{2}}}{2} \omega_{n-1} I_n^{\frac{n}{2}} \left(1 - \frac{n+2}{6n(n-4)} \operatorname{Scal}_g(\xi) \delta_\varepsilon(t)^2 + \mathrm{o}\left(\delta_\varepsilon(t)^2\right)\right),$$

$$\frac{d}{dt}\left(\int_M \left|\nabla W_{\delta_\varepsilon(t),\xi}\right|_g^2 dv_g\right) \tag{4.3}$$
$$= -\frac{n^{\frac{n-4}{2}} (n-2)^{\frac{n+2}{2}} (n+2)}{6(n-4)} \omega_{n-1} I_n^{\frac{n}{2}} \operatorname{Scal}_g(\xi) \delta_\varepsilon'(t) \delta_\varepsilon(t) + \mathrm{o}\left(\delta_\varepsilon'(t) \delta_\varepsilon(t)\right),$$

and

$$\int_M h W_{\delta_\varepsilon(t),\xi}^2 dv_g = \frac{2n^{\frac{n-4}{2}} (n-1)(n-2)^{\frac{n}{2}}}{n-4} \omega_{n-1} I_n^{\frac{n}{2}} h(\xi) \delta_\varepsilon(t)^2 + \mathrm{o}\left(\delta_\varepsilon(t)^2\right), \tag{4.4}$$

$$\frac{d}{dt}\left(\int_M h W_{\delta_\varepsilon(t),\xi}^2 dv_g\right) = \frac{4n^{\frac{n-4}{2}} (n-1)(n-2)^{\frac{n}{2}}}{n-4} \omega_{n-1} I_n^{\frac{n}{2}} h(\xi) \delta_\varepsilon'(t) \delta_\varepsilon(t) + \mathrm{o}\left(\delta_\varepsilon'(t) \delta_\varepsilon(t)\right) \tag{4.5}$$

as $\varepsilon \to 0$, C^1-uniformly with respect to ξ in M and C^0-uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$. Taking into account that there hold

$$I_{n-\frac{n-2}{2}\varepsilon}^{\frac{n-2}{2}} = I_n^{\frac{n-2}{2}} + \frac{n-2}{2} \widetilde{I}_n^{\frac{n-2}{2}} \varepsilon + \mathrm{O}\left(\varepsilon^2\right)$$

and

$$I_{n-\frac{n-2}{2}\varepsilon}^{\frac{n}{2}} = I_n^{\frac{n}{2}} + \frac{n}{2} \widetilde{I}_n^{\frac{n-2}{2}} \varepsilon + \mathrm{O}\left(\varepsilon^2\right)$$

as $\varepsilon\to0$, we can also compute

$$\begin{aligned}&\frac{1}{2^*-\varepsilon}\int_M W_{\delta_\varepsilon(t),\xi}^{2^*-\varepsilon}dv_g\\&=\frac{(n(n-2))^{\frac{n}{2}-\frac{n-2}{4}\varepsilon}}{2(2^*-\varepsilon)}\omega_{n-1}\delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}\\&\times\left(I_n^{\frac{n-2}{2}}+\frac{n-2}{2}\widetilde{I}_n^{\frac{n-2}{2}}\varepsilon-\frac{1}{6n}I_n^{\frac{n}{2}}\operatorname{Scal}_g(\xi)\delta_\varepsilon(t)^2+\mathrm{o}\left(\delta_\varepsilon(t)^2\right)\right)\\&=\frac{n^{\frac{n-2}{2}}(n-2)^{\frac{n+2}{2}}}{4}\omega_{n-1}\Bigg(I_n^{\frac{n-2}{2}}+\frac{n-2}{2}I_n^{\frac{n-2}{2}}\varepsilon\ln\delta_\varepsilon(t)\\&+\frac{n-2}{2n}\left(n\widetilde{I}_n^{\frac{n-2}{2}}+\left(1-n\ln\sqrt{n(n-2)}\right)I_n^{\frac{n-2}{2}}\right)\varepsilon\\&-\frac{1}{6n}I_n^{\frac{n}{2}}\operatorname{Scal}_g(\xi)\delta_\varepsilon(t)^2+\mathrm{o}\left(\delta_\varepsilon(t)^2\right)\Bigg)\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}&\frac{1}{2^*-\varepsilon}\frac{d}{dt}\left(\int_M W_{\delta_\varepsilon(t),\xi}^{2^*-\varepsilon}dv_g\right)\\&=\frac{(n(n-2))^{\frac{n}{2}-\frac{n-2}{4}\varepsilon}}{2(2^*-\varepsilon)}\omega_{n-1}\delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}\delta_\varepsilon'(t)\\&\times\left(\frac{n-2}{2}I_n^{\frac{n-2}{2}}\frac{\varepsilon}{\delta_\varepsilon(t)}-\frac{1}{3n}I_n^{\frac{n}{2}}\operatorname{Scal}_g(\xi)\delta_\varepsilon(t)+\mathrm{o}(\delta_\varepsilon(t))\right)\\&=\frac{n^{\frac{n-2}{2}}(n-2)^{\frac{n+2}{2}}}{4}\omega_{n-1}\delta_\varepsilon'(t)\Bigg(\frac{n-2}{2}I_n^{\frac{n-2}{2}}\frac{\varepsilon}{\delta_\varepsilon(t)}\\&-\frac{1}{3n}I_n^{\frac{n}{2}}\operatorname{Scal}_g(\xi)\delta_\varepsilon(t)+\mathrm{o}(\delta_\varepsilon(t))\Bigg)\end{aligned}\tag{4.7}$$

as $\varepsilon\to0$, C^1-uniformly with respect to ξ in M and C^0-uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$. By (4.2)–(4.7), taking into account that

$$\frac{n-2}{n}I_n^{\frac{n}{2}}=I_n^{\frac{n-2}{2}}=\frac{\omega_n}{2^{n-1}\omega_{n-1}}\,,$$

we finally get (4.1). □

We can now give the asymptotic expansion as $\varepsilon\to0$ of the function $\widetilde{J}_\varepsilon$ defined as in (2.16).

Lemma 4.2. *If $n\ge6$ and $\delta_\varepsilon(t)$ is as in (2.12), then there holds*

$$\widetilde{J}_\varepsilon(t,\xi)=J_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)+\mathrm{o}(\varepsilon)\tag{4.8}$$

as $\varepsilon\to0$, C^1-uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$.

Proof. We first compute

$$\begin{aligned}\widetilde{J}_\varepsilon(t,\xi)-J_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)&=\int_M\left(\Delta_gW_{\delta_\varepsilon(t),\xi}+hW_{\delta_\varepsilon(t),\xi}-f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)\phi_{\delta_\varepsilon(t),\xi}dv_g\\-\int_M\left(F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)-F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)-f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}\right)dv_g&+\frac12\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_h^2\,,\end{aligned}\tag{4.9}$$

where $F_\varepsilon(u) = \int_0^u f_\varepsilon(s)\,ds$. By Hölder's inequality, we get

$$\left|\int_M \left(\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)\phi_{\delta_\varepsilon(t),\xi}dv_g\right| \\ \le \left\|\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right\|_{\frac{2n}{n+2}}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{2^*}. \quad (4.10)$$

In the proof of Lemma 3.2, we have shown in particular that, for any θ in $(0,1)$, there holds

$$\left\|\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right\|_{\frac{2n}{n+2}} = \mathrm{o}\left(|\varepsilon|^\theta\right) \quad (4.11)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. By (4.10), (4.11), and by Proposition 2.1, for any θ in $(0,1)$, we get

$$\int_M \left(\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)\phi_{\delta_\varepsilon(t),\xi}dv_g = \mathrm{o}\left(|\varepsilon|^{2\theta}\right) \quad (4.12)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. We now estimate the second term in the right hand side of (4.9). As is easily checked, there exists a positive constant C such that for ε small, there holds

$$|F_\varepsilon(x+y) - F_\varepsilon(x) - f_\varepsilon(x)y| \le C|y|^2\left(x^{2^*-2-\varepsilon} + |y|^{2^*-2-\varepsilon}\right)$$

for all positive real numbers x and all real numbers y. By Hölder's inequality, it follows that

$$\left|\int_M \left(F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right) - F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}\right)dv_g\right| \\ \le C\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{2^*-\varepsilon}^2\left(\left\|W_{\delta_\varepsilon(t),\xi}\right\|_{2^*-\varepsilon}^{2^*-2-\varepsilon} + \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{2^*-\varepsilon}^{2^*-2-\varepsilon}\right).$$

By (4.6), by Proposition 2.1, and since $2^* - \varepsilon < s_\varepsilon$, for any θ in $(0,1)$, we then get

$$\int_M \left(F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right) - F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}\right)dv_g = \mathrm{o}\left(|\varepsilon|^{2\theta}\right) \quad (4.13)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. The C^0-uniform estimate (4.8) follows from (4.9), (4.12), (4.13), and Proposition 2.1. Now, we recall that by Proposition 2.1, for ε small, for any point ξ in M and any positive real number t, there holds

$$DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right) = \sum_{i=0}^n \lambda^i_{\delta_\varepsilon(t),\xi}\left\langle Z^i_{\delta_\varepsilon(t),\xi}, \cdot\right\rangle_h \quad (4.14)$$

for some real numbers $\lambda^0_{\delta_\varepsilon(t),\xi}, \dots, \lambda^n_{\delta_\varepsilon(t),\xi}$, where $Z^i_{\delta_\varepsilon(t),\xi}$ is as in (2.11). As a first step in the proof of the C^1-uniform estimate (4.8), we claim that for any θ in $(0,1)$, there holds

$$\sum_{i=0}^n \left|\lambda^i_{\delta_\varepsilon(t),\xi}\right| = \mathrm{O}\left(|\varepsilon|^\theta\right) \quad (4.15)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. In order to prove this claim, we have to estimate $DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right)\left[Z^i_{\delta_\varepsilon(t),\xi}\right]$ as $\varepsilon \to 0$ for $i = 0, \dots, n$. As is easily checked, for $i, j = 0, \dots, n$, there holds

$$\left\langle Z^i_{\delta_\varepsilon(t),\xi}, Z^j_{\delta_\varepsilon(t),\xi}\right\rangle_h \longrightarrow \begin{cases} \|V_i\|^2_{D^{1,2}(\mathbb{R}^n)} & \text{if } i = j, \\ 0 & \text{if } i \ne j, \end{cases} \quad (4.16)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$, where the function V_i is as in (2.9)–(2.10). On the one hand, it follows from (4.14) and (4.16) that for $i = 0, \dots, n$, there holds

$$DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right)\left[Z^i_{\delta_\varepsilon(t),\xi}\right] = \lambda^i_{\delta_\varepsilon(t),\xi}\left\|V_i\right\|^2_{D^{1,2}(\mathbb{R}^n)} + \mathrm{o}\left(\sum_{j=0}^n\left|\lambda^j_{\delta_\varepsilon(t),\xi}\right|\right) \tag{4.17}$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. On the other hand, for ε small, since the function $\phi_{\delta_\varepsilon(t),\xi}$ belongs to $K^\perp_{\delta_\varepsilon(t),\xi}$, we get

$$\begin{aligned}DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right)\left[Z^i_{\delta_\varepsilon(t),\xi}\right] &= \int_M\left(\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)Z^i_{\delta_\varepsilon(t),\xi}dv_g\\&\quad - \int_M\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)Z^i_{\delta_\varepsilon(t),\xi}dv_g\,.\end{aligned} \tag{4.18}$$

By Hölder's inequality, by (4.11) and (4.16), for any θ in $(0,1)$, we then get

$$\begin{aligned}&\left|\int_M\left(\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)Z^i_{\delta_\varepsilon(t),\xi}dv_g\right|\\&\quad\le\left\|\Delta_g W_{\delta_\varepsilon(t),\xi} + hW_{\delta_\varepsilon(t),\xi} - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right\|_{\frac{2n}{n+2}}\left\|Z^i_{\delta_\varepsilon(t),\xi}\right\|_{2^*} = \mathrm{o}\left(\left|\varepsilon\right|^\theta\right)\end{aligned} \tag{4.19}$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. As is easily checked, there exists a positive constant C such that for ε small,

$$\left|f_\varepsilon(x+y) - f_\varepsilon(x)\right| \le C\left|y\right|\left(x^{2^*-2-\varepsilon} + \left|y\right|^{2^*-2-\varepsilon}\right)$$

for all positive real numbers x and all real numbers y. By Hölder's inequality, it follows that

$$\begin{aligned}&\left|\int_M\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)Z^i_{\delta_\varepsilon(t),\xi}dv_g\right|\\&\qquad\le C\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{2^*-\varepsilon}\left\|Z^i_{\delta_\varepsilon(t),\xi}\right\|_{2^*-\varepsilon}\left(\left\|W_{\delta_\varepsilon(t),\xi}\right\|^{2^*-2-\varepsilon}_{2^*-\varepsilon} + \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^{2^*-2-\varepsilon}_{2^*-\varepsilon}\right).\end{aligned}$$

By (4.6), (4.16), by Proposition 2.1, and since $2^* - \varepsilon < s_\varepsilon$, for any θ in $(0,1)$, we then get

$$\int_M\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right) - f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right)Z^i_{\delta_\varepsilon(t),\xi}dv_g = \mathrm{o}\left(\left|\varepsilon\right|^\theta\right) \tag{4.20}$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. (4.15) follows from (4.17)–(4.20). In order to get the C^1-uniform estimate (4.8) with respect to t, we can easily check that there holds

$$\frac{d\left(W_{\delta_\varepsilon(t),\xi}\right)}{dt} = \frac{1}{2t}Z^0_{\delta_\varepsilon(t),\xi}\,, \tag{4.21}$$

and we then compute

$$\begin{aligned}
&\frac{d\big(\widetilde{J}_\varepsilon\left(t,\xi\right)\big)}{dt}-\frac{d\big(J_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\big)}{dt}\\
&=\frac{1}{2t}\bigg(\int_M\left(\Delta_g Z^0_{\delta_\varepsilon(t),\xi}+hZ^0_{\delta_\varepsilon(t),\xi}-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)Z^0_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}dv_g\\
&\quad-\int_M\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)-f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}\right)Z^0_{\delta_\varepsilon(t),\xi}dv_g\bigg)\\
&\quad+DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)\left[\frac{d\left(\phi_{\delta_\varepsilon(t),\xi}\right)}{dt}\right].
\end{aligned}\tag{4.22}$$

Similarly, for $i=1,\dots,n$, we get

$$\left.\frac{\partial\big(W_{\delta_\varepsilon(t),\exp_\xi(y)}\big)}{\partial y_i}\right|_{y=0}=\frac{1}{\delta_\varepsilon\left(t\right)}Z^i_{\delta_\varepsilon(t),\xi}+R_{\delta_\varepsilon(t),\xi}\,,\tag{4.23}$$

where $\left\|R_{\delta_\varepsilon(t),\xi}\right\|_{h,s_\varepsilon}=\mathrm{o}\left(|\varepsilon|^{\frac{\theta}{2}}\right)$ as $\varepsilon\to0$ for all θ in $(0,1)$, and we then compute

$$\begin{aligned}
&\left.\frac{\partial\big(\widetilde{J}_\varepsilon\left(t,\exp_\xi\left(y\right)\right)\big)}{\partial y_i}\right|_{y=0}-\left.\frac{\partial\big(J_\varepsilon\big(W_{\delta_\varepsilon(t),\exp_\xi(y)}\big)\big)}{\partial y_i}\right|_{y=0}\\
&=\frac{1}{\delta_\varepsilon\left(t\right)}\bigg(\int_M\left(\Delta_g Z^i_{\delta_\varepsilon(t),\xi}+hZ^i_{\delta_\varepsilon(t),\xi}-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)Z^i_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}dv_g\\
&\quad-\int_M\left(f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)-f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}\right)Z^i_{\delta_\varepsilon(t),\xi}dv_g\bigg)\\
&\quad+DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)\left[\left.\frac{\partial\big(\phi_{\delta_\varepsilon(t),\exp_\xi(y)}\big)}{\partial y_i}\right|_{y=0}\right]+\mathrm{o}\left(|\varepsilon|^{\frac{3\theta}{2}}\right)
\end{aligned}\tag{4.24}$$

as $\varepsilon\to0$. We begin with estimating the first terms in the right hand sides of (4.22) and (4.24). By Hölder's inequality, for $i=0,\dots,n$, we get

$$\begin{aligned}
&\left|\int_M\left(\Delta_g Z^i_{\delta_\varepsilon(t),\xi}+hZ^i_{\delta_\varepsilon(t),\xi}-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)Z^i_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}dv_g\right|\\
&\qquad\le\left\|\Delta_g Z^i_{\delta_\varepsilon(t),\xi}+hZ^i_{\delta_\varepsilon(t),\xi}-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)Z^i_{\delta_\varepsilon(t),\xi}\right\|_{\frac{2n}{n+2}}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{2^*}
\end{aligned}\tag{4.25}$$

as $\varepsilon\to0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}^*_+$. Proceeding in the same way as in the proof of Lemma 3.2, we can prove that for $i=0,\dots,n$ and for any θ in $(0,1)$, there holds

$$\left\|\Delta_g Z^i_{\delta_\varepsilon(t),\xi}+hZ^i_{\delta_\varepsilon(t),\xi}-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)Z^i_{\delta_\varepsilon(t),\xi}\right\|_{\frac{2n}{n+2}}=\mathrm{o}\left(|\varepsilon|^{\theta}\right)\tag{4.26}$$

as $\varepsilon\to0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}^*_+$. By (4.25), (4.26), and Proposition 2.1, for $i=0,\dots,n$ and for any θ in $(0,1)$, we get

$$\int_M\left(\Delta_g Z^i_{\delta_\varepsilon(t),\xi}+hZ^i_{\delta_\varepsilon(t),\xi}-f'_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)Z^i_{\delta_\varepsilon(t),\xi}\right)\phi_{\delta_\varepsilon(t),\xi}dv_g=\mathrm{o}\left(|\varepsilon|^{2\theta}\right)\tag{4.27}$$

as $\varepsilon\to0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}^*_+$. We then estimate the second terms in the right hand sides of (4.22) and (4.24). By (3.30) and by

Hölder's inequality, for $i = 0, \dots, n$, we get that if $n = 6$ and $\varepsilon < 0$, then there holds

$$\left| \int_M \left(f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} \right) - f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - f'_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \phi_{\delta_\varepsilon(t),\xi} \right) Z^i_{\delta_\varepsilon(t),\xi} dv_g \right| \\ \le C \left\| \phi_{\delta_\varepsilon(t),\xi} \right\|^2_{3-\varepsilon} \left(\left\| W^{-\varepsilon}_{\delta_\varepsilon(t),\xi} Z^i_{\delta_\varepsilon(t),\xi} \right\|_{\frac{3-\varepsilon}{1-\varepsilon}} + \left\| \phi_{\delta_\varepsilon(t),\xi} \right\|^{-\varepsilon}_{3-\varepsilon} \left\| Z^i_{\delta_\varepsilon(t),\xi} \right\|_{3-\varepsilon} \right). \quad (4.28)$$

and otherwise, if $n \ge 7$ or $\varepsilon > 0$, then there holds

$$\left| \int_M \left(f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} \right) - f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - f'_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \phi_{\delta_\varepsilon(t),\xi} \right) Z^i_{\delta_\varepsilon(t),\xi} dv_g \right| \\ \le C \left\| \phi_{\delta_\varepsilon(t),\xi} \right\|^2_{2^*-\varepsilon} \left\| W^{2^*-3-\varepsilon}_{\delta_\varepsilon(t),\xi} Z^i_{\delta_\varepsilon(t),\xi} \right\|_{\frac{2^*-\varepsilon}{2^*-2-\varepsilon}} \quad (4.29)$$

One can easily check that for $i = 0, \dots, n$, there holds

$$\left\| W^{2^*-3-\varepsilon}_{\delta_\varepsilon(t),\xi} Z^i_{\delta_\varepsilon(t),\xi} \right\|_{\frac{2^*-\varepsilon}{2^*-2-\varepsilon}} \longrightarrow \left\| U^{2^*-3} V_i \right\|_{L^{\frac{2^*}{2^*-2}}(\mathbb{R}^n)} < +\infty \quad (4.30)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}^*_+$. By (4.6), (4.16), (4.28), (4.29), (4.30), by Proposition 2.1, and since $2^* - \varepsilon < s_\varepsilon$, for $i = 0, \dots, n$ and for any θ in $(0, 1)$, we get

$$\int_M \left(f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} \right) - f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - f'_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \phi_{\delta_\varepsilon(t),\xi} \right) Z^i_{\delta_\varepsilon(t),\xi} dv_g = \mathrm{o}\left(|\varepsilon|^{2\theta} \right) \quad (4.31)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}^*_+$. It remains to estimate the last terms in the right hand sides of (4.22) and (4.24). For ε small, by (4.14) and since the function $\phi_{\delta_\varepsilon(t),\xi}$ belongs to $K^\perp_{\delta_\varepsilon(t),\xi}$, we get

$$DJ_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} \right) \left[\frac{d\left(\phi_{\delta_\varepsilon(t),\xi} \right)}{dt} \right] = \sum_{j=0}^n \lambda^j_{\delta_\varepsilon(t),\xi} \left\langle Z^j_{\delta_\varepsilon(t),\xi}, \frac{d\left(\phi_{\delta_\varepsilon(t),\xi} \right)}{dt} \right\rangle_h \\ = - \sum_{j=0}^n \lambda^j_{\delta_\varepsilon(t),\xi} \left\langle \frac{d\left(Z^j_{\delta_\varepsilon(t),\xi} \right)}{dt}, \phi_{\delta_\varepsilon(t),\xi} \right\rangle_h \quad (4.32)$$

and

$$DJ_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} \right) \left[\left. \frac{\partial\left(\phi_{\delta_\varepsilon(t),\exp_\xi(y)} \right)}{\partial y_i} \right|_{y=0} \right] = \sum_{j=0}^n \lambda^j_{\delta_\varepsilon(t),\xi} \left\langle Z^j_{\delta_\varepsilon(t),\xi}, \left. \frac{\partial\left(\phi_{\delta_\varepsilon(t),\exp_\xi(y)} \right)}{\partial y_i} \right|_{y=0} \right\rangle_h \\ = - \sum_{j=0}^n \lambda^j_{\delta_\varepsilon(t),\xi} \left\langle \left. \frac{\partial\left(Z^j_{\delta_\varepsilon(t),\exp_\xi(y)} \right)}{\partial y_i} \right|_{y=0}, \phi_{\delta_\varepsilon(t),\xi} \right\rangle_h \quad (4.33)$$

for $i = 1, \dots, n$. As is easily checked, there hold

$$\left\| \frac{d\left(Z^j_{\delta_\varepsilon(t),\xi} \right)}{dt} \right\|_h \longrightarrow \frac{1}{2t} \left\| \left. \frac{d\left(\delta^{\frac{2-n}{2}} V_j \left(\delta^{-1} y \right) \right)}{d\delta} \right|_{\delta=1} \right\|_{D^{1,2}(\mathbb{R}^n)} = \mathrm{O}\left(1 \right), \quad (4.34)$$

and

$$\left\| \left. \frac{\partial\left(Z^j_{\delta_\varepsilon(t),\exp_\xi(y)} \right)}{\partial y_i} \right|_{y=0} \right\|_h \sim \frac{1}{\delta_\varepsilon(t)} \left\| \frac{\partial V_j}{\partial y_i} \right\|_{D^{1,2}(\mathbb{R}^n)} = \mathrm{O}\left(\frac{1}{\delta_\varepsilon(t)} \right) \quad (4.35)$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. By (4.32), (4.34), and by Proposition 2.1, for any θ in $(0,1)$, we get

$$DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)\left[\frac{d\left(\phi_{\delta_\varepsilon(t),\xi}\right)}{dt}\right]=\mathrm{o}\left(|\varepsilon|^\theta\sum_{j=0}^n\left|\lambda^j_{\delta_\varepsilon(t),\xi}\right|\right) \tag{4.36}$$

Similarly, by (4.33), (4.35), and Proposition 2.1, for $i=1,\dots,n$ and for any θ in $(0,1)$, we get

$$DJ_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}\right)\left[\left.\frac{\partial\left(\phi_{\delta_\varepsilon(t),\exp_\xi(y)}\right)}{\partial y_i}\right|_{y=0}\right]=\mathrm{o}\left(|\varepsilon|^{\frac{2\theta-1}{2}}\sum_{j=0}^n\left|\lambda^j_{\delta_\varepsilon(t),\xi}\right|\right) \tag{4.37}$$

as $\varepsilon \to 0$, uniformly with respect to ξ in M and to t in compact subsets of $\mathbb{R}_+^*$. By (4.15), (4.22), (4.24), (4.27), (4.31), (4.36), and (4.37), we then get the C^1-uniform estimate (4.8). This ends the proof of Lemma 4.2. □

By Lemmas 4.1 and 4.2, we get (2.17). We now prove the second part of Proposition 2.2.

End of proof of Proposition 2.2. It remains to prove that given two real numbers a and b satisfying $0<a<b$, for ε small, if $(t_\varepsilon,\xi_\varepsilon)\in[a,b]\times M$ is a critical point of the function $\widetilde{J}_\varepsilon$, then $W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of (2.5). By (4.14), (4.17), (4.21), and (4.36), we get

$$\left.\frac{d\left(\widetilde{J}_\varepsilon\left(t_\varepsilon,\xi_\varepsilon\right)\right)}{dt}\right|_{t=t_\varepsilon}=\frac{1}{2t_\varepsilon}\|V_0\|^2_{D^{1,2}(\mathbb{R}^n)}\left(\lambda^0_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\mathrm{o}\left(\sum_{j=0}^n\left|\lambda^j_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}\right|\right)\right)$$

as $\varepsilon\to0$. Similarly, by (4.14), (4.17), (4.23), and (4.37), for $i=1,\dots,n$, we get

$$\left.\frac{\partial\left(\widetilde{J}_\varepsilon\left(t_\varepsilon,\exp_{\xi_\varepsilon}(y)\right)\right)}{\partial y_i}\right|_{y=0}=\frac{1}{\delta_\varepsilon\left(t_\varepsilon\right)}\|V_i\|^2_{D^{1,2}(\mathbb{R}^n)}\left(\lambda^i_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\mathrm{o}\left(\sum_{j=0}^n\left|\lambda^j_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}\right|\right)\right)$$

as $\varepsilon\to0$. If $(t_\varepsilon,\xi_\varepsilon)$ is a critical point of $\widetilde{J}_\varepsilon$ for ε small, then it follows that

$$\lambda^i_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}=\mathrm{o}\left(\sum_{j=0}^n\left|\lambda^j_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}\right|\right)$$

as $\varepsilon\to0$, for $i=0,\dots,n$, and thus, for ε small, there holds

$$\lambda^0_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}=\dots=\lambda^n_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}=0.$$

By (4.14), we then get that $W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a critical point of the functional J_ε for ε small. This ends the proof of Proposition 2.2. □

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